FAN-EXTENSIONS IN FRAGILE MATROIDS

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ABSTRACT. If $\mathcal{S}$ is a set of matroids, then the matroid M is $\mathcal{S}$-fragile if, for every element $e \in E(M)$, either $M \backslash e$ or M/e has no minor isomorphic to a member of $\mathcal{S}$. Excluded-minor characterizations often depend, implicitly or explicitly, on understanding classes of fragile matroids. In certain cases, when $\mathcal{M}$ is a minor-closed class of $\mathcal{S}$-fragile matroids, and $N \in \mathcal{M}$, the only members of $\mathcal{M}$ that contain N as a minor are obtained from N by increasing the length of fans. We prove that if this is the case, then we can certify it with a finite case-analysis. The analysis involves examining matroids that are at most two elements larger than N.

1. INTRODUCTION

Let $\mathcal{S}$ be a set of matroids. When we say that a matroid has an $\mathcal{S}$-*minor*, we mean that it has a minor isomorphic to a member of $\mathcal{S}$. The matroid M is $\mathcal{S}$-*fragile* if, for every element $e \in E(M)$, either $M \backslash e$ or M/e has no $\mathcal{S}$-minor. Note that every minor of an $\mathcal{S}$-fragile matroid is also $\mathcal{S}$-fragile. Fragility has been studied at various times under different names: Oxley examined non-binary $\{U_{2,4}\}$-fragile matroids [11]; Truemper proved a constructive characterization of a class of binary $\{F_7, F_7^*\}$-fragile matroids [13]; and Kingan and Lemos have made a study of binary $\{F_7, F_7^*, M^*(K_{3,3}), M^*(K_5)\}$-fragile matroids [6, 7].

Our study of fragile matroids is motivated by the goal of finding new excluded-minor characterizations. The matroid S is a *strong stabilizer* for the partial field $\mathbb{P}$, if, roughly speaking, every $\mathbb{P}$-representation of S extends uniquely to a $\mathbb{P}$-representation of any $\mathbb{P}$-representable matroid that contains S as a minor. More information on strong stabilizers can be found in [3] or [12]. Understanding $\mathcal{S}$-fragile matroids, where $\mathcal{S}$ is a set of strong stabilizers, has been important in excluded-minor characterizations. For example, $U_{2,4}$ is a strong stabilizer for both GF(4) and the near-regular partial field. The excluded-minor characterizations of GF(4)-representable [4] and near-regular matroids [5] both implicitly use the fact that a non-binary 3-connected GF(4)-representable matroid is $\{U_{2,4}\}$-fragile if and only if it is a whirl. Geelen, Gerards, and Whittle conjecture that, for any prime power q, and any GF(q)-representable matroid N, there is an integer k such that every GF(q)-representable $\{N\}$-fragile matroid has branch width at most k

[2, Conjecture 5.9]. In another application of the link between fragility and excluded-minor results, Mayhew, Van Zwam, and Whittle [8] have shown that this conjecture implies that there are only finitely many excluded minors for GF(5)-representability.

In our study of $\mathcal{S}$-fragile matroids, $\mathcal{S}$ will typically be a set of excluded minors for representability over some partial field. This allows us to assume certain properties of $\mathcal{S}$. In particular, we can assume that the members of $\mathcal{S}$ are 3-connected and contain at least four elements. We say that the matroid N is 3-*connected up to series and parallel sets* if it is connected, and $\min\{r(X), r(Y)\} = 1$ or $\min\{r^*(X), r^*(Y)\} = 1$ for every 2-separation (X, Y) of N. The next result follows immediately from [6, Proposition 3.1].

Proposition 1.1. *Let $\mathcal{S}$ be a collection of 3-connected matroids, each of which has at least four elements. Assume that N is $\mathcal{S}$-fragile and that N has an $\mathcal{S}$-minor. Then N is 3-connected up to series and parallel sets.*

It is a fairly easy exercise to show that wheels and whirls are representable over every partial field, except that whirls are not representable over GF(2) and the regular partial field. Therefore we will henceforth assume that $\mathcal{S}$ contains no wheels or whirls. This implies that any matroid with an $\mathcal{S}$-minor is neither a wheel nor a whirl.

A *fan* is a sequence of elements, where consecutive sets of three elements alternately form triads and triangles. In some cases, the only way to construct $\mathbb{P}$-representable $\mathcal{S}$-fragile matroids by building from a matroid N is to increase the length of fans in N. For example, Figure 1(i) shows the rank-6 binary matroid N_{12}. This matroid is obtained by gluing three copies of $M(K_4)$ to F_7 along three lines that contain a common point p, and then deleting the points of intersection, apart from three that lie in a common line avoiding p. The matroid in Figure 1(ii) is obtained from N_{12} by lengthening the fan (u_1, u_2, u_3, u_4) to $(u_1, a, b, u_2, u_3, u_4)$, and by lengthening (v_1, v_2, v_3, v_4) to (v_1, v_2, v_3, v_4, c). Any matroid obtained by lengthening fans in this way is a *fan-extension* of N_{12}. (We delay the formal definition of fan-extensions until Section 2.) Any 3-connected binary matroid that is $\{F_7, F_7^*\}$-fragile and contains N_{12} as a minor is obtained from N_{12} by lengthening the three disjoint 4-element fans. The resulting family of matroids is essentially the same as the family $\mathcal{F}_1(m, n, r)$, as described in [6]. This result, and other applications of our main theorem, will be described in Section 4. These applications bring the excluded-minor characterisations for matroids representable over the partial fields $\mathbb{H}_5$ and $\mathbb{U}_2$ within reach.

Suppose we are given a minor-closed class, $\mathcal{M}$, and a matroid $N \in \mathcal{M}$. We would like to know whether $M \in \mathcal{M}$ being 3-connected with N as a minor implies that M is a fan-extension of N. Our main theorem allows us to use a finite case-analysis to check whether this implication is true. From now on we make no mention of fragility. Theorem 1.2 instead uses the conditions implied by Proposition 1.1, and the assumption that $\mathcal{S}$ does not contain any wheels or whirls.
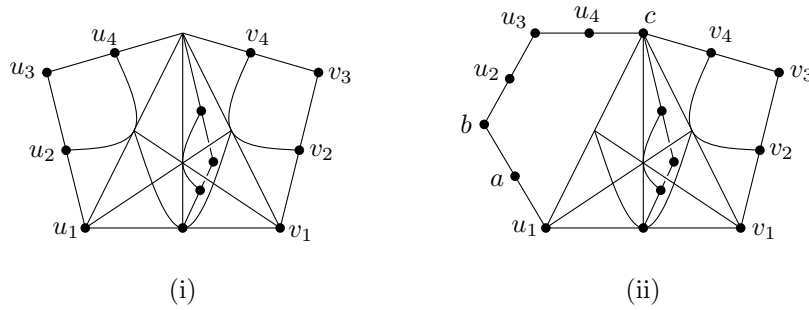
FIGURE 1. N_{12}, and one of its fan-extensions.

Theorem 1.2. *Let $\mathcal{M}$ be a set of matroids closed under isomorphism and minors. Let $N \in \mathcal{M}$ be a 3-connected matroid such that $|E(N)| \geq 4$ and N is neither a wheel nor a whirl. Assume that any member of $\mathcal{M}$ with N as a minor is 3-connected up to series and parallel sets. If there is a 3-connected matroid in $\mathcal{M}$ with N as a minor that is not a fan-extension of N, then there exists such a matroid, M, satisfying $|E(M)| - |E(N)| \leq 2$.*

This is restated as Theorem 6.10 later in the paper. The assumptions on $\mathcal{M}$ and N are justified if $\mathcal{M}$ is a minor-closed class of $\mathcal{S}$-fragile matroids, and N is a member of $\mathcal{M}$ with an $\mathcal{S}$-minor, where we make the assumption that the members of $\mathcal{S}$ are 3-connected with at least four elements, and $\mathcal{S}$ contains no wheel or whirl.

To see that the bound $|E(M)| - |E(N)| \leq 2$ is best possible, we let G be one of the graphs drawn schematically in Figure 2. If G is the graph on the left, let $N = M(G) \backslash x \backslash y$. If G is the graph on the right, let $N = M(G)/x \backslash y$. In either case, let $M = M(G)$. Now set $\mathcal{M}$ to be the class containing all minors of M and their isomorphs. We note that M is not a fan-extension of N relative to the fans (u_1, u_2, u_3) and (v_1, v_2, v_3). However, any member of $\mathcal{M}$ that has N as a minor and that is at most one element larger than N is a fan-extension.
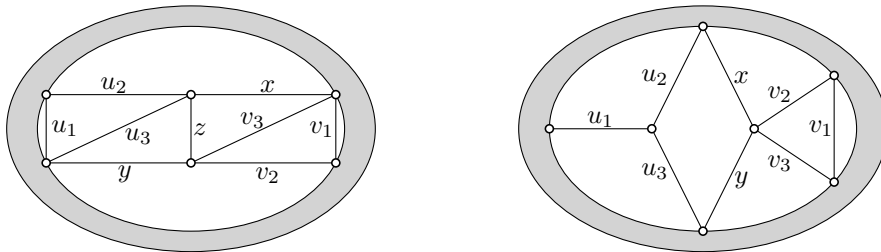

FIGURE 2. Schematic drawings of two graphs.

In Section 2 we state some definitions, including that of a fan-extension. Section 3 is dedicated to an alternative formulation of fan-extensions, based

upon the idea of gluing wheels to a 'core' matroid. We use this alternative formulation in Section 4, where we sketch some applications of our theorem to binary $\{F_7, F_7^*\}$-fragile matroids and $\{U_{2,5}, U_{3,5}\}$-fragile matroids that are representable over the partial field $\mathbb{H}_5$. The proof of the main theorem is contained in Sections 5 and 6.

2. Fan-extensions

Any unexplained terminology or notation that we use can be found in Oxley [9].

Definition 2.1. Let M be a matroid. A *fan* of M is an ordered sequence, $(e_1, \ldots, e_n)$, of $n \geq 3$ distinct elements such that

$$\{e_1, e_2, e_3\}, \{e_2, e_3, e_4\}, \ldots, \{e_{n-2}, e_{n-1}, e_n\}$$

is an alternating sequence of triangles and triads.

If $\{e_1, e_2, e_3\}$ is a triangle, then the elements with odd indices in $(e_1, \ldots, e_n)$ are *spoke* elements, and elements with even indices are *rim* elements. These labels are reversed if $\{e_1, e_2, e_3\}$ is a triad. We sometimes blur the distinction between ordered and unordered sets where doing so creates no problems. So for example, we may talk about fans being disjoint. Note that $(e_1, \ldots, e_n)$ is a fan of M if and only if it is a fan of M^*. If $1 < i < n$, then e_i is an *internal* element of the fan, otherwise it is a *terminal* element. In a 3-connected matroid with at least five elements, no triangle can be a triad, so the partitioning into spoke and rim elements is unambiguous. In particular, in a 3-connected matroid with at least four elements that is not a whirl, an element in a fan is either a spoke element or a rim element, and not both. We frequently replace the ordering $(e_1, \ldots, e_n)$ with $(e_n, \ldots, e_1)$. We call this process *reversing*. A *contiguous subsequence* of $(e_1, \ldots, e_n)$ is a subsequence of the form $(e_s, e_{s+1}, \ldots, e_{t-1}, e_t)$, where $1 \leq s \leq t \leq n$. If $F = (e_1, \ldots, e_n)$ and $F' = (e_1', \ldots, e_m')$ are two sequences, then we say that F is *consistent* with F' if $(e_1, \ldots, e_n)$ is a subsequence (not necessarily contiguous) of either $(e_1', \ldots, e_m')$ or $(e_m', \ldots, e_1')$. If $F = (e_1, \ldots, e_n)$ is an ordered sequence of elements, and $X \subseteq \{e_1, \ldots, e_n\}$, then $F - X$ is the subsequence produced from F by omitting all elements in X. As usual, we abbreviate the singleton set $\{x\}$ to x. We say that the fan $F = (e_1, \ldots, e_n)$ is *maximal* if there is no fan $(e_1', \ldots, e_m')$ such that $\{e_1', \ldots, e_m'\}$ properly contains $\{e_1, \ldots, e_n\}$.

Now we formally define fan-extensions. To avoid disrupting the exposition, we will relegate some technical lemmas until later sections. Let M be a 3-connected matroid with a fan $(e_1, \ldots, e_n)$, where $n \geq 4$. If e_1 is a spoke element, and $M \backslash e_1$ is 3-connected, then clearly $(e_2, \ldots, e_n)$ is a fan of $M \backslash e_1$, and M is said to be obtained from $M \backslash e_1$ by a *fan-lengthening* move on this fan. Similarly, if e_1 is a rim element, and M/e_1 is 3-connected, then M is obtained from M/e_1 by a fan-lengthening move on $(e_2, \ldots, e_n)$. If $n \geq 5$, and e_i is a rim element, where $1 \leq i \leq n-1$, and $M/e_i \backslash e_{i+1}$ is 3-connected,

then Proposition 5.2 shows $(e_1, \ldots, e_{i-1}, e_{i+2}, \ldots, e_n)$ is a fan of $M/e_i \backslash e_{i+1}$, and M is said to be obtained from $M/e_i \backslash e_{i+1}$ by a fan-lengthening move on this fan. Note that M is obtained by a fan-lengthening move on M' if and only if M^* is obtained by a fan-lengthening move on $(M')^*$ (applied to the reversed fan, in the case that $|E(M)| = |E(M')| + 2$). Moreover, M is necessarily 3-connected.

Let N be a 3-connected matroid with at least four elements. Let $\mathcal{F}_N$ be a collection of pairwise disjoint fans in N. Note that we do not require fans in $\mathcal{F}_N$ to be maximal. A fan in N could potentially contain many fans in $\mathcal{F}_N$ as subsequences. If M' has N as a minor, and $\mathcal{F}'$ is a family of fans of M', then we say that $\mathcal{F}'$ is a *covering family* of M' (relative to N and $\mathcal{F}_N$) if the following conditions are satisfied:

(i) the fans in $\mathcal{F}'$ are pairwise disjoint,
(ii) $|\mathcal{F}'| = |\mathcal{F}_N|$,
(iii) for every $F_N \in \mathcal{F}_N$, there is a fan $F' \in \mathcal{F}'$ such that F_N is consistent with F',
(iv) every element in $E(M') - E(N)$ is contained in one of the fans in $\mathcal{F}'$.

Observe that the fan F' in condition (iii) may contain elements of $E(N)$ that are not in F_N. If we reverse any fan in a covering family, we obtain another covering family. Informally, a fan-extension of N is obtained by finding a covering family, applying a fan-lengthening move to one of the fans in that family, and then reiterating this process. More formally, we have:

Definition 2.2. We define N to be a fan-extension of N. We recursively define the set of fan-extensions of N (relative to $\mathcal{F}_N$) to be the smallest set satisfying the following condition:

- if M' is a fan-extension of N, and $\mathcal{F}'$ is a covering family of M' containing the fan F', then any matroid obtained from M' by a fan-lengthening move on F' is a fan-extension of N (relative to $\mathcal{F}_N$).

If M is obtained from M' by lengthening F' into the fan F, then $(\mathcal{F}' - \{F'\}) \cup \{F\}$ is a covering family of M. Therefore the next result follows easily from the definition.

Proposition 2.3. *Let N be a 3-connected matroid with at least four elements and let $\mathcal{F}_N$ be a collection of pairwise disjoint fans in N. If M is a fan-extension of N, then M is 3-connected, has N as a minor, and has a covering family.*

Note that M is a fan-extension of N relative to $\mathcal{F}_N$ if and only if M^* is a fan-extension of N^* relative to $\mathcal{F}_N$. The converse of Proposition 2.3 need not hold. In Lemma 6.11, we essentially construct a 3-connected matroid that is not a fan-extension, although it does have N as a minor and a covering family. However, we will have occasion to use the partial converse in Corollary 2.5. If $\mathcal{F}$ is a covering family of M, then we say that $\mathcal{F}$ admits a *fan-shortening move* if M is obtained from M' by using a fan-lengthening

move on $F' \in \mathcal{F}'$ to produce the fan F, where $F \in \mathcal{F}$ and M' has N as a minor.

Proposition 2.4. *Assume N is neither a wheel nor a whirl, and that no fan in N contains two distinct fans in $\mathcal{F}_N$ (considered as unordered sets). Let M be a 3-connected matroid with N as a minor, and assume that every minor of M that has N as a minor is 3-connected up to series and parallel sets. Let $\mathcal{F}$ be a covering family of M. If $M \neq N$, then $\mathcal{F}$ admits a fan-shortening move.*

Proof. Let $(e_1, \ldots, e_n)$ be an arbitrary fan in $\mathcal{F}$, and assume that the internal element e_i belongs to $E(M) - E(N)$. By duality, we can assume that e_i is a rim element. If N is a minor of $M \backslash e_i$, then $n \leq 4$, for otherwise $M \backslash e_i$ contains a triangle that contains a series pair, and this contradicts the hypotheses of the proposition (see Proposition 6.2). But if $n \leq 4$, the three elements in $\{e_1, e_2, e_3, e_4\} - e_i$ all belong to a fan in $\mathcal{F}_N$, and as this set contains a series pair in $M \backslash e_i$, it follows that N is not cosimple, a contradiction. Therefore N is a minor of M / e_i. As $\{e_{i-1}, e_{i+1}\}$ is a parallel pair in M / e_i, we can reverse $(e_1, \ldots, e_n)$ as necessary, and assume that N is a minor of $M / e_i \backslash e_{i+1}$. Thus $n \geq 5$, as $(e_1, \ldots, e_n)$ must contain three elements from $E(N)$. Proposition 6.4 implies that $M / e_i \backslash e_{i+1}$ is 3-connected, and we can set M' to be $M / e_i \backslash e_{i+1}$. Now Proposition 5.2 implies $(e_1, \ldots, e_{i-1}, e_{i+2}, \ldots, e_n)$ is a fan of M'. As M is obtained from M' by performing a fan-lengthening move on this fan, we are done. Therefore we will now assume that the internal elements of fans in $\mathcal{F}$ all belong to $E(N)$.

As $M \neq N$, we can reverse as necessary, and let $(e_1, \ldots, e_n)$ be a fan in $\mathcal{F}$ where e_1 belongs to $E(M) - E(N)$. By duality, we can assume that e_1 is a spoke element. If $M \backslash e_1$ is 3-connected, then we set M' to be $M \backslash e_1$ and we are done. Therefore we assume that $M \backslash e_1$ is not 3-connected, and thus contains a series pair. Now e_1 is contained in a triad, T^*, of M. Orthogonality with the triangle $\{e_1, e_2, e_3\}$ means that T^* contains e_2 or e_3. Let x be the element in $T^* - \{e_1, e_2, e_3\}$. Because $(e_1, \ldots, e_n)$ contains at least three elements of $E(N)$, $n \geq 4$, so e_2 and e_3 are internal elements, and hence belong to $E(N)$. As either $\{x, e_2\}$ or $\{x, e_3\}$ is a series pair in $M \backslash e_1$, we see N is a minor of $M \backslash e_1 / x$. Because $\mathcal{F}$ is a covering family, x is contained in a fan in $\mathcal{F}$. Because x is not in $E(N)$, it is not an internal element, so x is a terminal element of a fan in $\mathcal{F}$. Orthogonality with T^* shows that it is a rim element. It cannot be the case that x is in $(e_1, \ldots, e_n)$, for then $x = e_n$, so this fan would contain a triad that does not consist of three consecutive elements. The dual of Proposition 5.6 shows that this is a contradiction. Assume that $x = f_m$, where $(f_1, \ldots, f_m)$ is a fan in $\mathcal{F}$. If M / f_m is 3-connected we are done, so we assume f_m is in a triangle, T. This triangle must contain either f_{m-2} or f_{m-1}, and an element from T^*. Orthogonality with the triad $\{e_2, e_3, e_4\}$ shows that T cannot contain e_2 or e_3, so it contains e_1.

If $T = \{f_{m-1}, f_m, e_1\}$ and $T^* = \{f_m, e_1, e_2\}$, then $(f_1, \ldots, f_m, e_1, \ldots, e_n)$ is a fan of M that contains two fans in $\mathcal{F}_N$. Let $(x_1, \ldots, x_p)$ be the subsequence obtained from $(f_1, \ldots, f_m, e_1, \ldots, e_n)$ by omitting the elements not in $E(N)$. (Any such elements have to be in $\{f_1, f_m, e_1, e_n\}$.) It follows from [9, Corollary 8.2.5] that any three consecutive elements in $(x_1, \ldots, x_p)$ are 3-separating in N, and therefore form either a triangle or a triad in N. It is not too difficult to see, using orthogonality, that either $(x_1, \ldots, x_p)$ is a fan in N, or $\min\{r_N(\{x_1, \ldots, x_p\}), r_N^*(\{x_1, \ldots, x_p\})\} \leq 2$. In the former case, N has a fan that contains two distinct fans from $\mathcal{F}_N$, so we have a contradiction to the hypotheses. Therefore $\{x_1, \ldots, x_p\}$ is a line in either N or N^*. However, $\{e_2, e_3, e_4\}$ is a triad of M, and of N also (since $\{e_2, \ldots, e_{n-1}\} \subseteq E(N)$ and $\{e_2, \ldots, e_n\}$ contains at least three elements of $E(N)$). Similarly, $\{f_{m-3}, f_{m-2}, f_{m-1}\}$ is a triangle of N, so $\{f_{m-3}, f_{m-2}, f_{m-1}, e_2, e_3, e_4\}$ cannot be contained in a line of N or N^*.

This disposes of the case that $T = \{f_{m-1}, f_m, e_1\}$ and $T^* = \{f_m, e_1, e_2\}$. Next we assume that $T \neq \{f_{m-1}, f_m, e_1\}$. A symmetrical argument will then deal with the case that $T^* \neq \{f_m, e_1, e_2\}$. Because T is not $\{f_{m-1}, f_m, e_1\}$, it is $\{f_{m-2}, f_m, e_1\}$ instead. This means that $m = 4$, for otherwise we violate orthogonality between T and $\{f_{m-4}, f_{m-3}, f_{m-2}\}$. If $T^* = \{f_m, e_1, e_2\}$, then $(f_1, f_3, f_2, f_4, e_1, e_2, \ldots, e_n)$ is a fan of M that contains two fans of $\mathcal{F}_N$. If T^* is not $\{f_m, e_1, e_2\}$, then it is $\{f_m, e_1, e_3\}$, which implies that $n = 4$. In this case $(f_1, f_3, f_2, f_4, e_1, e_3, e_2, e_4)$ is a fan of M that contains two fans in $\mathcal{F}_N$. In either case, we can obtain a contradiction to the hypotheses of the proposition, exactly as in the previous paragraph. This completes the proof. $\qquad\square$

Corollary 2.5. *Let M and N be as described in* Proposition 2.4. *If M contains a covering family, then it is a fan-extension of N relative to $\mathcal{F}_N$.*

Proof. The proof is by induction on $|E(M)| - |E(N)|$. If $M = N$, then M is a fan-extension of N, as desired. Therefore we assume $M \neq N$. Let $\mathcal{F}$ be a covering family of M. By Proposition 2.4, M is obtained from some matroid M' by a performing a fan-lengthening move on F' to obtain $F \in \mathcal{F}$. As $(\mathcal{F} - \{F\}) \cup \{F'\}$ is a covering family of M', it follows by induction that M' is a fan-extension of N. Now the result is immediate. $\qquad\square$

3. A wheel-gluing lemma

In this section we develop an alternative description of fan-extensions that will be of use in Section 4, where we describe some applications of our main theorem. Intuitively, a family of fans in N can be seen as the result of gluing wheels along a family of triangles in a matroid that is smaller than N. A fan-extension of N can be obtained in the same way, except that we glue on larger wheels. The object of this section is to make these ideas formal. Our focus here will be on the case that N is representable, although it would be possible to extend these results to arbitrary matroids.

Proposition 3.1. *Let M_1 and M_2 be matroids on the same ground set. Assume (e_1, e_2, e_3, e_4) is a fan with e_1 as a spoke element in both M_1 and M_2. If $M_1\backslash e_1 = M_2\backslash e_1$, then $M_1 = M_2$.*

Proof. Assume that $M_1 \neq M_2$. Without loss of generality, we can assume that X is a circuit in M_1, but an independent set in M_2. Then e_1 must be contained in X. As $M_1\backslash e_1/e_2 = M_2\backslash e_1/e_2$, and M_i/e_2 is obtained from $M_i\backslash e_1/e_2$ by adding the element e_1 parallel to e_3 (for $i = 1, 2$), it follows that $M_1/e_2 = M_2/e_2$. Therefore e_2 is not in X. Now $M_1/e_2\backslash e_3 = M_2/e_2\backslash e_3$, and $M_i\backslash e_3$ is obtained from $M_i/e_2\backslash e_3$ by adding e_2 in series to e_4. Therefore $M_1\backslash e_3 = M_2\backslash e_3$, so $e_3 \in X$. By strong circuit-exchange between X and $\{e_1, e_2, e_3\}$, there is a circuit, C, in M_1 contained in $(X - e_1) \cup e_2$ that contains e_2. As C does not contain e_1, it is also a circuit of M_2. Now C and $\{e_1, e_2, e_3\}$ are distinct circuits of M_2, both of which are contained in $X \cup e_2$ and both of which contain e_2. As X is independent in M_2, this is a contradiction. $\qquad\square$

Let M_1 and M_2 be matroids with $E(M_1) \cap E(M_2) = T$. The *generalised parallel connection* of M_1 and M_2 is defined if $M_1|T = M_2|T$, and T is a modular flat of M_2. In this case, we use the notation $M_1 \boxtimes_T M_2$ to denote the generalized parallel connection. Note that a triangle in a simple binary matroid is a modular flat. The flats of $M_1 \boxtimes_T M_2$ are precisely the subsets $F \subseteq E(M_1) \cup E(M_2)$ such that $F \cap E(M_i)$ is a flat of M_i, for $i = 1, 2$ (see [9, Proposition 11.4.13]). Assume M_1, M_2, and M_3 are matroids with $E(M_2) \cap E(M_3) \subseteq E(M_1)$. Let $T_i = E(M_1) \cap E(M_i)$ for $i = 2, 3$. If $M_1 \boxtimes_{T_2} M_2$ and $M_1 \boxtimes_{T_3} M_3$ are both defined, then it follows easily from the definition that $(M_1 \boxtimes_{T_2} M_2) \boxtimes_{T_3} M_3$ and $(M_1 \boxtimes_{T_3} M_3) \boxtimes_{T_2} M_2$ are defined and equal.

Let N_0 be a matroid and let $\mathcal{T} = \{T_i\}_{i \in I}$ be a multiset of triangles of N_0, indexed by the set $I = \{1, \ldots, t\}$. Note that triangles in $\mathcal{T}$ need not be disjoint, nor indeed distinct. These are the triangles to which we will glue wheels. For each $i \in I$, let $T_i = \{a_i, b_i, c_i\}$. The end points of the fan that we generate by the gluing operation will be a_i and c_i. Let r be a function from I to $\{2, 3, 4, 5, \ldots\}$. This function determines the rank of the wheels that we glue to the triangles in $\mathcal{T}$. Finally, let X be a subset of $\cup_{k \in I} T_k$ such that, for all $i \in I$, $b_i \notin X$ implies $b_i = a_j$ or $b_i = c_j$ for some $j \in I$. After gluing the wheels to the triangles in $\mathcal{T}$, we delete the set X. We call the tuple $(N_0, \mathcal{T}, r, X)$ a *blueprint*.

Assume that $(N_0, \mathcal{T}, r, X)$ is a blueprint. For each $i \in I$, we let the matroid W_i be a copy of a wheel with rank $r(i)$. The ground set of W_i will be $\{x_1^i, \ldots, x_{r(i)}^i, y_1^i, \ldots, y_{r(i)}^i\}$, where we make the identifications $x_1^i = a_i$, $y_{r(i)}^i = b_i$, and $x_{r(i)}^i = c_i$. The ground set of W_i is labeled in such a way that $(x_1^i, y_1^i, x_2^i, y_2^i, \ldots, x_{r(i)}^i, y_{r(i)}^i)$ is a fan with x_1^i as a spoke element. Moreover $\{x_1^i, y_{r(i)}^i, x_{r(i)}^i\}$ is a triangle. Thus $E(W_i) \cap E(W_j) \subseteq \cup_{k \in I} T_k$ when $i \neq j$. For each $i \in I = \{1, \ldots, t\}$, we recursively define N_i to be

$N_{i-1} \boxtimes_{T_i} W_i$. By an earlier observation, the ordering of the indices in I makes no difference to the definition of N_t. We define $\boxtimes(N_0, \mathcal{T}, r, X)$ to be $N_t \backslash X$, and we say that this matroid is obtained by *gluing wheels* to N_0. From the definition of generalized parallel connection, it is straightforward to show that $(x_1^i, y_1^i, x_2^i, \ldots, y_{r(i)-1}^i, x_{r(i)}^i) - X$ is a fan in $\boxtimes(N_0, \mathcal{T}, r, X)$, for every $i \in I$. We call such a fan a *canonical fan* of $\boxtimes(N_0, \mathcal{T}, r, X)$.

Henceforth we take N to be a 3-connected representable matroid that is neither a wheel nor a whirl. Let E be the ground of N. We assume that $|E| \geq 4$, so that N is simple. If X and Y are disjoint subsets of $E(N)$, then $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$. Given an embedding of N in a projective geometry, the parameter $\sqcap(X, Y)$ tells us the rank of the maximal subspace spanned by both X and Y. The next two results are standard, and not difficult to prove. We omit their proofs.

Proposition 3.2. *Let $F = (e_1, \ldots, e_n)$ be a fan of N. If e_1 is a rim element, then $\sqcap(\{e_1, e_2\}, E - F) = 1$.*

Proposition 3.3. *Let $F = (e_1, \ldots, e_n)$ be a fan of N, and let R be the set of rim elements in F. Then $\sqcap(R, E - F) = 1$.*

Recall that we have required N to be representable, so henceforth we identify elements of E with points in a projective geometry, P. Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a family of pairwise disjoint fans in N indexed by the set $I = \{1, \ldots, t\}$. As N is neither a wheel nor a whirl, Proposition 5.4 implies that the complement of any fan F_i contains at least two elements. We are going to apply Propositions 3.2 and 3.3 to each fan in $\mathcal{F}$ and its reversal. This identifies three points in P that are distinguished by the fan. We add three elements in parallel to these points. To be more precise, we let $F_i = (e_1, \ldots, e_n)$ be a fan in $\mathcal{F}$. If e_1 is a spoke element, then we add a_i in parallel to e_1. If e_1 is a rim element, then we add a_i in parallel to the single point in P that is in the closure of $E - F_i$ and $\{e_1, e_2\}$. Note that if e_1 is a spoke element, and e_1 is not in the closure of $E - F_i$, then it is in the closure and the coclosure of $\{e_2, \ldots, e_n\}$. This implies $\lambda(F_i) \leq 1$, which contradicts the 3-connectivity of N. Therefore a_i is in the closure of $E - F_i$ in any case. Similarly, we add c_i in parallel to e_n if e_n is a spoke element, and otherwise we add c_i in parallel to the point of P that lies in the closure of both $E - F_i$ and $\{e_{n-1}, e_n\}$. Let R be the set of rim elements in F_i. Add b_i in parallel to the point of P that is in the closure of both R and $E - F_i$. Proposition 5.1 implies that $\lambda(F_i) = 2$, so the maximal subspace of P that lies in the closure of F_i and $E - F_i$ has rank two. The points a_i, b_i, and c_i are all parallel to points in this subspace.

Define N^+ to be the matroid consisting of the points in $E(N)$ and $\cup_{i \in I}\{a_i, b_i, c_i\}$. (We have defined N^+ relative to a representation of N. In fact, any two representations of N will lead to the same matroid N^+, but we will not make use of this fact.) We define $\mathrm{Core}(N)$ to be the matroid obtained from N^+ by deleting $\cup_{i \in I} F_i$ and any other point of $E(N)$ that is

parallel to a point a_i or c_i for some $i \in I$. Figure 3 shows schematic representations of the matroids N, N^+, and $\mathrm{Core}(N)$. Proposition 3.4 shows that $\mathcal{T} = \{\{a_i, b_i, c_i\}\}_{i \in I}$ is a family of pairwise disjoint triangles in $\mathrm{Core}(N)$. We say that $\mathrm{Core}(N)$ and $\mathcal{T}$ are *determined* by $\mathcal{F}$.
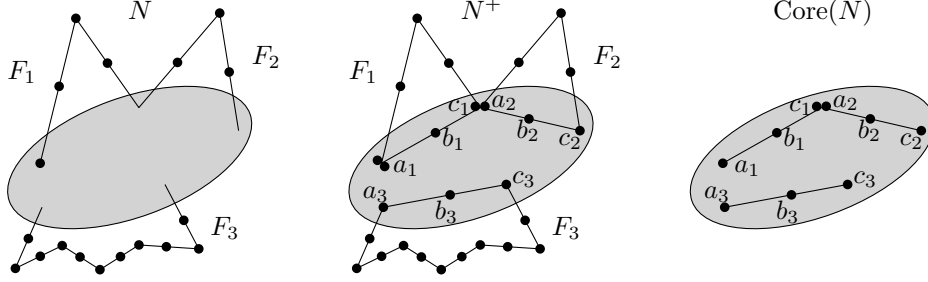

FIGURE 3. Geometric illustrations of N, N^+, and $\mathrm{Core}(N)$.

If $F_i = (e_1, \ldots, e_n)$ is a fan in $\mathcal{F}$, then we define F_i^+ to be the sequence $(x_1, \ldots, x_m)$, formed from $(e_1, \ldots, e_n)$ by prepending a_i if e_1 is a rim element, and appending c_i if e_n is a rim element. Note that if e_1 is a rim element, then orthogonality and our choice of a_i mean that a_i, e_1, and e_2 are pairwise distinct. Thus $\{x_1, x_2, x_3\}$ is a triangle of N^+ in any case. Similarly, $\{x_{m-2}, x_{m-1}, x_m\}$ is a triangle of N^+. Any triangle of N contained in F_i is also a triangle in N^+. If T^* is a triad of N contained in F_i, then every point in $\cup_{i \in I}\{a_i, b_i, c_i\}$ is in the closure of the complement of T^*. Therefore T^* is a triad in N^+. We have just shown that F_i^+ is a fan in N^+, and that both x_1 and x_m are spoke elements. This implies that m is odd.

Proposition 3.4. *For each $i \in I$, $\{a_i, b_i, c_i\}$ is a triangle in N^+.*

Proof. We have already noted that $r_{N^+}(\{a_i, b_i, c_i\}) \leq 2$. Let $F_i = (e_1, \ldots, e_n)$, and let F_i^+ be $(x_1, \ldots, x_m)$. Assume that a_i and c_i are parallel. This means that x_1 and x_m are parallel. As $\{x_1, x_2, x_3\}$ and $\{x_{m-2}, x_{m-1}, x_m\}$ are triangles, there is a circuit contained in $\{x_2, x_3, x_{m-2}, x_{m-1}\}$ that contains x_{m-1}. From this it follows that if $x_{m-1} = e_k$ (where k is either $n-1$ or n), then e_k is in the closure of $\{e_1, \ldots, e_{k-1}\}$. However, we also know that $\{e_{k-2}, e_{k-1}, e_k\}$ is a triad, so e_k is in the coclosure of $\{e_1, \ldots, e_{k-1}\}$ in N. From this we can deduce that $\lambda_N(\{e_1, \ldots, e_k\}) \leq 1$, and this leads to a contradiction to the fact that N is 3-connected. Therefore a_i and c_i are not parallel. Next we will show that a_i is not parallel to b_i. A symmetrical argument can be used to prove that c_i is not parallel to b_i, and that therefore $\{a_i, b_i, c_i\}$ is a triangle, as desired. Assume that a_i and b_i are parallel. The set of rim elements in $(e_1, \ldots, e_n)$ is equal to $\{x_2, x_4, \ldots, x_{m-1}\}$. Thus a_i is in the closure of this set and of $\{x_2, x_3\}$. Hence $\{x_2, x_3, x_4, x_6, \ldots, x_{m-1}\}$ contains a circuit, and this circuit must contain at least three elements. If k is the largest integer such that e_k is

in this circuit, then e_k is in the closure and the coclosure of $\{e_1, \ldots, e_{k-1}\}$ in N. From this we can derive a contradiction to the 3-connectivity of N. $\square$

If $F_1 = (e_1, \ldots, e_m)$ and $F_2 = (e'_1, \ldots, e'_n)$ are fans, we will say that F_1 is *enclosed* in F_2 if either $(e_1, \ldots, e_m)$ or $(e_m, \ldots, e_1)$ is equal to $(e'_i, \ldots, e'_{i+m-1})$, for some $i \in \{1, \ldots, n - m + 1\}$. Let $\mathcal{F}$ and $\mathcal{F}'$ be collections of fans. We will say that $\mathcal{F}$ is *enclosed* in $\mathcal{F}'$ if there is an bijective function from $\mathcal{F}$ to $\mathcal{F}'$ such that every fan in $\mathcal{F}$ is enclosed in its image.

The next result shows that a fan-extension of N can be constructed by gluing wheels to $\mathrm{Core}(N)$, where $\mathrm{Core}(N)$ is defined relative to some pairwise disjoint family of fans in N. Note that Corollary 2.5 and the hypotheses of the lemma imply that M is a fan-extension of N.

Lemma 3.5. *Let N be a 3-connected representable matroid, where $|E(N)| \geq 4$ and N is neither a wheel nor a whirl. Let $\mathcal{F}_N$ be a pairwise disjoint family of fans in N. Assume there is no fan, F, in N such that two distinct fans in $\mathcal{F}_N$ (considered as unordered sets) are contained in F. Assume that M is a 3-connected matroid with N as a minor, and every minor of M with N as a minor is 3-connected up to series and parallel sets. Let $\mathcal{F}$ be a covering family in M relative to $\mathcal{F}_N$. There exists a pairwise disjoint family of fans, $\mathcal{F}_0$, in N, such that $\mathcal{F}_0$ encloses $\mathcal{F}_N$, and moreover, up to relabeling, $M = \boxtimes(\mathrm{Core}(N), \mathcal{T}, r, X)$, for some blueprint $(\mathrm{Core}(N), \mathcal{T}, r, X)$ where $\mathrm{Core}(N)$ and $\mathcal{T}$ are determined by $\mathcal{F}_0$. Furthermore, $\mathcal{F}$ is enclosed in the family of canonical fans of $\boxtimes(\mathrm{Core}(N), \mathcal{T}, r, X)$.*

Proof. Let $E = E(N)$. The proof will be by induction on $|E(M)| - |E|$. First we assume $|E(M)| = |E|$. Then $M = N$. Now $\mathcal{F}$ is a covering family of N, relative to $\mathcal{F}_N$. This does not mean that $\mathcal{F} = \mathcal{F}_N$. The fans in $\mathcal{F}_N$ may be enclosed in larger fans, and these larger fans may be members of $\mathcal{F}$. Nonetheless, $\mathcal{F}$ is a pairwise disjoint family of fans in N. Because each fan in $\mathcal{F}_N$ is consistent with a fan in $\mathcal{F}$, it follows without difficulty from Lemma 5.8 that each fan in $\mathcal{F}_N$ must be enclosed by a fan in $\mathcal{F}$. Thus we can define $\mathcal{F}_0$ to be $\mathcal{F}$. Let $\mathcal{F} = \{F_i\}_{i \in I}$, where $I = \{1, \ldots, t\}$, and let $\mathrm{Core}(N)$ and $\mathcal{T} = \{\{a_i, b_i, c_i\}\}_{i \in I}$ be the matroid and family of triangles determined by $\mathcal{F}$. To prove the base case of the induction, we now need only check that $N = \boxtimes(\mathrm{Core}(N), \mathcal{T}, r, X)$ (up to relabeling).

Recall that N^+ is the matroid consisting of the points in E and $\cup_{i \in I}\{a_i, b_i, c_i\}$. Let F_t^+ be $(x_1, \ldots, x_m)$. This is a fan in N^+, and x_1 and x_m are spoke elements. The rim elements in F_t are $\{x_2, x_4, \ldots, x_{m-1}\}$. Thus b_t is in the closure of this set. Any circuit contained in $\{x_2, x_4, \ldots, x_{m-1}, b_t\}$ must contain $\{x_2, x_4, \ldots, x_{m-1}\}$, for otherwise we can find a violation of orthogonality with one of the triads in F_t. Thus $\{x_2, x_4, \ldots, x_{m-1}, b_t\}$ is a circuit in N^+, so $\{x_{m-1}, b_t\}$ is a circuit in $N^+/\{x_2, x_4, \ldots, x_{m-3}\}\backslash\{x_3, x_5, \ldots, x_{m-2}\}$. We are going to apply a result by Oxley and Wu [10, Theorem 1.8]. This theorem applies only to 3-connected matroids, but the simplification of N^+ is 3-connected, and it is easy to see that the result still holds. The theorem by Oxley and Wu tells us that, up to relabeling, N^+ is equal to

$(N^+\backslash F_t) \boxtimes_{\{a_t,b_t,c_t\}} W_t$, where W_t has rank $(m + 1)/2$. Moreover $N^+\backslash F_t$ is 3-connected up to parallel pairs. Now F_{t-1}^+ is a fan in N^+. Note that F_t is disjoint from F_{t-1}^+ by construction of $\text{Core}(N)$. As $\text{si}(N^+\backslash F_t)$ is 3-connected, it is easy to see that F_{t-1}^+ is a fan in $N^+\backslash F_t$ also. By again using [10, Theorem 1.8], we see that $N^+\backslash F_t$ can be obtained by using generalized parallel connection to glue a wheel of the correct rank to $N^+\backslash F_t\backslash F_{t-1}$ along the line $\{a_{t-1}, b_{t-1}, c_{t-1}\}$. We proceed inductively in this way, and conclude that N^+ is obtained from

$$N^+\backslash F_t\backslash F_{t-1}\backslash \cdots \backslash F_1$$

by attaching wheels via generalized parallel connections. Let S be the set of elements in $E - (\cup_{i\in I}F_i)$ that are parallel to some element a_i or c_i. Then $\text{Core}(N) = N^+\backslash(\cup_{i\in I}F_i)\backslash S$. This means that $N^+\backslash S$ can be obtained from $\text{Core}(N)$ by attaching wheels to the lines $\{\{a_i, b_i, c_i\}\}_{i\in I}$. This in turn means that N can be obtained from $\text{Core}(N)$ in the following way: attach wheels to the lines $\{\{a_i, b_i, c_i\}\}_{i\in I}$, and then, for each element $s \in S$, distinguish an element in $\{\{a_i, b_i, c_i\}\}_{i\in I}$ that is parallel to it, and relabel that element as s, and then finally delete all other elements in $\{\{a_i, b_i, c_i\}\}_{i\in I}$. In other words, up to relabeling, we can obtain N from $\text{Core}(N)$ by gluing on wheels. This is exactly what we aimed to prove, and it establishes the base case of our induction.

Now we can assume that $M \neq N$. Proposition 2.4 implies that there is a fan $F \in \mathcal{F}$ such that M and F are obtained from a 3-connected matroid M' by performing a fan-lengthening move on the fan F'. It is clear that $\mathcal{F}' = (\mathcal{F} - \{F\}) \cup \{F'\}$ is a covering family in M', and we can apply the inductive hypothesis. There is a family, $\mathcal{F}_0$, of pairwise disjoint fans in N such that $\mathcal{F}_0$ encloses $\mathcal{F}_N$. We relabel M' in such a way that it is obtained by gluing wheels to $\text{Core}(N)$ (relative to $\mathcal{F}_0$), and $\mathcal{F}'$ is enclosed in the family of canonical fans. Let X be the set of elements that we delete after attaching wheels to $\text{Core}(N)$. This means that F' is enclosed in the canonical fan $(x_1^i, y_1^i, x_2^i, \ldots, y_{r(i)-1}^i, x_{r(i)}^i) - X$, for some i. Let F be $(e_1, \ldots, e_m)$.

Assume that M' is obtained from M by deleting a terminal spoke element of F. By reversing, we may assume that $M' = M\backslash e_1$. Now $F' = (e_2, \ldots, e_m)$ is enclosed in $(x_1^i, y_1^i, x_2^i, \ldots, y_{r(i)-1}^i, x_{r(i)}^i) - X$. As $\{e_2, e_3, e_4\}$ is a triad in M', by reversing $(x_1^i, y_1^i, x_2^i, \ldots, y_{r(i)-1}^i, x_{r(i)}^i)$ as necessary, we can assume $e_2 = y_k^i$ and $e_3 = x_{k+1}^i$. Assume for a contradiction that either (i) $k > 1$, or (ii) $k = 1$ and $x_1^i = e_1$, and hence $x_1^i \in E(N)$, or (iii) $k = 1$ and $x_1^i = a_i$, but $a_i \notin X$. If any of these situations hold, then $\{x_k^i, y_k^i, x_{k+1}^i\}$ and $\{e_1, e_2, e_3\}$ are both triangles in M, and they intersect in the elements e_2 and e_3. This means that $\{x_k^i, e_1, e_2\}$ is a triangle in M that intersects the triad $\{e_2, e_3, e_4\}$ in a single element. This contradiction means that $k = 1$, $x_1^i = a_i$, and $a_i \in X$. Now let M'' be the matroid obtained by gluing the same wheels to $\text{Core}(N)$, except that instead of deleting X, we delete $X - x_1^i$. Obviously $M''\backslash x_1^i = M' = M\backslash e_1$. We relabel M so that

it inherits the relabeling of M' and e_1 is relabeled as x_1^i. This means that $(x_1^i, y_1^i, x_2^i, y_2^i)$ is a fan in both M and M''. Now Proposition 3.1 implies that $M'' = M$. It is clear that after the relabeling F is enclosed in the canonical fan $(x_1^i, y_1^i, x_2^i, \ldots, y_{r(i)-1}^i, x_{r(i)}^i) - (X - x_1^i)$, so the result holds.

Next we will assume that $M' = M/e_1$, where e_1 is a rim element in F. We can assume that $e_2 = x_k^i$ and $e_3 = y_k^i$. If $k > 1$, then $\{y_{k-1}^i, x_k^i, y_k^i\}$ is a triad in M', and hence in M. As $\{e_1, e_2, e_3\}$ is also a triad of M, it follows that $\{y_{k-1}^i, e_1, e_2\}$ is a triad in M. As $\{e_2, e_3, e_4\}$ is a triangle, this is a contradiction, so $k = 1$. Therefore a_i is parallel to $x_1^i = e_2$. This means that we can assume that e_2 is relabeled as a_i when M' is relabeled in such a way that it becomes equal to $\boxtimes(\mathrm{Core}(N), \mathcal{T}, r, X)$. Thus $a_i \notin X$. Let W_i'' be a copy of a wheel with rank $r(i) + 1$. Let the ground set of W_i'' be $E(W_i) \cup \{x, y\}$, labeled in such a way that $(x, y, x_1^i, y_1^i, x_2^i, \ldots, y_{r(i)-1}^i, x_{r(i)}^i, y_{r(i)}^i)$ is a fan and $\{x, y_{r(i)}^i, x_{r(i)}^i\}$ is a triangle. Therefore x is identified with a_i. Now let M'' be the matroid obtained from $\mathrm{Core}(N)$ by gluing on the same wheels as before, except that we use W_i'' instead of W_i, and we then delete $X \cup a_i$. Now $W_i''/y\backslash x = W_i$. From this it follows easily that $M''/y = M'$. We relabel M in such a way that it inherits the labeling we applied to M', and e_1 is labeled as y. Now (y, x_1^i, y_1^i, x_2^i) is a fan in both M and M'', and $M/y = M' = M''/y$. Therefore the dual of Proposition 3.1 implies that $M'' = M$, as desired.

For the final case, we assume $|E(M')| = |E(M)| - 2$. Since $(\mathcal{F} - \{F\}) \cup \{F'\}$ is a covering family in M', it follows that $m \geq 5$. By reversing F as necessary, we can assume that $M' = M/e_j\backslash e_{j+1}$, where e_j is a rim element of F. We will assume that $j > 1$. An almost identical argument will hold in the case that $j = 1$. Now $(e_1, \ldots, e_{j-1}, e_{j+2}, \ldots, e_m)$ is a fan of M' that is enclosed in $(x_1^i, y_1^i, x_2^i, \ldots, y_{r(i)-1}^i, x_{r(i)}^i)$. By reversing $(x_1^i, y_1^i, x_2^i, \ldots, y_{r(i)-1}^i, x_{r(i)}^i)$ as necessary, we can assume that $e_{j-1} = x_k^i$ and $e_{j+2} = y_k^i$ (if $j + 2 \leq m$). Let W_i'' be a copy of a wheel with rank $r(i) + 1$, and let the ground set of W_i'' be $E(W_i) \cup \{x, y\}$. We label W_i'' in such a way that $(x_1^i, y_1^i, \ldots, x_k^i, y, x, y_k^i, \ldots, x_{r(i)}^i)$ is a fan. Now let M'' be the matroid obtained from $\mathrm{Core}(N)$ by gluing on the same wheels used to obtain M', except that we use W_i'' instead of W_i. After attaching these wheels to $\mathrm{Core}(N)$ we delete exactly the same set X. Now $W_i''/y\backslash x = W_i$, and it follows that $M''/y\backslash x = M'$. We relabel M so that it inherits the relabeling of M', while e_i is labeled as y and e_{i+1} is labeled as x. After this relabeling, $M/y\backslash x = M' = M''/y\backslash x$. We claim that this implies $M = M''$. Note that M/y and M''/y are both obtained by adding x parallel to x_k^i in $M/y\backslash x = M''/y\backslash x$. Thus $M/y = M''/y$. If either $(x_{k-1}^i, y_{k-1}^i, x_k^i, y)$ or (y, x, y_k^i, x_{k+1}^i) is a fan in both M and M'', then the dual of Proposition 3.1 implies that $M = M''$ as desired. If neither of these sequences is a fan in M and M'', then since $m \geq 5$, it is not difficult to see that $m = 5$, and $e_j = e_3$. Thus $(x_1^i, x_2^i, y, x, x_3^i)$ is a fan in both M and M''. In this case

$M\backslash x$ and $M''\backslash x$ are both obtained from $M/y\backslash x = M''/y\backslash x$ by adding y in series to x_3^i. Thus $M\backslash x = M''\backslash x$, and we can use Proposition 3.1 to show that $M = M''$, as claimed. Obviously F is contained in the canonical fan associated with gluing W_i'' to $\mathrm{Core}(N)$, so the proof is complete. $\qquad\square$

4. Applications

Now we consider some applications of Theorem 1.2. We omit the proofs, since Theorem 1.2 and Lemma 3.5 reduce them to computational case checking. Details can be found in [1]. Our first application concerns the binary matroids that are $\{F_7, F_7^*\}$-fragile. We describe these via *grafts* (see [9, p. 386]). Recall that N_{12} is illustrated in Figure 1(i). Figure 4 gives graft representations of the matroids N_{11}^+ and N_{12}. In both case, the distinguished vertices in the graft are those vertices with squares around them. In N_{12}, let w_4 be the matroid element that corresponds to the set of distinguished vertices.
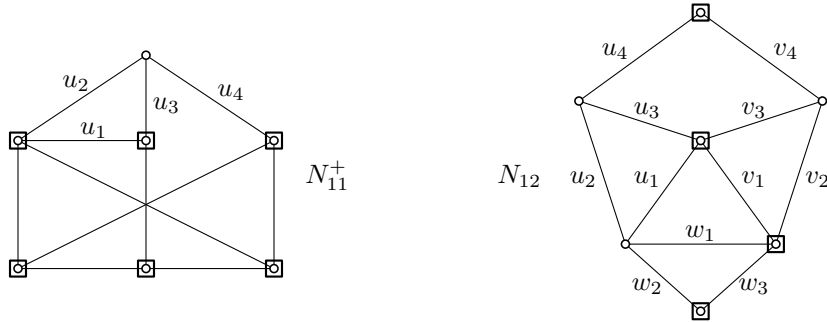

FIGURE 4. Graft representations of N_{11}^+ and N_{12}.

Now we have the tools to characterize binary $\{F_7, F_7^*\}$-fragile matroids that have N_{11}^+ or N_{12} as a minor. The following result is known to Truemper [13] and Kingan and Lemos [6]. It is a straightforward application of Theorem 1.2 and Lemma 3.5.

Theorem 4.1. *Let M be a 3-connected binary $\{F_7, F_7^*\}$-fragile matroid. If M has N_{11}^+ as a minor, then M is a fan-extension of N_{11}^+ relative to $\{(u_1, u_2, u_3, u_4)\}$, and M is a restriction of a member of the family illustrated in the lefthand diagram in* Figure 5. *If M has N_{12} as a minor, then M is a fan-extension of N_{12} relative to $\{(u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4), (w_1, w_2, w_3, w_4)\}$, and M is a restriction of a member of the family illustrated in the righthand diagram in* Figure 5.

Next we consider fragile classes of matroid representable over the partial fields $\mathbb{U}_2$ and $\mathbb{H}_5$. Every $\mathbb{U}_2$-representable matroid is also $\mathbb{H}_5$-representable, so it suffices to consider only the latter partial field. We currently have a solid understanding of the $\mathbb{H}_5$-representable $\{U_{2,5}, U_{3,5}\}$-fragile matroids,
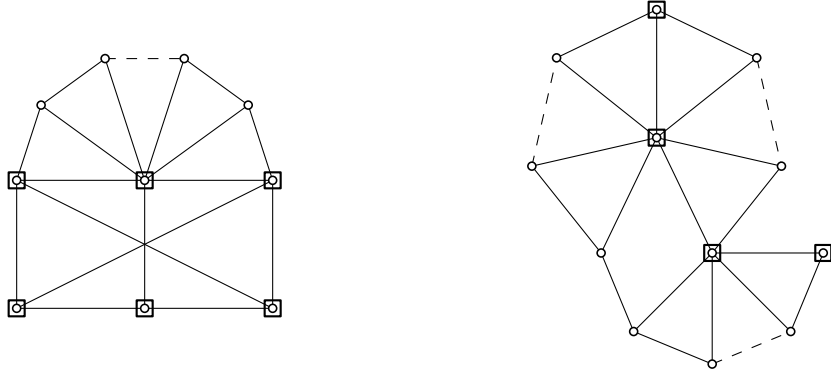
FIGURE 5. The N_{11}^+ and N_{12} families.

and this brings excluded-minor characterizations for these two partial fields within grasp. Proofs and definitions can all be found in [1].

Proposition 4.2. *Let M be a 3-connected $\mathbb{H}_5$-representable $\{U_{2,5}, U_{3,5}\}$-fragile matroid. Assume that M has N as minor, where N is in $\{M_{9,0}, M_{9,2}, M_{9,15}, M_{9,18}\}$. Then M is a fan-extension of N.*

Theorem 4.3. *Let M be a 3-connected $\mathbb{H}_5$-representable matroid. Up to relabeling, one of the following statements holds:*

  (i) *M has one of X_8, Y_8, or Y_8^* as a minor,*
  (ii) *M is in $\{U_{2,6}, U_{3,6}, U_{4,6}, P_6, M_{9,9}, M_{9,9}^*\}$,*
  (iii) *M or M^* can be obtained from $U_{2,5}$ (with groundset $\{a, b, c, d, e\}$) by gluing wheels to $\{(a, c, b), (a, d, b), (a, e, b)\}$,*
  (iv) *M or M^* can be obtained from $U_{2,5}$ (with groundset $\{a, b, c, d, e\}$) by gluing wheels to $\{(a, b, c), (c, d, e)\}$,*
  (v) *M or M^* can be obtained from $M_{7,1}$ by gluing on a wheel.*

5. RESULTS ON FANS

Now we start moving towards a proof of our main theorem. This section collects some results that we will need. The next proposition can be proved with an easy inductive argument.

Proposition 5.1. *Let $(e_1, \ldots, e_n)$ be a fan of the matroid M. Then $\{e_1, \ldots, e_n\}$ is 3-separating in M.*

Proposition 5.2. *Let M be a 3-connected matroid, and let $F_1 = (e_1, \ldots, e_n)$ be a fan of M such that $n \geq 5$. Let e_i be a rim element, for some $i \in \{1, \ldots, n-1\}$. Then $F_2 = (e_1, \ldots, e_{i-1}, e_{i+2}, \ldots, e_n)$ is a fan of $M/e_i \backslash e_{i+1}$, and for all j, e_j is a spoke element in F_1 if and only if it is a spoke element in F_2*

Proof. Any triad of M contained in $\{e_1, \ldots, e_{i-2}\}$ remains a triad in $M/e_i \backslash e_{i+1}$, by orthogonality with the triangle $\{e_{i-1}, e_i, e_{i+1}\}$. Similarly, any

triad of M in $\{e_{i+4}, \ldots, e_n\}$ is a triad in $M/e_i \backslash e_{i+1}$ by orthogonality with $\{e_{i+1}, e_{i+2}, e_{i+3}\}$. In the same way, the triad $\{e_i, e_{i+1}, e_{i+2}\}$ implies that any triangle in $\{e_1, \ldots, e_{i-1}\}$ or $\{e_{i+3}, \ldots, e_n\}$ is also a triangle in $M/e_i \backslash e_{i+1}$.

Now the only sets we need check are $\{e_{i-2}, e_{i-1}, e_{i+2}\}$ and $\{e_{i-1}, e_{i+2}, e_{i+3}\}$. Strong cocircuit-exchange between $\{e_{i-2}, e_{i-1}, e_i\}$ and $\{e_i, e_{i+1}, e_{i+2}\}$ gives a cocircuit contained in $\{e_{i-2}, e_{i-1}, e_{i+1}, e_{i+2}\}$ that contains e_{i+1}. This cocircuit contains e_{i-1}, by orthogonality with $\{e_{i-1}, e_i, e_{i+1}\}$. If e_{i-2} is not in the cocircuit, then $\{e_{i-1}, e_{i+1}, e_{i+2}\}$ and $\{e_i, e_{i+1}, e_{i+2}\}$ are triads, so $\{e_{i-1}, e_i, e_{i+1}\}$ is a triad and a triangle. This contradicts the fact that M is 3-connected with at least 5 elements. Hence e_{i-2} is in the cocircuit, and a similar argument shows that $\{e_{i-2}, e_{i-1}, e_{i+1}, e_{i+2}\}$ is a cocircuit of M. Thus $\{e_{i-2}, e_{i-1}, e_{i+2}\}$ is a triad in $M/e_i \backslash e_{i+1}$, as desired. We can show that $\{e_{i-1}, e_{i+2}, e_{i+3}\}$ is a triangle of $M/e_i \backslash e_{i+1}$ by a dual argument.   □

Proposition 5.3. *Let $(e_1, \ldots, e_n)$ be a fan in the 3-connected matroid M, and assume that $n \geq 4$. Let $e \in E(M) - \{e_1, \ldots, e_n\}$ be such that there is a triangle, T, satisfying $\{e\} \subseteq T \subseteq \{e, e_1, \ldots, e_n\}$. Then either:*

  (i) *$T = \{e, e_1, e_2\}$, e_1 is a rim element, and $(e, e_1, \ldots, e_n)$ is a fan,*
  (ii) *$T = \{e, e_{n-1}, e_n\}$, e_n is a rim element, and $(e_1, \ldots, e_n, e)$ is a fan,*
  (iii) *$T = \{e, e_1, e_n\}$, and e_1 and e_n are spoke elements,*
  (iv) *$T = \{e, e_2, e_4\}$, e_2 is a rim element, and $n \leq 5$, or*
  (v) *$T = \{e, e_{n-1}, e_{n-3}\}$, e_{n-1} is a rim element, and $n \leq 5$.*

Proof. Let $T = \{e, e_x, e_y\}$, where $1 \leq x < y \leq n$. If $x = 1$ and $y = n$, then statement (iii) must hold, since orthogonality with T requires that $\{e_1, e_2, e_3\}$ and $\{e_{n-2}, e_{n-1}, e_n\}$ are both triangles. We will assume that $x > 1$, and show that statement (ii) or statement (iv) holds. If $x = 1$, then $y < n$, so we can replace $(e_1, \ldots, e_n)$ with $(e_n, \ldots, e_1)$, and swap labels on x and y, and then apply exactly the same arguments to deduce that (i) or (v) holds.

Now we assume that $x > 1$. Assume also that $y = x + 1$. If $y < n$, then either $\{e_{x-1}, e_x, e_{x+1}\}$ is a triangle and $\{e_x, e_{x+1}, e_{x+2}\}$ is a triad, or $\{e_x, e_{x+1}, e_{x+2}\}$ is a triangle and $\{e_{x-1}, e_x, e_{x+1}\}$ is a triad. As M is 3-connected, in the first case $\{e, e_{x-1}, e_x\}$ is a triangle, and in the second case $\{e, e_{x+1}, e_{x+2}\}$ is a triangle. In either case we have a contradiction to orthogonality. Therefore $y = n$, and e_y is a rim element, by orthogonality between $T = \{e, e_{n-1}, e_n\}$ and $\{e_{n-3}, e_{n-2}, e_{n-1}\}$. Now it is clear that statement (ii) holds. Hence we will assume that $y > x + 1$. Thus e_x is a rim element, by orthogonality between T and $\{e_{x-1}, e_x, e_{x+1}\}$. If $2 < x$, then we have a contradiction to orthogonality between T and $\{e_{x-2}, e_{x-1}, e_x\}$. Therefore $x = 2$. If $y > x + 2$, then T intersects the triad $\{e_x, e_{x+1}, e_{x+2}\}$ in a single element. Therefore $y = x + 2 = 4$. Finally, $n \leq 5$, for otherwise T intersects the triad $\{e_y, e_{y+1}, e_{y+2}\}$ in a single element. We have shown that statement (iv) holds, so the proof is complete.   □

Proposition 5.4. *Assume that $(e_1, \ldots, e_n)$ is a fan of the 3-connected matroid M, and that $4 \leq |E(M)| \leq n + 1$. Then M is a wheel or a whirl.*

Proof. If $|E(M)| = 4$, then M is isomorphic to the whirl $U_{2,4}$, and we are done. Therefore we assume $|E(M)| > 4$. Since wheels and whirls are self-dual, we can switch to M^* as required, and assume that e_1 is a spoke element of $(e_1, \ldots, e_n)$. If $|E(M)| = n$, then result follows immediately from [10, Lemma 4.8]. Therefore we assume that $|E(M)| = n + 1$. Let e be the single element in $E(M) - \{e_1, \ldots, e_n\}$.

Suppose that e_n is a rim element, so that n is even. Then $\{e_1, e_3, \ldots, e_{n-3}, e_{n-1}\}$ spans $E(M) - \{e, e_n\}$. Since M is 3-connected, it also spans $E(M)$. Let C be a circuit contained in $\{e_1, e_3, \ldots, e_{n-3}, e_{n-1}\} \cup \{e\}$ that contains e. If i is odd and $3 \leq i \leq n-1$, then $e_i \notin C$, or else C intersects the triad $\{e_{i-1}, e_i, e_{i+1}\}$ in a single element. Therefore $C \subseteq \{e, e_1\}$, which is impossible as M is 3-connected. This implies that e_n is a spoke element and n is odd. Suppose that $\{e_3, e_5, \ldots, e_{n-2}, e_n\}$ spans $E(M)$. Let C be a circuit in $\{e_3, e_5, \ldots, e_{n-2}, e_n\} \cup \{e_1\}$ that contains e_1. Then $C \subseteq \{e_1, e_n\}$, for otherwise C intersects a triad in a single element. This is a contradiction, so $\{e_3, e_5, \ldots, e_{n-2}, e_n\}$ does not span $E(M)$. Since it spans $E(M) - \{e, e_1, e_2\}$, it follows that $\{e, e_1, e_2\}$ is a triad. Thus $(e, e_1, \ldots, e_n)$ is a fan, and we can again apply [10, Lemma 4.8] to deduce that M is a wheel or a whirl. $\square$

Proposition 5.5. *Let M be a 3-connected matroid with $|E(M)| \geq 4$. Let $(e_1, \ldots, e_n)$ be a fan of M. If there is an element $e \in E(M) - \{e_1, \ldots, e_n\}$, and elements $x, y, z \in \{e_1, \ldots, e_n\}$ such that $\{e, x, y\}$ is a triangle and $\{e, y, z\}$ is a triad, then M is a wheel or a whirl.*

Proof. Let $F = \{e_1, \ldots, e_n\}$. Then $\lambda(F) \leq 2$, and as e is in the closure and coclosure of F, $\lambda(F \cup e) \leq 1$. Therefore the complement of $F \cup e$ contains at most one element. If $n = 3$, then either F and $\{e, x, y\}$ are both triangles, or F and $\{e, y, z\}$ are both triads. In the first case $M|(F \cup e) \cong U_{2,4}$, and in the second $M^*|(F \cup e) \cong U_{2,4}$. In either case, $F \cup e$ contains a triangle that is a triad. Since M is 3-connected, this means $|E(M)| = 4$, and M is isomorphic to $U_{2,4}$, so we are done. Therefore we assume $n \geq 4$.

We apply Proposition 5.3. If statement (i) or (ii) holds, then we can apply Proposition 5.4 to $(e_1, \ldots, e_n, e)$ or $(e, e_1, \ldots, e_n)$ and conclude that M is a wheel or a whirl. Next assume that statement (iii) holds. Then $n \geq 5$. By reversing $(e_1, \ldots, e_n)$ as necessary, we assume that $y = e_n$. Then orthogonality between the triad $\{e, y, z\}$ and the triangle $\{e_{n-2}, e_{n-1}, e_n\}$ requires that z is in $\{e_{n-2}, e_{n-1}\}$. If $z = e_{n-1}$, then $(e_1, \ldots, e_n, e)$ is a fan, and we can again apply Proposition 5.4. Therefore we assume $z = e_{n-2}$. This means that the triangle $\{e_{n-4}, e_{n-3}, e_{n-2}\}$ and the triad $\{e, y, z\}$ meet in a single element, a contradiction. Therefore statement (iv) or (v) in Proposition 5.3 holds. By reversing, we can assume that $\{x, y\} = \{e_2, e_4\}$, and e_2 is a rim element.

If $n = 4$, then we apply Proposition 5.4 to the fan (e_1, e_3, e_2, e_4, e). Therefore we assume $n = 5$. If $z = e_1$, then $y = e_2$, by orthogonality between

the triangles $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$, and the triad $\{e, y, z\}$. Therefore $(e, e_1, e_2, e_3, e_4, e_5)$ is a fan. The same argument disposes of the case that $z = e_5$. Therefore we assume $z = e_3$. But in this case either $\{e_1, e_2, e_3\}$ or $\{e_3, e_4, e_5\}$ is a triangle that intersects the triad $\{e, y, z\}$ in a single element. This contradiction completes the proof. $\qquad\square$

Proposition 5.6. *Let M be a 3-connected matroid such that $|E(M)| \geq 4$, and M is not a wheel or a whirl. Let $(e_1, \ldots, e_n)$ be a fan of M. If $T \subseteq \{e_1, \ldots, e_n\}$ is a triangle, then $T = \{e_i, e_{i+1}, e_{i+2}\}$ for some $i \in \{1, \ldots, n-2\}$ such that e_i is a spoke element.*

Proof. Let i, j, and k be such that $1 \leq i < j < k \leq n$, and $T = \{e_i, e_j, e_k\}$. If e_i is a rim element of $(e_1, \ldots, e_n)$, then $\{e_i, e_{i+1}, e_{i+2}\}$ is a triad. Since T is a triangle, it follows that e_i is in the closure and the coclosure of $\{e_{i+1}, \ldots, e_n\}$. As $\lambda(\{e_{i+1}, \ldots, e_n\}) \leq 2$, it follows that $\lambda(\{e_i, \ldots, e_n\}) \leq 1$. As M is 3-connected, we deduce that the complement of $\{e_i, \ldots, e_n\}$ contains at most one element. Thus M is a wheel or a whirl by Proposition 5.4, and this is a contradiction. Therefore e_i is a spoke element. The same argument shows that e_k is a spoke element.

Assume that $j > i+1$. If e_j is a rim element of $(e_1, \ldots, e_n)$, then $j \geq i+3$, so T intersects the triad $\{e_{j-2}, e_{j-1}, e_j\}$ in a single element, violating orthogonality. Therefore e_j is a spoke element. If $k > j+1$, then T intersects the triad $\{e_{j-1}, e_j, e_{j+1}\}$ in a single element. Therefore $k = j+1$, implying that e_k is a rim element. Since this contradicts our earlier conclusion, we see that $j = i+1$. Now $k = i+2$, by orthogonality between T and $\{e_{i+1}, e_{i+2}, e_{i+3}\}$, and the fact that e_k is a spoke element. $\qquad\square$

Proposition 5.7. *Let M be a 3-connected matroid such that $|E(M)| \geq 4$ and M is neither a wheel nor a whirl. Let $(e_1, \ldots, e_n)$ be a fan of M. If $X \subseteq \{e_1, \ldots, e_n\}$ is a 3-separating set of M such that $|X| \geq 3$, then there are integers $1 \leq x < y \leq n$ such that $X = \{e_x, e_{x+1}, \ldots, e_y\}$.*

Proof. Assume that the result fails for X, and that X is as large as possible with respect to this assumption. If $|E(M)| = 4$, then M is $U_{2,4}$, contradicting the fact that M is not a whirl. If $|E(M)| = 5$, then M is $U_{2,5}$ or $U_{3,5}$. In the first case M has no triads, and in the second, M has no triangles. In either case, $n = 3$, and $X = \{e_1, e_2, e_3\}$. Therefore we may as well assume that $|E(M)| \geq 6$. Let i be the least index such that $|X \cap \{e_1, \ldots, e_i\}| = 3$. Then $X \cup \{e_1, \ldots, e_i\}$ is contained in $\{e_1, \ldots, e_n\}$, so the complement of $X \cup \{e_1, \ldots, e_i\}$ contains at least two elements, or else M is a wheel or a whirl by Proposition 5.4. As M is 3-connected, it follows that $X \cup \{e_1, \ldots, e_i\}$ is not 2-separating. As $\lambda(\{e_1, \ldots, e_i\}) \leq 2$, $\lambda(X) \leq 2$, and $\lambda(X \cup \{e_1, \ldots, e_i\}) \geq 2$, it follows that $X' = X \cap \{e_1, \ldots, e_i\}$ is a 3-separating set with cardinality three, by the submodularity of λ ([9, Lemma 8.2.9]). Since $|E(M)| \geq 6$, it follows that $(X', E(M) - X')$ is a 3-separation of M. Thus X' is a triad or a triangle ([9, Corollary 8.2.2]). By

duality, we will assume that X' is a triangle of M. Proposition 5.6 implies that $X' = \{e_{i-2}, e_{i-1}, e_i\}$.

There must be an index $p > i$ such that $e_p \notin X$, or else X is not a counterexample. Let p be the least such index. Note that e_{p-2} and e_{p-1} are in X. Therefore e_p is in either $\mathrm{cl}_M(X)$ or $\mathrm{cl}_M^*(X)$, so $X \cup e_p$ is 3-separating. Because $X \cup e_p$ is strictly larger than X, it follows that $X \cup e_p$ is not a counterexample, so $X \cup e_p = \{e_x, e_{x+1}, \dots, e_y\}$ for some indices x and y. Clearly $p < y$, or else $X = \{e_x, e_{x+1}, \dots, e_{y-1}\}$. Thus e_{p+1} is in X. Now $\{e_{p-2}, e_{p-1}, e_p\}$ and $\{e_{p-1}, e_p, e_{p+1}\}$ show that e_p is in the closure and the coclosure of X. Therefore $\lambda(X \cup e_p) \leq 1$, so the complement of $X \cup e_p$ contains at most one element. Hence M is a wheel or a whirl, and this contradiction completes the proof. □

Lemma 5.8. *Let M be a 3-connected matroid such that $|E(M)| \geq 4$ and M is not a wheel or a whirl. Let $(f_1, \dots, f_m)$ and $(g_1, \dots, g_n)$ be fans of M. Assume that $|\{f_1, \dots, f_m\} \cap \{g_1, \dots, g_n\}| \geq 3$. Then there are integers $1 \leq x < y \leq m$ and $1 \leq x' < y' \leq n$ such that*

$$\{f_1, \dots, f_m\} \cap \{g_1, \dots, g_n\} = \{f_x, f_{x+1}, \dots, f_y\} = \{g_{x'}, g_{x'+1}, \dots, g_{y'}\}.$$

Proof. The proof is by induction on $m + n$. The hypotheses imply that $m + n \geq 6$. If $m + n = 6$, then obviously $m = n = 3$ and

$$\{f_1, f_2, f_3\} \cap \{g_1, g_2, g_3\} = \{f_1, f_2, f_3\} = \{g_1, g_2, g_3\},$$

so there is nothing left to prove. Let us assume that $m + n > 6$, and that the result does not hold for $(f_1, \dots, f_m)$ and $(g_1, \dots, g_n)$. We will make the inductive assumption that the result is true for any pair of fans with combined length less than $m + n$.

Let $F = \{f_1, \dots, f_m\}$, and let $G = \{g_1, \dots, g_n\}$.

5.8.1. *$m, n > 3$.*

Proof. By symmetry, it suffices to prove that $m > 3$. If $m = 3$, then $F \cap G = \{f_1, f_2, f_3\}$. Moreover, F is a triangle or a triad contained in G. Proposition 5.6 or its dual tells us that $F = F \cap G$ consists of three consecutive elements from $(g_1, \dots, g_n)$. Therefore the result holds, contrary to our choice of $(f_1, \dots, f_m)$ and $(g_1, \dots, g_n)$. □

5.8.2. *$f_1, f_m \in G$ and $g_1, g_n \in F$.*

Proof. Suppose that $f_1 \notin G$. Then $(f_2, \dots, f_m)$ and $(g_1, \dots, g_n)$ are fans by 5.8.1. Since $|\{f_2, \dots, f_m\} \cap \{g_1, \dots, g_n\}| \geq 3$, the inductive assumption implies that there are integers $1 \leq x < y \leq m$ and $1 \leq x' < y' \leq n$ such that

$$\begin{aligned}
\{f_1, \dots, f_m\} \cap \{g_1, \dots, g_n\} &= \{f_2, \dots, f_m\} \cap \{g_1, \dots, g_n\} \\
&= \{f_x, f_{x+1}, \dots, f_y\} = \{g_{x'}, g_{x'+1}, \dots, g_{y'}\},
\end{aligned}$$

and the result holds, contrary to assumption. Exactly the same argument shows that $f_m \in G$ and $g_1, g_n \in F$. □

Let the ground set of M be E.

5.8.3. $|E - (F \cup G)| \leq 1$.

Proof. Assume that $|E - (F \cup G)| \geq 2$. As $|F \cup G| \geq 2$, and M is 3-connected, it follows that $\lambda(F \cup G) \geq 2$. Since $\lambda(F), \lambda(G) \leq 2$, the submodularity of λ shows that $\lambda(F \cap G) \leq 2$. Then we can set $X = F \cap G$ and apply Proposition 5.7 to see that

$$F \cap G = \{f_x, f_{x+1}, \ldots, f_y\} = \{g_{x'}, g_{x'+1}, \ldots, g_{y'}\}$$

for integers $1 \leq x < y \leq m$ and $1 \leq x' < y' \leq n$. This contradiction completes the proof of the claim. $\qquad\square$

5.8.4. $|F - G|, |G - F| \geq 2$.

Proof. We have complete symmetry between F and G, so it suffices to prove that $|F - G| \geq 2$. Assume that $|F - G| \leq 1$. If $|F - G| = 0$, then $|E - G| \leq 1$ by 5.8.3, so Proposition 5.4 implies that M is a wheel or a whirl. This contradiction means that $|F - G| = 1$. Let f_i be the unique element in $F - G$. By 5.8.2, we deduce that $1 < i < m$. Either $\{f_{i-1}, f_i, f_{i+1}\}$ and $\{f_i, f_{i+1}, f_{i+2}\}$ are a triangle and a triad that are contained in $G \cup f_i$, or we can make the same statement about $\{f_{i-1}, f_i, f_{i+1}\}$ and $\{f_{i-2}, f_{i-1}, f_i\}$. This means that we can apply Proposition 5.5, and deduce that M is a wheel or a whirl. This contradiction completes the proof of 5.8.4. $\qquad\square$

5.8.5. *The elements of $F - G$ form a consecutive subsequence of $(f_1, \ldots, f_m)$. The elements of $G - F$ form a consecutive subsequence of $(g_1, \ldots, g_n)$.*

Proof. It suffices to prove that the elements in $F - G$ form a consecutive subsequence of $(f_1, \ldots, f_m)$. Certainly $F \cup (E - G)$ contains at least two elements. Its complement, $G - F$, contains at least two elements by 5.8.4. Therefore $\lambda(F \cup (E - G)) \geq 2$. As $\lambda(F), \lambda(E - G) \leq 2$, it follows that $\lambda(F \cap (E - G)) = \lambda(F - G) \leq 2$.

If $|F - G| \geq 3$, then we can let $X = F - G$, and apply Proposition 5.7 to the fan $(f_1, \ldots, f_m)$. Therefore we assume $|F - G| = 2$. Let $i, j \in \{1, \ldots, m\}$ be chosen so that $i < j$, and $\{f_i, f_j\} = F - G$. By 5.8.2, $1 < i < j < m$. If $j = i + 1$, there is nothing left to prove, so $j > i + 1$. Assume that $j > i + 2$. Then $\{f_{i-1}, f_i, f_{i+1}\}$ and $\{f_i, f_{i+1}, f_{i+2}\}$ are a triangle or triad contained in $G \cup f_i$. Proposition 5.5 implies that M is a wheel or a whirl. Hence $j = i + 2$.

By duality, we can assume that f_i is a rim element of $(f_1, \ldots, f_m)$. Since $\{f_{i-1}, f_i, f_{i+1}\}$ is a triangle that contains f_i and is otherwise contained in G, we can apply Proposition 5.3. From 5.8.4 and the fact that $|F \cap G| \geq 3$, we see that $m, n \geq 5$. Thus $\{f_{i-1}, f_{i+1}\}$ is $\{g_1, g_2\}$, $\{g_{n-1}, g_n\}$, $\{g_1, g_n\}$, or $\{g_2, g_4\}$. But $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triangle, and we can apply the same arguments to show that $\{f_{i+1}, f_{i+3}\}$ is also one of the same four sets. We deduce that $\{\{f_{i-1}, f_{i+1}\}, \{f_{i+1}, f_{i+3}\}\}$ is one of $\{\{g_1, g_2\}, \{g_1, g_n\}\}$, $\{\{g_1, g_n\}, \{g_{n-1}, g_n\}\}$, $\{\{g_1, g_2\}, \{g_2, g_4\}\}$, or $\{\{g_2, g_4\}, \{g_4, g_5\}\}$. Proposition 5.3 implies that, in the first case, g_1 is both a spoke element and a

rim element. In the second case it implies g_n is a spoke element and a rim element. In the third case, g_2 is both a rim element and a spoke element, and in the fourth, g_4 is a rim element and a spoke element. Thus we have a contradiction in any case. $\qquad\square$

By 5.8.2, 5.8.4, and 5.8.5, there are indices $1 \le i < i+3 \le j \le m$ such that $F \cap G = \{f_1, \ldots, f_i\} \cup \{f_j \ldots, f_m\}$. Note that $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle or a triad that contains exactly one element of G. If $f_i = g_k$, then $k = 1$ or $k = n$, for otherwise, since $n \ge 4$, it follows that g_k is contained in a triangle and a triad that are both contained in G. This leads to a contradiction to orthogonality. By applying the same arguments to $\{f_{j-2}, f_{j-1}, f_j\}$, we see that f_j is equal to either g_1 or g_n. By reversing $(g_1, \ldots, g_n)$ as necessary, we can assume that $f_i = g_n$ and that $f_j = g_1$.

Assume that $|F \cap G| \ge 4$. Either $1 < i$ or $j < m$. Let us first assume $1 < i$. The fans $(f_1, \ldots, f_m)$ and $(g_1, \ldots, g_{n-1})$ intersect in $\{f_1, \ldots, f_{i-1}\} \cup \{f_j, \ldots, f_m\}$, and this set contains at least three elements. The inductive assumption implies that $(f_1, \ldots, f_m)$ and $(g_1, \ldots, g_{n-1})$ should meet in consecutive subsequences of $(f_1, \ldots, f_m)$ and $(g_1, \ldots, g_{n-1})$, but $G - g_n$ contains f_{i-1} and f_j, and no element between them. Similarly, if $j < m$, then $(f_1, \ldots, f_m)$ and $(g_2, \ldots, g_n)$ intersect in $\{f_1, \ldots, f_i\} \cup \{f_{j+1}, \ldots, f_m\}$, and this set is not consecutive in $(f_1, \ldots, f_m)$. From this contradiction to the inductive hypothesis, we deduce that $|F \cap G| = 3$. Either $i = 1$ and $j = m - 1$, or $i = 2$ and $j = m$. We can reverse $(f_1, \ldots, f_m)$ and $(g_1, \ldots, g_n)$, and assume that the former case holds. Thus $f_1 = g_n$ and $f_{m-1} = g_1$. Since $\{f_{m-2}, f_{m-1}, f_m\}$ is a triangle or a triad contained in $G \cup f_{m-2}$ that contains g_1 but not g_n, Proposition 5.3 implies that $f_m = g_2$. Hence $F \cap G = \{f_1, f_{m-1}, f_m\} = \{g_n, g_1, g_2\}$.

By replacing M with M^*, we can assume that $f_m = g_2$ is a spoke element of $(f_1, \ldots, f_m)$, so that $\{f_{m-2}, f_{m-1}, f_m\}$ is a triangle. Note that $\{g_1, g_2, g_3\}$ is not a triangle, or else it intersects the triad $\{f_{m-3}, f_{m-2}, f_{m-1}\}$ in the single element $f_{m-1} = g_1$. Therefore g_1 is a rim element of $(g_1, \ldots, g_n)$. It follows that

$$(f_1, \ldots, f_{m-2}, f_{m-1}, f_m, g_3, \ldots, g_{n-1}) = (f_1, \ldots, f_{m-2}, g_1, g_2, g_3, \ldots, g_{n-1})$$

is a fan of M. By 5.8.3, there is at most one element of E not contained in this fan, so M is a wheel or a whirl. This contradiction completes the proof of Lemma 5.8. $\qquad\square$

6. A FINITE CASE-CHECK THEOREM

From this point onwards, $\mathcal{M}$ will be a class of matroids closed under isomorphism and minors, and $N \in \mathcal{M}$ will be a fixed 3-connected matroid such that $|E(N)| \ge 4$, and N is neither a wheel nor a whirl. We let $\mathcal{F}_N$ be a family of disjoint fans in N. Assume that whenever $M' \in \mathcal{M}$ has N as a minor, M' is 3-connected up to series and parallel sets. Note that we can

replace $\mathcal{M}$ with the dual class $\{M^* \mid M \in \mathcal{M}\}$ and replace N with N^* and the same hypotheses will hold.

In the subsequent results we state some consequences of these definitions. First of all, the only 3-connected minors of wheels and whirls are wheels and whirls. This has the following implication.

Proposition 6.1. *No matroid having N as a minor is a wheel or a whirl.*

We are going to make continuous use of the next two results.

Proposition 6.2. *Let M' be a matroid in $\mathcal{M}$ that has N as a minor. Every triangle in M' is coindependent, and every triad is independent.*

Proof. By duality, it suffices to assume that T is a codependent triangle in M'. Then $r_{M'}(T) = 2$, and $r_{M'}^*(T) < 3$, so $\lambda_{M'}(T) \le 1$. Clearly T is not contained in a parallel class. If T is contained in a series class, then $\lambda_{M'}(T) = 0$, so $E(M') = T$ since M' is connected. This contradicts the fact that N is a minor of M'. As M' is 3-connected up to series and parallel sets, the complement of T is contained in a series or parallel class. This implies that a 3-connected minor of M' with at least four elements has precisely four elements. Hence $|E(N)| = 4$, and N is isomorphic to $U_{2,4}$, which contradicts the assumption that N is not a whirl. $\square$

The next result follows from Proposition 6.2 by orthogonality.

Corollary 6.3. *Let M' be a matroid in $\mathcal{M}$ that has N as a minor. If X is a $U_{2,4}$-restriction of M', and T^* is a triad of M', then $X \cap T^* = \emptyset$.*

Proposition 6.4. *Let M' be a 3-connected matroid in $\mathcal{M}$. Assume that $(e_1, \ldots, e_n)$ is a fan of M' such that $n \ge 4$, and N is a minor of $M'/e_i \backslash e_{i+1}$ for some $i \in \{1, \ldots, n-1\}$. Then $M'/e_i \backslash e_{i+1}$ is 3-connected.*

Proof. If $M'/e_i \backslash e_{i+1}$ is not 3-connected, then, as it is 3-connected up to series and parallel sets, there is a triangle that contains e_i but not e_{i+1}, or a triad that contains e_{i+1} but not e_i. We assume T^* is a triad satisfying $T^* \cap \{e_i, e_{i+1}\} = \{e_{i+1}\}$.

First assume that e_{i+1} is a rim element. If $i > 2$, then $\{e_{i-2}, e_{i-1}, e_i\}$ is a codependent triangle in $M' \backslash e_{i+1}$, contradicting Proposition 6.2. Similarly, if $i < n-2$, then $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a dependent triad in M'/e_i. Therefore $n - 2 \le i \le 2$, implying that $n = 4$ and $i = 2$. Since $e_i \notin T^*$, orthogonality with $\{e_i, e_{i+1}, e_{i+2}\}$ implies T^* contains the parallel pair $\{e_{i+1}, e_{i+2}\}$ in M'/e_i, and we have another contradiction to Proposition 6.2.

Thus e_{i+1} is a spoke element. If $\{e_i, e_{i+1}\}$ is contained in a triangle, then T^* contains the third element of this triangle, and T^* is a triad of M'/e_i that contains a parallel pair, contradicting Proposition 6.2. Therefore there is no triangle containing $\{e_i, e_{i+1}\}$, so $i = 1$. In this case T^* contains two elements of the triangle $\{e_{i+1}, e_{i+2}, e_{i+3}\}$. In a matroid without series pairs, two triads that intersect in two elements form a $U_{2,4}$-corestriction. Therefore, if T^* contains e_{i+2}, then $T^* \cup e_i$ is a $U_{2,4}$-corestriction of M'

that intersects the triangle $\{e_{i+1}, e_{i+2}, e_{i+3}\}$, contradicting Corollary 6.3. Therefore $T^* \cap \{e_{i+1}, e_{i+2}, e_{i+3}\} = \{e_{i+1}, e_{i+3}\}$. Let M'' be produced from M' by swapping labels on e_i and e_{i+2}. As $\{e_i, e_{i+2}\}$ is a series pair in $M' \backslash e_{i+1}$, it follows that $M'' \backslash e_{i+1} = M' \backslash e_{i+1}$, so $M'' \backslash e_{i+1}/e_i$ has N as a minor. However, M''/e_i contains the parallel pair $\{e_{i+1}, e_{i+3}\}$, which is contained in the triad T^*. This contradicts Proposition 6.2.

To complete the proof, we must assume that there is a triangle containing e_i but not e_{i+1}. In this case we replace M', N, and $\mathcal{M}$ by their duals, and we reverse the fan $(e_1, \ldots, e_n)$, relabeling e_{n-i} as e_i. After the relabeling, there is a triad that contains e_{i+1} but not e_i, so we make the same argument as before. □

Proposition 6.5. *Let M' be a matroid in $\mathcal{M}$ with N as a minor. Assume M' has a covering family, and that $F = (e_1, \ldots, e_n)$ is a fan in that family, where $\{e_i, e_{i+1}\} \subseteq E(M') - E(N)$ for some rim element e_i. Then N is a minor of $M'/e_i \backslash e_{i+1}$.*

Proof. Note that $n \geq 5$, since F contains a fan in $\mathcal{F}_N$ as well as $\{e_i, e_{i+1}\}$. If N is a minor of $M' \backslash e_i$, then $n = 5$ and $i = 3$, for otherwise $\{e_{i-3}, e_{i-2}, e_{i-1}\}$ or $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a codependent triangle in $M' \backslash e_i$. But now $F - \{e_i, e_{i+1}\} \subseteq E(N)$, so N contains the series pair $\{e_{i-2}, e_{i-1}\}$. Therefore N is a minor of M'/e_i. If $i > 1$, then N is a minor of $M'/e_i \backslash e_{i+1}$, as $\{e_{i-1}, e_{i+1}\}$ is a parallel pair in M'/e_i. If $i = 1$, then $\{e_3, e_4, e_5\}$ is a dependent triad in $M'/e_i/e_{i+1}$, so N' is a minor of $M'/e_i \backslash e_{i+1}$ in either case. □

Proposition 6.6. *Let M' be a 3-connected matroid in $\mathcal{M}$ with N as a minor. Assume that $M'/x \backslash y$ is 3-connected and has N as a minor. Let F be a fan of M' such that a fan in $\mathcal{F}_N$ is consistent with F. If either x or y is an internal element of F, then they are consecutive in F, and x is a rim element.*

Proof. Let $F = (e_1, \ldots, e_m)$, and assume x or y is an internal element in F. In the latter case, we swap M', N, and $\mathcal{M}$ for their duals, and swap labels on x and y. Therefore we lose no generality in assuming x is an internal element, e_i, of F. Assume x is a spoke element. If $n > 5$, then $\{e_{i-3}, e_{i-2}, e_{i-1}\}$ or $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a dependent triad in M'/x, which is a contradiction. Therefore $n \leq 5$. If $n = 4$, then $F - x \subseteq E(N)$, as F contains a fan in $\mathcal{F}_N$, as well as x. But $F - x$ also contains a parallel pair in M'/x. This leads to the contradiction that N contains a parallel pair. Therefore $n = 5$, and $x = e_3$, for otherwise M'/x contains a dependent triad. Now either $\{e_1, e_2\}$ or $\{e_4, e_5\}$ is contained in $E(N)$, and as both these sets are parallel pairs in M'/x, this is a contradiction.

Hence x is a rim element in F. Because $\{e_{i-1}, e_{i+1}\}$ is a parallel pair in M'/x, and $M'/x \backslash y$ is 3-connected, it follows that $y \in \{e_{i-1}, e_{i+1}\}$, and we are done. □

Proposition 6.7. *Let M' be a 3-connected member of $\mathcal{M}$, and assume that $(e_1, \ldots, e_n)$ is a fan of M' containing x, y, and z, where $\{x, y, z\}$ is a triangle of M', and N is a minor of $M'/x\backslash y$. If $n \geq 5$, then x and y are consecutive elements in $(e_1, \ldots, e_n)$.*

Proof. By Proposition 5.6, there is some $i \in \{1, \ldots, n-2\}$ such that $\{x, y, z\} = \{e_i, e_{i+1}, e_{i+2}\}$, where e_i is a spoke element. By reversing $(e_1, \ldots, e_n)$ as necessary, we can assume that $(x, y) = (e_i, e_{i+2})$, for otherwise there is nothing left to prove. If $i + 2 < n$, then $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a dependent triad of $M'/e_i = M'/x$, since $\{e_{i+1}, e_{i+2}\}$ is a parallel pair. This is impossible, as M'/x has N as a minor. Therefore $n = i + 2$.

Let M'' be the matroid obtained from M' by swapping the labels on $e_{i+1} = z$ and $e_{i+2} = y$. Since $\{y, z\}$ is a parallel pair in M'/x, it follows that $M''/x\backslash y = M'/x\backslash y$. Therefore N is a minor of $M''\backslash y$. But $\{e_{i-2}, e_{i-1}, e_i\}$ is a triangle of $M''\backslash y$ that contains a series pair. This contradiction completes the proof. $\square$

Proposition 6.8. *Let $M' \in \mathcal{M}$ be a 3-connected matroid with N as a minor. If $e \in E(M') - E(N)$, then for some (M_1, N_1) in $\{(M', N), ((M')^*, N^*)\}$ one of the following statements holds:*

   (i) *M_1/e is 3-connected and has N_1 as a minor,*
  (ii) *there is a triangle $\{x, e, f\}$ of M_1 such that N_1 is a minor of $M_1/e\backslash f$ and $M_1\backslash f$ is 3-connected,*
 (iii) *there is a fan (x, e, f, y) of M_1, where $\{x, e, f\}$ is a triangle, and $M_1/e\backslash f$ is 3-connected with N_1 as a minor.*

Proof. By duality, we can assume N_1 is a minor of M_1/e. We assume M_1/e is not 3-connected. Since M_1 is 3-connected, and M_1/e is 3-connected up to series and parallel sets, M_1 has a triangle containing e. Let this triangle be $\{x, e, f\}$, where N_1 is a minor of $M_1/e\backslash f$. We assume $M_1\backslash f$ is not 3-connected, so f is in a triad of M_1. This triad contains x or e by orthogonality, and does not contain both by Proposition 6.2. Let y be the third element in the triad. If $\{x, f, y\}$ is a triad, then it is a dependent triad in M_1/e, which is impossible, so $\{e, f, y\}$ is a triad, and statement (iii) holds by Proposition 6.4. $\square$

Proposition 6.9. *Let N' be a matroid in $\mathcal{M}$ such that (e_1, e_2, e_3, e_4) is a fan of N' with e_2 as a rim element, and $N'/e_2\backslash e_3 = N$. Then N' is 3-connected.*

Proof. Because N' has N as a minor, it is 3-connected up to series and parallel sets, so it is connected. In particular, it has no loops or coloops. If e_2 is in a parallel pair, then N'/e_2 has N as a minor and a loop, which is impossible. If e_2 is in a series pair, then orthogonality requires that the pair is contained in $\{e_1, e_2, e_3\}$, so this set is a codependent triangle in N', a contradiction to Proposition 6.2. By a dual argument, e_3 is not contained in any series or parallel pair in N. As $N = N'/e_2\backslash e_3$ is simple and cosimple, it follows that N' is simple and cosimple.

Assume N' is not 3-connected. As it is connected, we can let (X, Y) be a 2-separation. Since N' is simple and cosimple, it follows that $|X|, |Y| \geq 3$. If $X \cap \{e_2, e_3\}$ and $Y \cap \{e_2, e_3\}$ are both non-empty, then $(X - \{e_2, e_3\}, Y - \{e_2, e_3\})$ is a 2-separation in N, a contradiction. Therefore we can assume $e_2, e_3 \in X$. If $|X| > 3$, then $(X - \{e_2, e_3\}, Y)$ is a 2-separation in N. Therefore $|X| \leq 3$. If $|E(N)| = 4$, then N is isomorphic to $U_{2,4}$, contradicting the hypothesis that it is not a whirl. Hence $|E(N')| = |E(N)| + 2 \geq 7$, so $|Y| \geq 4$. Now $(X \cup \{e_1, e_4\}, Y - \{e_1, e_4\})$ is a 2-separation in N' that induces a 2-separation in N, so we have a contradiction that completes the proof. □

Now we can begin the proof of our main theorem.

Theorem 6.10. *Let $\mathcal{M}$ be a set of matroids that is closed under minors and isomorphism. Let $N \in \mathcal{M}$ be a 3-connected matroid such that $|E(N)| \geq 4$ and N is neither a wheel nor a whirl. Assume that any member of $\mathcal{M}$ with N as a minor is 3-connected up to series and parallel sets. Let $\mathcal{F}_N$ be a family of pairwise disjoint fans of N. If there is a 3-connected matroid in $\mathcal{M}$ with N as a minor that is not a fan-extension of N relative to $\mathcal{F}_N$, then there exists such a matroid, M, satisfying $|E(M)| - |E(N)| \leq 2$.*

For the remainder of the paper, we let $\mathcal{M}$, N, and $\mathcal{F}_N$ be as in the statement of Theorem 6.10. We let $M \in \mathcal{M}$ be a 3-connected matroid with N as a minor such that M is not a fan-extension of N, and, subject to these constraints, $|E(M)|$ is as small as possible. We assume that $|E(M)| - |E(N)| > 2$, and ultimately derive a contradiction from this, thereby proving Theorem 6.10.

Lemma 6.11. *Let M_0 be isomorphic to M, and assume that M_0 has N as a minor, but is not a fan-extension of N. Then M_0 does not have a covering family (relative to N and $\mathcal{F}_N$).*

Proof. We assume for a contradiction that M_0 has a covering family. Note $M_0 \neq N$, as M_0 is not a fan-extension of N. The fans in any covering family of M_0 contain the elements of $E(M_0) - E(N)$.

6.11.1. *In any fan belonging to a covering family of M_0, there are no two consecutive elements in $E(M_0) - E(N)$.*

Proof. Assume there is a covering family, $\mathcal{F}$, of M_0, and a fan, $F = (e_1, \ldots, e_n)$, in $\mathcal{F}$, such that e_i and e_{i+1} are not in $E(N)$ for some $i \in \{1, \ldots, n-1\}$. If e_i is a spoke element, then we can replace M, M_0, N, and $\mathcal{M}$ by their duals, and then apply the forthcoming arguments. Thus we assume that e_i is a rim element. Then N is a minor of $M_0/e_i \backslash e_{i+1}$ by Proposition 6.5. Propositions 5.2 and 6.4 show that $M_0/e_i \backslash e_{i+1}$ is a 3-connected matroid containing the fan $F - \{e_i, e_{i+1}\}$. The minimality of M implies $M_0/e_i \backslash e_{i+1}$ is a fan-extension of N. It is easy to see that $(\mathcal{F} - \{F\}) \cup \{F - \{e_i, e_{i+1}\}\}$ is a covering family in $M_0/e_i \backslash e_{i+1}$, and M_0

is obtained from $M_0/e_i \backslash e_{i+1}$ by a fan-lengthening move on $F - \{e_i, e_{i+1}\}$. Therefore M_0 is a fan-extension of N, a contradiction. $\qquad \square$

6.11.2. *If F is a fan in a covering family of M_0, then all internal elements of F belong to $E(N)$.*

Proof. Assume $F = (e_1, \ldots, e_n)$ is a fan in a covering family, and e_i is an internal element in $E(M_0) - E(N)$, so $n \geq 4$. By replacing M, M_0, N, and $\mathcal{M}$ with their duals if necessary, we can assume e_i is a rim element. If N is a minor of M_0/e_i, we contradict 6.11.1, as $\{e_{i-1}, e_{i+1}\}$ is a parallel pair in M_0/e_i, so N is a minor of $M_0/e_i \backslash e_{i-1}$ or $M_0/e_i \backslash e_{i+1}$. Therefore N is a minor of $M_0 \backslash e_i$, so $n \leq 5$, and if $n = 5$, then $i = 3$, for otherwise $\{e_{i-3}, e_{i-2}, e_{i-1}\}$ or $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a codependent triangle in $M_0 \backslash e_i$. By reversing F as required, we will assume that $i < n - 1$. Now $\{e_{i+1}, e_{i+2}\}$ is a series pair in $M_0 \backslash e_i$, so 6.11.1 means that N is a minor of $M_1 \backslash e_i / e_{i+2}$. Thus $n = 5$, since F contains $\{e_i, e_{i+2}\}$ and at least three elements of $E(N)$. In this case, $\{e_{i-2}, e_{i-1}, e_{i+1}\} \subseteq E(N)$, which leads to a contradiction as $\{e_{i-2}, e_{i-1}\}$ is is a series pair in $M_0 \backslash e_i$. $\qquad \square$

Fix the covering family $\mathcal{F}$, and let $F = (e_1, \ldots, e_n)$ be a fan in $\mathcal{F}$ such that $e_1 \notin E(N)$. This implies $n \geq 4$. By duality, we assume e_1 is a rim element, and this easily implies that N is a minor of M_0/e_1. If M_0/e_1 is 3-connected, then M_0 is obtained from M_0/e_1 by a fan-lengthening move on $F - e_1$, and $(\mathcal{F} - \{F\}) \cup \{F - e_1\}$ is a covering family in M_0/e_1. By minimality, M_0/e_1 is a fan-extension of N, and therefore, so is M_0. Hence M_0/e_1 is not 3-connected. As M_0/e_2 contains a parallel pair, it is not 3-connected. The dual of Tutte's Triangle Lemma (see [9, Lemma 8.7.7]) implies there is a triangle, T, of M_0 containing e_1 and either e_2 or e_3. It follows from Proposition 5.6 that T cannot be contained in F, for otherwise $\{e_1, e_2, e_3\}$ is a triad and a triangle. Let f be the element in $T - \{e_1, e_2, e_3\}$. Then N is a minor of $M_0/e_1 \backslash f$, by 6.11.2. Since $\mathcal{F}$ is a covering family, there is a fan, $F_f \in \mathcal{F}$, such that f is in F_f. As F_f contains at least three elements of $E(N)$, any internal element of F_f is contained in a triad. Orthogonality with the triangle T now implies that f is a terminal spoke element of F_f. Clearly $F_f - f$ is a fan of M_0.

If e_2 is in T, then $F + f = (f, e_1, \ldots, e_n)$ is a fan of M_0, and it is straight-forward to see that $(\mathcal{F} - \{F, F_f\}) \cup \{F + f, F_f - f\}$ is a covering family of M_0. But now e_1 is an internal element of $F + f$ that is not in $E(N)$, contradicting 6.11.2. Hence $T = \{f, e_1, e_3\}$. This means that $n = 4$, for otherwise T violates orthogonality with the triad $\{e_3, e_4, e_5\}$. Thus (e_2, e_3, e_4) or its reversal is in $\mathcal{F}_N$.

Let $F_f = (f_1, \ldots, f_m)$, where $f = f_1$. By applying the duals of the previous arguments, we see that $\{f_1, f_3\}$ is contained in a triad of M_0, that $m = 4$, and that either (f_2, f_3, f_4) or its reversal is in $\mathcal{F}_N$. The triad containing $\{f_1, f_3\}$ contains either e_1 or e_3, by orthogonality with T, and the latter case cannot occur, by orthogonality with $\{e_2, e_3, e_4\}$. Thus $\{f_3, f_1, e_1\}$ is a triad, and $(f_4, f_2, f_3, f_1, e_1, e_3, e_2, e_4)$ is a fan of M_0.

Let I and I^* be, respectively, independent and coindependent sets such that $I \cap I^* = \emptyset$ and $N = M_0/I \backslash I^*$. Let $N' = M_0/(I - \{e_1, f_1\}) \backslash (I^* - \{e_1, f_1\})$. Note that $\{f_4, f_2, f_3\}$ and $\{e_3, e_2, e_4\}$ are, respectively, a triad and a triangle in N', for otherwise N contains a circuit or cocircuit with at most two elements. Since $\{f_2, f_3, f_1\}$ is a union of circuits in N', it is easy to see it is a circuit of N', for otherwise we can use orthogonality with $\{f_4, f_2, f_3\}$ to show that $\{f_2, f_3\}$ contains a circuit in N', and hence in N. Hence f_1 is not a loop of N'. Similarly, $\{e_1, e_3, e_2\}$ is a triad in N' and e_1 is not a coloop. Since $\{f_3, f_1, e_1\}$ is a union of cocircuits, orthogonality with $\{f_2, f_3, f_1\}$ now implies that $\{f_3, f_1, e_1\}$ is a triad in N'. Orthogonality with $\{e_1, e_3, e_2\}$ implies $\{f_1, e_1, e_3\}$ is a triangle in N'. Therefore $(f_4, f_2, f_3, f_1, e_1, e_3, e_2, e_4)$ is a fan of N'.

Because $N' \backslash e_1$ contains the codependent triangle $\{f_1, f_3, f_2\}$, we see that N is not a minor of $N' \backslash e_1$. Therefore it is a minor of N'/e_1, and this matroid contains the parallel pair $\{f_1, e_3\}$, so $N = N'/e_1 \backslash f_1$. By applying Proposition 6.9 to the fan (e_3, e_1, f_1, f_3), we see that N' is 3-connected. Since $|E(M)| - |E(N)| > 2 = |E(N')| - |E(N)|$, N' is a proper minor of M_0. Therefore N' is a fan-extension of N.

As $|E(N')| - |E(N)| = 2$, it follows that N' is obtained from N by either one or two fan-lengthening moves. However, none of N'/e_1, $N' \backslash e_1$, N'/f_1, $N' \backslash f_1$ is 3-connected, so N' is obtained from N by a single fan-lengthening move. Hence there is a covering family, $\mathcal{F}'$, of N' containing a fan, F', such that e_1 and f_1 are consecutive elements of F'. Assume that e_3 is in F'. Because $\mathcal{F}'$ is a covering family and (e_2, e_3, e_4) or its reversal is in $\mathcal{F}_N$, it follows that (e_2, e_3, e_4) is consistent with F'. This means that F' contains $\{f_1, e_1, e_3, e_2, e_4\}$. By applying Proposition 5.6 to the triangle $\{f_1, e_1, e_3\}$, the triad $\{e_1, e_3, e_2\}$, and the triangle $\{e_3, e_2, e_4\}$, we see that $(f_1, e_1, e_3, e_2, e_4)$ is a contiguous subsequence of F'. Hence (e_2, e_3, e_4) is not consistent with F' after all. This contradiction shows that e_3 is not in F'. As $\{e_3, e_1, f_1\}$ is a triangle, Proposition 5.3 implies that, up to reversing, e_1 and f_1 are the first two elements in F', and F' starts with a rim element. If f_3 is not in F', then we can apply Proposition 5.3 to the triad $\{f_3, f_1, e_1\}$ and get the contradiction that F' starts with a spoke element. Therefore f_3 is in F', and therefore (f_2, f_3, f_4) is consistent with F'. As $\{e_1, f_1, f_3\}$, $\{f_1, f_3, f_2\}$, and $\{f_3, f_2, f_4\}$ are triangles or triads of N' contained in F', Proposition 5.6 implies that $(e_1, f_1, f_3, f_2, f_4)$ is a contiguous subsequence of F', so (f_2, f_3, f_4) is not consistent with F' after all. This contradiction completes the proof. $\square$

Lemma 6.12. *Let M_0 be a matroid isomorphic to M such that N is a minor of M_0, but M_0 is not a fan-extension of N. Assume that $(e_1, \ldots, e_n)$ is a fan of M_0, where $n \geq 5$, and $\{e_1, e_2, e_3\}$ is a triangle. If N is a minor of $M_0/e_2 \backslash e_3$ and some ordering of $\{e_1, e_4, \ldots, e_n\}$ is in $\mathcal{F}_N$, then either $(e_1, e_4, \ldots, e_n)$ or its reversal is in $\mathcal{F}_N$.*

Proof. We assume that some ordering of $\{e_1, e_4, \ldots, e_n\}$ is in $\mathcal{F}_N$. Let I and I^* be independent and coindependent sets such that $I \cap I^* = \emptyset$ and $N = M_0/I\backslash I^*$ and let $N' = M_0/(I-\{e_2, e_3\})\backslash(I^*-\{e_2, e_3\})$. If $4 \le i \le n-2$ and i is odd (respectively, even), then $\{e_i, e_{i+1}, e_{i+2}\} \subseteq E(N)$, and $\{e_i, e_{i+1}, e_{i+2}\}$ is a circuit (cocircuit) in M_0, and is therefore a union of circuits (cocircuits) in N'. Hence it is a circuit (cocircuit) in N', or else N contains a circuit (cocircuit) of at most two elements, which contradicts the 3-connectivity of N. Thus $(e_4, \ldots, e_n)$ is a fan of N'. As $\{e_3, e_4, e_5\}$ is a union of circuits in N', it is a circuit, for otherwise $\{e_4, e_5\}$ contains a circuit in N', and hence in N. Since $\{e_1, e_2, e_3\}$ is a union of circuits in N', e_2 is not a coloop of N'. Now, as $\{e_2, e_3, e_4\}$ is a union of cocircuits, orthogonality with $\{e_3, e_4, e_5\}$ shows that $\{e_2, e_3, e_4\}$ is a triad of N'. Finally, e_1 is not a loop of N, and hence not a loop in N'. Neither e_2 nor e_3 is a loop in N', as $\{e_2, e_3, e_4\}$ is a triad. If $\{e_1, e_2, e_3\}$ is not a circuit, then it is contained in a parallel class, and this leads to a contradiction to orthogonality with $\{e_2, e_3, e_4\}$. Thus we have shown $(e_1, \ldots, e_n)$ is a fan in N'.

As $N'\backslash e_2$ contains a codependent triangle, it cannot have N as a minor. Therefore N'/e_2 has N as a minor, as well as the parallel pair $\{e_1, e_3\}$. Therefore $N = N'/e_2\backslash e_3$. Proposition 6.9 implies that N' is 3-connected. As $|E(M_0)| - |E(N)| > 2 = |E(N')| - |E(N)|$, we see that N' is a fan-extension of N.

Let F' be the fan in a covering family of N' such that some ordering of $\{e_1, e_4, \ldots, e_n\}$ is consistent with F'. As $(e_1, \ldots, e_n)$ is a fan of N' that intersects F' in at least three elements, including e_1 and e_n, we see from Lemma 5.8 that there is a contiguous subsequence of F' comprising the elements $\{e_1, \ldots, e_n\}$. Now, repeatedly applying Proposition 5.6, it follows that $(e_1, \ldots, e_n)$ is a contiguous subsequence of F'. Thus $(e_1, e_4, \ldots, e_n)$ is consistent with F', so this fan, or its reversal, is in $\mathcal{F}_N$. □

If we apply an arbitrary permutation to the element labels of M, we do not know if the resulting matroid will be a fan-extension of N, or, indeed, if it will have N as a minor. In the next lemma, and in Lemma 6.15, we look for circumstances under which the permuted matroid is guaranteed to have N as a minor, but not to be a fan-extension of N. This will allow us to relabel elements of M without losing generality.

Lemma 6.13. *Let M_0 be isomorphic to M, and assume that M_0 has N as a minor, but is not a fan-extension of N. Assume that (e_1, e_2, e_3, e_4) is a fan of M_0 with e_2 as a rim element, and N is a minor of $M_0/e_2\backslash e_3$. Let M' be the matroid obtained from M_0 by swapping the labels on e_1 and e_3. Then M' contains N as a minor, but is not a fan-extension of N.*

Proof. Because $\{e_1, e_3\}$ is a parallel pair of M_0/e_2, it follows that $M'/e_2\backslash e_3 = M_0/e_2\backslash e_3$, so N is a minor of $M'/e_2\backslash e_3$. Proposition 6.4 implies $M_0/e_2\backslash e_3$, and hence $M'/e_2\backslash e_3$, is 3-connected. Note that (e_3, e_2, e_1, e_4) is a fan of M', where e_2 is a rim element. Assume for a contradiction that M'

is a fan-extension of N. As $e_2, e_3 \notin E(N)$, there is a covering family of M' that contains fans that contain e_2 and e_3.

6.13.1. *If $\mathcal{F}'$ is a covering family of M', then no fan in $\mathcal{F}'$ contains $\{e_1, e_2, e_3\}$.*

Proof. Assume $F' \in \mathcal{F}'$ contains $\{e_1, e_2, e_3\}$. Then Proposition 5.6 implies that $\{e_1, e_2, e_3\}$ is a set of three consecutive elements in F'. As $e_2, e_3 \notin E(N)$, it now is obvious that we can swap the labels on e_1 and e_3 and obtain a covering family of M_0, contradicting Lemma 6.11. $\qquad\square$

6.13.2. *Let $\mathcal{F}'$ be a covering family of M' and let F_2 be a fan in $\mathcal{F}'$ that contains e_2. Then e_2 is a terminal rim element of F_2.*

Proof. Assume that e_2 is in a triangle, T, contained in F_2. If e_3 is not in T, then $M'/e_2 \backslash e_3$ contains a parallel pair, and this is impossible as $M'/e_2 \backslash e_3$ is 3-connected with at least four elements in its ground set. Therefore $e_2, e_3 \in T$. Orthogonality with the triad $\{e_2, e_1, e_4\}$ shows that $e_1 \in T$ or $e_4 \in T$. The former case is not true by 6.13.1, so $T = \{e_2, e_3, e_4\}$. As $\{e_1, e_2, e_3\}$ is also a triangle, $\{e_1, e_2, e_3, e_4\}$ is a $U_{2,4}$-restriction of M' that intersects the triad $\{e_2, e_1, e_4\}$, contradicting Corollary 6.3. Therefore e_2 is contained in no triangle in F_2. Note that F_2 contains a fan in $\mathcal{F}_N$, as well as the element e_2, so it contains at least four elements. Therefore e_2 is a terminal rim element in F_2, as desired. $\qquad\square$

6.13.3. *There is a covering family, $\mathcal{F}'$, of M', and a fan $F' \in \mathcal{F}'$, such that e_2 and e_3 are consecutive elements in F'.*

Proof. Assume that this is not true. Let $\mathcal{F}'$ be an arbitrary covering family of M'. Let $F_2 = (f_1, \ldots, f_m)$ be a fan in $\mathcal{F}'$ where $e_2 = f_1$ and $\{f_1, f_2, f_3\}$ is a triad. Orthogonality with the triangle $\{e_1, e_2, e_3\}$ shows that e_3 or e_1 is in $\{f_2, f_3\}$. Assume that e_3 is in $\{f_2, f_3\}$, so $e_1 \notin F_2$ by 6.13.1. Because we have assumed that e_2 and e_3 are not consecutive in F_2, we deduce that $e_3 = f_3$. Now $m \leq 4$, or else we have a contradiction to orthogonality between $\{e_1, e_2, e_3\} = \{e_1, f_1, f_3\}$ and the triad $\{f_3, f_4, f_5\}$. However, F_2 contains a fan in $\mathcal{F}_N$ as well as the elements e_2 and e_3, so $m \geq 5$. This contradiction shows that e_1 is in $\{f_2, f_3\}$, and therefore $e_3 \notin F_2$.

Now $\{f_1, f_2, f_3\} = \{e_1, e_2, z\}$ for some element z, where $z \neq e_3$. If $z \neq e_4$, then $\{e_2, e_1, e_4, z\}$ is a $U_{2,4}$-corestriction of M' that intersects the triangle $\{e_1, e_2, e_3\}$, contradicting the dual of Corollary 6.3. Therefore $z = e_4$, so $\{f_1, f_2, f_3\} = \{e_1, e_2, e_4\}$. If $(f_1, f_2, f_3) = (e_2, e_1, e_4)$, then $F_2 + e_3 = (e_3, f_1, \ldots, f_m)$ is a fan of M'. Let F_3 be the fan in $\mathcal{F}'$ that contains e_3. Then F_3 contains a fan in $\mathcal{F}_N$ as well as e_3, so $|F_3| \geq 4$. If e_3 is an internal element in F_3, then it is contained in a triad that is contained in F_3, and such a triad violates orthogonality with $\{e_1, e_2, e_3\}$. Therefore e_3 is a terminal spoke element of F_3. Now we easily see that $(\mathcal{F}' - \{F_2, F_3\}) \cup \{F_2 + e_3, F_3 - e_3\}$ is a covering family of M', and e_3 and e_2 are consecutive in $F_2 + e_3$. This contradicts our assumption, so $(f_1, f_2, f_3) = (e_2, e_4, e_1)$.

Now $m \leq 4$, or else we have a contradiction to orthogonality between the triangle $\{e_1, e_2, e_3\}$ and the triad $\{f_3, f_4, f_5\}$ (recall e_3 is not in F_2). In fact, $m = 4$, since F_2 contains one of the fans in $\mathcal{F}_N$, as well as e_2. Therefore $F_2 = (f_1, f_2, f_3, f_4) = (e_2, e_4, e_1, f_4)$, and (e_4, e_1, f_4) or (f_4, e_1, e_4) is in $\mathcal{F}_N$. However, $(e_3, f_1, f_3, f_2, f_4) = (e_3, e_2, e_1, e_4, f_4)$ is also a fan of M', so $(e_1, e_2, e_3, e_4, f_4)$ is a fan of M_0. Now $\{e_1, e_2, e_3\}$ is a triangle, and N is a minor of $M_0/e_2\backslash e_3$. By Lemma 6.12, (e_1, e_4, f_4) or (f_4, e_4, e_1) is in $\mathcal{F}_N$. Thus $\mathcal{F}_N$ contains two distinct fans that are non-disjoint. This is a contradiction. $\square$

By 6.13.2 and 6.13.3 we can let $\mathcal{F}'$ be a covering family of M' and let $F' = (f_1, \ldots, f_m)$ in $\mathcal{F}'$ be such that $(f_1, f_2) = (e_2, e_3)$ where $m \geq 5$ and $\{f_1, f_2, f_3\}$ is a triad. Now 6.13.1 implies $e_1 \notin F'$. Observe that $F' + e_1 = (e_1, f_1, f_2, \ldots, f_m)$ is a fan of M'. By 6.13.1, $(\mathcal{F}' - \{F'\}) \cup \{F' + e_1\}$ cannot be a covering family in M'. Therefore there is a fan, $G = (g_1, \ldots, g_t)$, in $\mathcal{F}' - \{F'\}$, that contains e_1. Orthogonality with $\{e_1, e_2, e_3\}$ shows that e_1 is not contained in a triad that is contained in G. Let G_N be the fan in $\mathcal{F}_N$ that is consistent with G. If e_1 is not in G_N, then G contains at least four elements, so e_1 must be a terminal element of G. In this case, it is easy to see that $(\mathcal{F}' - \{F', G\}) \cup \{F' + e_1, G - e_1\}$ is a covering family of M', which leads to a contradiction with 6.13.1. Therefore e_1 is contained in G_N.

Assume that $t = 3$. Then, by reversing, we can assume that (g_1, g_2, g_3) is in $\mathcal{F}_N$. Orthogonality with $\{e_1, e_2, e_3\}$ implies $\{g_1, g_2, g_3\}$ is a triangle. But $\{e_2, e_1, e_4\}$ is a triad in M', so orthogonality requires that e_4 is in $\{g_1, g_2, g_3\}$. Let g be the element in $\{g_1, g_2, g_3\} - \{e_1, e_4\}$. Then (e_3, e_2, e_1, e_4, g) is a fan in M', so (e_1, e_2, e_3, e_4, g) is a fan in M_0. As N is a minor of $M_0/e_2\backslash e_3$, Lemma 6.12 implies that, up to reversing (e_1, e_4, g) is in $\mathcal{F}_N$, so $(e_1, e_4, g) = (g_1, g_2, g_3)$. This means that (e_3, e_2, e_1, e_4, g) is a fan in M' that is consistent with (g_1, g_2, g_3), and

$$(\mathcal{F}' - \{F', (e_1, e_4, g)\}) \cup \{F' - \{f_1, f_2\}, (e_3, e_2, e_1, e_4, g)\}$$

is a covering family in M', contradicting 6.13.1. Therefore $t \geq 4$.

Because e_1 is not contained in any triad that is contained in G, we see that e_1 is a terminal spoke element in $(g_1, \ldots, g_t)$. By reversing as necessary, we can assume that $e_1 = g_1$ and $\{g_1, g_2, g_3\}$ is a triangle.

Orthogonality between $\{g_1, g_2, g_3\}$ and $\{e_1, e_2, e_4\}$ requires that e_4 is in $\{g_2, g_3\}$. Assume that $e_4 = g_2$. Then

$$G' = (e_3, e_2, g_1, g_2, g_3, \ldots, g_t) = (e_3, e_2, e_1, e_4, g_3, \ldots, g_t)$$

is a fan in M'. It is easy to see that $(\mathcal{F}' - \{F', G\}) \cup \{F' - \{f_1, f_2\}, G'\}$ is a covering family in M', contradicting 6.13.1. Therefore $g_3 = e_4$.

Next we observe that $t = 4$, for otherwise $\{g_3, g_4, g_5\}$ is a triangle and $\{e_1, e_2, e_4\}$ is a triad that intersects it in the element $e_4 = g_3$. Now G_N is a subsequence of at least three elements from (g_1, g_2, g_3, g_4) or (g_4, g_3, g_2, g_1) that contains e_1.

Assume that g_4 is not in G_N. Then G_N is equal, up to reversing, to (g_1, g_2, g_3). Since $(e_3, e_2, g_1, g_3, g_2) = (e_3, e_2, e_1, e_4, g_2)$ is a fan in M', $(e_1, e_2, e_3, e_4, g_2)$ is a fan in M_0. As N is a minor of $M_0/e_2 \backslash e_3$, we see from Lemma 6.12 that $(e_1, e_4, g_2) = (g_1, g_3, g_2)$, or its reversal, is in $\mathcal{F}_N$. Therefore $\mathcal{F}_N$ contains two distinct fans of three elements that are non-disjoint. This contradiction shows that g_4 is in G_N.

Assume that exactly one of g_2 and g_3 is in G_N. Then G_N is consistent with the fan $G' = (e_3, e_2, g_1, g_3, g_2, g_4) = (e_3, e_2, e_1, e_4, g_2, g_4)$ in M', and

$$(\mathcal{F}' - \{F', G\}) \cup \{F' - \{f_1, f_2\}, G'\}$$

is a covering family of M', violating 6.13.1. We conclude that G_N is equal, up to reversing, to (g_1, g_2, g_3, g_4).

Now $(e_3, e_2, g_1, g_3, g_2, g_4) = (e_3, e_2, e_1, e_4, g_2, g_4)$ is a fan in M', so $(e_1, e_2, e_3, e_4, g_2, g_4)$ is a fan in M_0, and some ordering of $\{e_1, e_4, g_2, g_4\}$ is in $\mathcal{F}_N$. By Lemma 6.12, $(e_1, e_4, g_2, g_4) = (g_1, g_3, g_2, g_4)$ or its reversal is in $\mathcal{F}_N$. This shows that $\mathcal{F}_N$ contains two distinct fans that are non-disjoint. We have a contradiction that completes the proof of the lemma. □

Now we have some control over the ways in which we may permute the element labels on M. The next result shows a consequence of this: we can assume that there is an element in M whose removal is 3-connected with N as a minor.

Lemma 6.14. *There exists a matroid M_0, isomorphic to M, such that N is a minor of M_0, but M_0 is not a fan-extension of N. Moreover, M_0 has a 3-connected single-element deletion or contraction that has N as a minor.*

Proof. We will assume that the lemma is false, so that no such matroid M_0 exists. Let e be an element in $E(M) - E(N)$. We use Proposition 6.8. By possibly replacing M, N, and $\mathcal{M}$ with their duals, we can assume that $(e_1, \ldots, e_n)$ is a fan of M such that N is a minor of $M/e_i \backslash e_{i+1}$ for some rim element e_i, where $i \in \{2, \ldots, n-2\}$. Moreover, $M/e_i \backslash e_{i+1}$ is 3-connected. We assume that amongst all such fans, $(e_1, \ldots, e_n)$ has been chosen so that n is as large as possible.

6.14.1. *If e_1 is a spoke element of $(e_1, \ldots, e_n)$, then $M \backslash e_1$ is 3-connected, and if e_1 is a rim element, then M/e_1 is 3-connected.*

Proof. Assume e_1 is a rim element. Then $n > 4$, for otherwise $i = 2$, and both e_1 and e_2 are rim elements. If M/e_1 is not 3-connected, then e_1 is in a triangle. It follows from Proposition 5.6 that this triangle is not contained in $\{e_1, \ldots, e_n\}$. Let $z \notin \{e_1, \ldots, e_n\}$ be an element in a triangle with e_1. Orthogonality with $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ shows $\{e_1, e_2, z\}$ is a triangle, and $(z, e_1, \ldots, e_n)$ is a fan. This contradicts the maximality of n. Similarly, if e_1 is a spoke element and $M \backslash e_1$ is not 3-connected, then e_1 is in a triad with an element $z \notin \{e_1, \ldots, e_n\}$, and either $(z, e_1, \ldots, e_n)$ is a fan, or $n = 4$, and $\{e_1, e_3, z\}$ is a triad. In the first case we have a contradiction to the maximality of n. In the latter case, $n = 4$ and $i = 2$, so N is a minor of

$M/e_2\backslash e_3$, but $\{e_1, e_3, z\}$ is a triad that contains the parallel pair $\{e_1, e_3\}$ in M/e_2, contradicting Proposition 6.2. $\qquad\square$

Define $M^{(0)}$ to be M. We obtain $M^{(1)}$ from $M^{(0)}$ by swapping labels on e_{i-1} and e_{i+1}. Then $M^{(1)}/e_i\backslash e_{i+1} = M^{(0)}/e_i\backslash e_{i+1}$, and by applying Lemma 6.13 to $(e_{i-1}, e_i, e_{i+1}, e_{i+2})$ we see that $M^{(1)}$ is not a fan-extension of N. Let $M^{(2)}$ be obtained from $M^{(1)}$ by swapping the labels on e_{i-2} and e_i. Then $M^{(2)}/e_i\backslash e_{i+1} = M^{(1)}/e_i\backslash e_{i+1}$, and the dual of Lemma 6.13 implies $M^{(2)}$ is not a fan-extension of N. In general, when $j \in \{1, \ldots, i-1\}$, we obtain $M^{(j)}$ from $M^{(j-1)}$ by swapping labels on e_{i-j} and e_{i+1} if j is odd, and on e_{i-j} and e_i if j is even. An obvious inductive argument establishes the following statement.

6.14.2. *For every $j \in \{0, 1, \ldots, i-1\}$, $M^{(j)}/e_i\backslash e_{i+1} = M/e_i\backslash e_{i+1}$, so $M^{(j)}/e_i\backslash e_{i+1}$ has N as a minor. Moreover, $M^{(j)}$ is not a fan-extension of N.*

Note that $M^{(i-1)}$ is obtained from M by relabeling $(e_1, \ldots, e_n)$ as

$$\begin{cases} (e_i, e_{i+1}, e_1, \ldots, e_{i-1}, e_{i+2}, \ldots, e_n) & \text{if } i \text{ is odd} \\ (e_{i+1}, e_i, e_1, \ldots, e_{i-1}, e_{i+2}, \ldots, e_n) & \text{if } i \text{ is even.} \end{cases}$$

Assume i is odd, so e_1 is a rim element of $(e_1, \ldots, e_n)$ in M. There is an isomorphism from M to $M^{(i-1)}$ that relabels e_1 as e_i. Therefore $M^{(i-1)}/e_i$ is 3-connected, by 6.14.1. Similarly, if i is even, then e_1 is a spoke element and $M^{(i-1)}\backslash e_{i+1}$ is 3-connected. In either case, $M^{(i-1)}$ is isomorphic to M, and has N as a minor, but is not a fan-extension of N. Since $M^{(i-1)}$ has a 3-connected single-element deletion or contraction that has N as a minor, the lemma is proved. $\qquad\square$

Now we can assume M has a 3-connected single-element deletion or contraction with N as a minor. Lemma 6.13 considered swapping labels on elements that belonged to a fan of M. In the next lemma we swap labels on elements that belong to a fan in a 3-connected single-element deletion.

Lemma 6.15. *Let M_0 be isomorphic to M, and assume that M_0 has N as a minor, but is not a fan-extension of N. Assume $M_0\backslash e$ is 3-connected and has (e_1, e_2, e_3, e_4) as a fan. Moreover, assume that either:*

    (i) *e_2 is a rim element and N is a minor of $M_0\backslash e/e_2\backslash e_3$, or*
    (ii) *e_2 is a spoke element, and N is a minor of $M_0\backslash e\backslash e_2/e_3$.*

Let M' be obtained from M_0 by swapping the labels on e_1 and e_3. Then M' has N as a minor, but is not a fan-extension of N.

Proof. Note that in case (i), $M'\backslash e/e_2\backslash e_3 = M_0\backslash e/e_2\backslash e_3$, and in case (ii), $M'\backslash e\backslash e_2/e_3 = M_0\backslash e\backslash e_2/e_3$, so M' certainly has N as a minor. If (e_1, e_2, e_3, e_4) is a fan of M_0, then we could apply Lemma 6.13 or its dual to (e_1, e_2, e_3, e_4), and deduce that M' is not a fan-extension of N. In this case there is nothing left to prove, so we assume that (e_1, e_2, e_3, e_4) is not

a fan of M_0. Thus, if statement (i) holds, $\{e, e_2, e_3, e_4\}$ is a cocircuit of M_0. In this case we set (x_1, x_2, x_3, x_4) to be (e_1, e_2, e_3, e_4). If statement (ii) holds, $\{e, e_1, e_2, e_3\}$ is a cocircuit, and in this case we set (x_1, x_2, x_3, x_4) to be (e_4, e_3, e_2, e_1). In either case, (x_1, x_2, x_3, x_4) is a fan of $M_0 \backslash e$ with x_1 as a spoke element, N is a minor of $M_0 \backslash e / x_2 \backslash x_3$, and $\{e, x_2, x_3, x_4\}$ is a cocircuit of M_0.

6.15.1. $M_0 \backslash e / x_2 \backslash x_3$ *is 3-connected.*

Proof. This follows by applying Proposition 6.4 to $M_0 \backslash e$. $\square$

6.15.2. $M_0 \backslash x_3$ *is 3-connected.*

Proof. Assume otherwise. Then x_3 is contained in a triad, T^*, of M_0. As $M_0 \backslash e$ is 3-connected, T^* is also a triad in $M_0 \backslash e$. Proposition 6.2 and orthogonality with $\{x_1, x_2, x_3\}$ shows that T^* contains exactly one of x_1 or x_2. If x_1 is in T^*, then T^* is a dependent triad in M_0 / x_2. Since N is a minor of M_0 / x_2 this contradicts Proposition 6.2. Therefore $x_2, x_3 \in T^*$. Note $T^* \neq \{x_2, x_3, x_4\}$, as $\{e, x_2, x_3, x_4\}$ is a cocircuit of M_0. Therefore $T^* \cup x_4$ is a $U_{2,4}$-corestriction in $M_0 \backslash e$ that intersects the triangle $\{x_1, x_2, x_3\}$. This contradiction to the dual of Corollary 6.3 completes the proof. $\square$

We will assume for a contradiction that M' is a fan-extension of N. Then there is a covering family of M' containing a fan, F_e, that contains e. As $M' \backslash e$ is 3-connected (since it is isomorphic to $M_0 \backslash e$) and F_e contains at least four elements, it follows that e is a terminal spoke element of a fan in M_0 that has at least four elements. Let T_e be the triangle in this fan that contains e. Orthogonality with $\{e, x_2, x_3, x_4\}$ shows that T_e contains x_2, x_3, or x_4. As $T_e - e$ is contained in a triad of M_0, it follows from 6.15.2 that x_3 is not in T_e. Assume that x_2 is in T_e. Orthogonality between $\{x_1, x_2, x_3\}$ and the triad containing $T_e - e$ shows that x_1 is contained in the triad. However, x_1 is not in T_e, for that would imply $T_e \cup x_3$ is a $U_{2,4}$-restriction of M_0 that intersects a triad, a contradiction. Now we can swap the labels on x_1 and x_3, and delete x_3. The resulting matroid has N as a minor, since x_1 and x_3 are parallel in M_0 / x_2, and contains the codependent triangle T_e. Because this is a contradiction, we conclude that x_2 is not in T_e, so $x_4 \in T_e$. Let z be the third element of T_e, so that $\{e, x_4, z\}$ is a triangle in M_0, and $\{x_4, z\}$ is contained in a triad, $\{x_4, z, z'\}$, of M_0.

6.15.3. *The elements in* $\{e, x_1, x_2, x_3, x_4, z, z'\}$ *are pairwise distinct.*

Proof. By the hypotheses of the lemma, e, x_1, x_2, x_3, and x_4 are pairwise distinct. Also, e, x_4, z, and z' are distinct members of a fan. If $z = x_1$, then orthogonality between $\{x_1, x_2, x_3\}$ and $\{x_4, z, z'\}$ shows that $z' \in \{x_2, x_3\}$. Then $\{x_4, z, z'\}$ and $\{x_2, x_3, x_4\}$ are distinct triads of $M_0 \backslash e$ that intersect in two elements. Hence $M_0 \backslash e$ has a $U_{2,4}$-corestriction intersecting a triangle, a contradiction. If $z = x_2$, then $\{e, x_2, x_4\}$ is a triangle and a triad in $M_0 \backslash x_3$, which is impossible by 6.15.2. Finally, if $z = x_3$, then $M_0 \backslash x_3$ contains the series pair $\{x_4, z'\}$, contradicting 6.15.2. This shows that e, x_1, x_2, x_3,

x_4, and z are pairwise distinct. If $z' \in \{x_1, x_2, x_3\}$, then we contradict orthogonality with $\{x_4, z, z'\}$. This completes the proof.   □

Now we subdivide into two cases.

Case 1. Statement (i) holds, so M' is obtained from M_0 by swapping the labels on x_1 and x_3. Since M' is a fan-extension of N, there is a covering family, $\mathcal{F}'$, of M' containing a fan, F_2, that contains x_2. Let $F_2 = (f_1, \ldots, f_m)$ Note that $m \geq 4$.

Assume that x_2 is contained in a triangle, T_2, that is contained in F_2. Suppose x_3 is not in T_2. From 6.15.1 we see that $M' \backslash e / x_2 \backslash x_3$ contains no parallel pair, so e is in T_2. This means $m \geq 5$, as F_2 contains e, x_2, and at least three elements of $E(N)$. Because $M' \backslash e$ is 3-connected, e is contained in no triad of M'. Therefore we can assume that $e = f_1$, and f_1 is a spoke element of F_2. Thus $T_2 = \{f_1, f_2, f_3\}$. As N is a minor of $M' / x_2 \backslash e$, Proposition 6.7 implies e and x_2 are consecutive in F_2, so $x_2 = f_2$. Proposition 6.4 applied to $(f_m, \ldots, f_1)$ implies that $M' / f_2 \backslash f_1 = M' / x_2 \backslash e$ is 3-connected. This is impossible, as this matroid contains the parallel pair $\{x_1, x_3\}$. Therefore we conclude that x_3 is in T_2.

If x_1 is not in T_2, then $T_2 \cup x_1$ is a $U_{2,4}$-restriction of $M' \backslash e$, and $\{x_2, x_1, x_4\}$ is a triad. This contradiction shows that $T_2 = \{x_1, x_2, x_3\}$. Now $\{x_1, x_2, x_3\}$ form a set of three consecutive elements in F_2, by Proposition 5.6. Let F_2' be the fan obtained from F_2 by swapping x_1 and x_3. As x_2 and x_3 are not in $E(N)$, it is clear that $(\mathcal{F}' - \{F_2\}) \cup \{F_2'\}$ is a covering family of M_0. This is a contradiction to Lemma 6.11, so we have to conclude that x_2 is contained in no triangle in F_2. Therefore we can assume that $x_2 = f_1$, and this is a rim element of F_2.

By orthogonality with $\{x_1, x_2, x_3\}$, and the fact that $M' \backslash x_1$ is 3-connected (by 6.15.2), we see that $\{f_1, f_2, f_3\}$ contains x_3, so $m \geq 5$. The dual of Proposition 6.7 shows that x_2 and x_3 are consecutive elements in F_2, so $x_3 = f_2$. If x_1 is in F_2, then we can apply Proposition 5.6 to deduce that $x_1 = f_3$. This implies $\{f_1, f_2, f_3\}$ is simultaneously a triad and a triangle of M', which is impossible. Hence x_1 is not in F_2. Let $(F_2 + x_1)'$ be the fan of M_0 obtained by appending x_1 to the end of F_2 and then swapping the locations of x_1 and x_3. Since $(\mathcal{F}' - \{F_2\}) \cup (F_2 + x_1)'$ cannot be a covering family in M_0, it follows that x_1 is in another fan, F_1, belonging to $\mathcal{F}'$. Orthogonality with $\{x_1, x_2, x_3\}$ shows that x_1 is a terminal spoke element in F_1. Because $(\mathcal{F}' - \{F_1, F_2\}) \cup \{F_1 - x_1, (F_2 + x_1)'\}$ is not a covering family of M_0, we deduce that x_1 is in the fan of $\mathcal{F}_N$ that is consistent with F_1.

Let T_1 be the triangle in F_1 that contains x_1. Orthogonality with $\{e, x_2, x_1, x_4\}$, implies that T_1 contains e or x_4. If $e \in T_1$, then $|F_1| \geq 4$, as e is not in $E(N)$. In this case, e is in a triad that is contained in F_1, which is a contradiction, as $M' \backslash e$ is 3-connected. Therefore T_1 contains x_4. Recall that $\{e, x_4, z\}$ is a triangle and $\{x_4, z, z'\}$ is a triad of M_0, and hence of M', for some element z'. Assume T_1 contains z. This implies that $\{e, x_1, x_4, z\}$

is a $U_{2,4}$-restriction of M' that meets the triad $\{x_4, z, z'\}$, which is impossible. Therefore T_1 contains z'. By swapping labels on x_1 and x_3 in M', we see that $\{x_3, x_4, z'\}$ is a triangle of M_0. Let M_1 be the matroid obtained from M_0 by swapping labels on x_2 and x_4. As $\{x_2, x_4\}$ is a series pair in $M_0 \backslash e \backslash x_3$, it follows that $M_1 \backslash e \backslash x_3 / x_2 = M_0 \backslash e \backslash x_3 / x_2$, so $M_1 \backslash e \backslash x_3 / x_2$ has N as a minor. Now $\{x_3, x_2, z'\}$ is a triangle of M_1, so $\{x_3, z'\}$ is a parallel pair in M_1/x_2. We let M_2 be the matroid obtained from M_1 by swapping labels on x_3 and z'. Then $M_2/x_2 \backslash x_3 \backslash e = M_1/x_2 \backslash x_3 \backslash e$, and hence $M_2/x_2 \backslash x_3 \backslash e$ has N as a minor. However, $\{e, x_2, z\}$ is a triangle in $M_2 \backslash x_3$, and $\{x_2, z\}$ is a series pair. This contradiction to Proposition 6.2 competes the analysis of Case 1.

Case 2. Statement (ii) holds, so M' is obtained from M_0 by swapping the labels on x_2 and x_4. Let $\mathcal{F}'$ be a covering family of M', and let $F_3 \in \mathcal{F}'$ be the fan that contains x_3. Note that x_3 is a terminal spoke element of F_3 as $M' \backslash x_3$ is 3-connected. Let T_3 be the triangle in F_3 that contains x_3. Orthogonality with the cocircuit $\{e, x_2, x_3, x_4\}$ shows that e, x_2, or x_4 is in T_3. Since $M' \backslash e$ is 3-connected, and $T_3 - x_3$ is contained in a triad, e is not in T_3. Assume x_2 is in T_3. Since $\{x_2, z, z'\}$ is a triad of M', either z or z' is in T_3. In the former case, $\{e, x_2, x_3, z\}$ is a $U_{2,4}$-restriction that intersects a triad, which is a contradiction. Hence $T_3 = \{x_2, x_3, z'\}$. As M'/x_2 contains the parallel pair $\{x_3, z'\}$ and N as a minor, we can swap labels on x_3 and z' in M' and then delete x_3. The resulting matroid contains N as a minor, a triangle $\{e, x_2, z\}$, and a series pair $\{x_2, z\}$. This contradicts Proposition 6.2, so we deduce that x_4 is in T_3. Now $T_3 = \{x_1, x_4, x_3\}$, for otherwise $T_3 \cup x_1$ is a $U_{2,4}$-restriction of $M' \backslash e$ that intersects a triad. Let $F_3 = (f_1, \ldots, f_s)$. By reversing, we can assume that $f_s = x_3$.

Proposition 5.6 implies that $\{x_1, x_4, x_3\} = \{f_{s-2}, f_{s-1}, f_s\}$. Assume that $x_4 \neq f_{s-1}$. If $s \geq 5$, then we let F_3' be the fan of M_0 obtained from F_3 by relabeling x_4 with x_2. As N is a minor of $M_0/x_2 \backslash x_3$, and $\{x_1, x_2, x_3\} \subseteq F_3'$, Proposition 6.7 implies x_2 and x_3 are consecutive in F_3', which implies x_4 and x_3 are consecutive in F_3, contrary to assumption. Therefore $s = 4$, and $F_3 = (f_1, x_4, x_1, x_3)$. Thus (f_1, x_4, x_1) or its reversal is in $\mathcal{F}_N$. Note that $\{f_1, x_1, x_4\}$ is a triad of M', so $\{f_1, x_1, x_2\}$ is a triad of M_0, and $(f_1, x_1, x_2, x_3, x_4)$ is a fan in $M_0 \backslash e$. As $M_0 \backslash e$ is a fan-extension of N, it contains a fan that has (f_1, x_4, x_1) as a subsequence. This fan intersects $(f_1, x_1, x_2, x_3, x_4)$ in at least three elements, so it contains $\{f_1, x_1, x_2, x_3, x_4\}$, by Lemma 5.8, and therefore contains $(f_1, x_1, x_2, x_3, x_4)$ as a contiguous subsequence, by three applications of Proposition 5.6 and its dual. Hence (f_1, x_4, x_1) is not a subsequence after all. From this contradiction we see that $x_4 = f_{s-1}$ and $(f_{s-2}, f_{s-1}, f_s) = (x_1, x_4, x_3)$.

Let F_N be the fan in $\mathcal{F}_N$ that is consistent with F_3. Let $\mathcal{F}$ be the family of fans in M_0 that is induced from $\mathcal{F}'$ by swapping the labels on x_2 and x_4. As the fans in $\mathcal{F}'$ contains all elements in $E(M') - E(N)$, it follows that the fans in $\mathcal{F}$ contain all elements in $E(M_0) - E(N)$, except possibly for x_2. However, x_4 is in F_3, so x_2 is in a fan of $\mathcal{F}$. Therefore the fans in $\mathcal{F}$

contain every element in $E(M_0) - E(N)$. As x_2 does not belong to a fan of $\mathcal{F}_N$, the only way $\mathcal{F}$ can fail to be a covering family of M_0 is if x_4 is in F_N. We deduce this is the case. Then x_4 is a terminal element of F_N, because $(f_{s-2}, f_{s-1}, f_s) = (x_1, x_4, x_3)$ and $x_3 \notin E(N)$.

We claim that $\{e, x_2, x_4\}$ is the unique triad of $M_0 \backslash x_3$ that contains e. Suppose T^* is another such triad. If x_2 is in T^*, then $T^* \cup x_4$ is a $U_{2,4}$-core-striction of $M_0 \backslash x_3$, and $\{e, x_4, z\}$ is a triangle that intersects it. This is a contradiction, so $x_2 \notin T^*$. This means that $T^* - e$ is a series pair of $M_0 \backslash e / x_2 \backslash x_3$, contradicting 6.15.1, and proving the claim.

As $M_0 \backslash x_3$ is 3-connected, it contains a covering family.

6.15.4. *There exists a covering family of $M_0 \backslash x_3$ containing a fan that contains e and x_2.*

Proof. Let $\mathcal{G}$ be an arbitrary covering family of $M_0 \backslash x_3$, and let $G = (g_1, \ldots, g_t)$ be the fan in $\mathcal{G}$ that contains e. If e is in a triad that is contained in G, then we are done, as this triad must be $\{e, x_2, x_4\}$, by the earlier claim. Therefore we assume that $e = g_1$ is a terminal spoke element of G, and x_2 is not in G. Orthogonality between $\{e, x_2, x_4\}$ and $\{g_1, g_2, g_3\}$ shows that $x_4 \in \{g_2, g_3\}$. If $x_4 = g_3$, then $t = 4$, for otherwise we contradict orthogonality between $\{e, x_2, x_4\}$ and $\{g_3, g_4, g_5\}$. In this case $(g_2, g_3, g_4) = (g_2, x_4, g_4)$ or its reversal is in $\mathcal{F}_N$. Recall F_N is the fan in $\mathcal{F}_N$ consistent with F_3, and x_4 is in F_N. Hence F_N is (g_2, x_4, g_4) or its reversal, and this is a contradiction, as we concluded x_4 is a terminal element of F_N. Therefore $x_4 = g_2$, and $G + x_2 = (x_2, g_1, \ldots, g_t)$ is a fan of $M_0 \backslash x_3$. Let G' be the fan in $\mathcal{G}$ that contains x_2. Then x_2 is a terminal rim element of G', by orthogonality with $\{e, x_2, x_4\}$. Now $(\mathcal{G} - \{G, G'\}) \cup \{G + x_2, G' - x_2\}$ is the desired covering family of $M_0 \backslash x_3$. $\square$

Next we claim $M_0 \backslash x_3 / x_2$ is 3-connected. Note that $(f_{s-3}, f_{s-2}, f_{s-1}, f_s) = (f_{s-3}, x_1, x_4, x_3)$ is a fan of M', so (f_{s-3}, x_1, x_2, x_3) is a fan in M_0. Now Proposition 6.4 tells us that $M_0 \backslash x_3 / x_2$ is 3-connected, as desired.

Let $\mathcal{G}$ be a covering family of $M_0 \backslash x_3$, and let $G = (g_1, \ldots, g_t)$ be a fan in $\mathcal{G}$ that contains e and x_2. Then $t \geq 5$. As $M_0 \backslash x_3 / x_2$ is 3-connected, x_2 is a terminal rim element of G. We can assume $x_2 = g_1$. The only triad of $M_0 \backslash x_3$ that contains e is $\{e, x_2, x_4\}$, so if x_4 is not in G then e is contained in no triad in G, implying $e = g_t$ is a spoke element. But then $\{e, x_2, x_4\}$ violates orthogonality with $\{g_{t-2}, g_{t-1}, g_t\}$. Hence G contains e, x_2, and x_4. The dual of Proposition 5.6 implies $\{e, x_2, x_4\} = \{g_1, g_2, g_3\}$. As $t \geq 5$, and e is not contained in the triad $\{g_3, g_4, g_5\}$, we see that $(g_1, g_2, g_3) = (x_2, e, x_4)$.

Since G contains x_4 it follows that F_N is consistent with G. Now consider the fans G and $(f_1, \ldots, f_{s-2}, x_2)$ in $M_0 \backslash x_3$. They both contain x_2, and the elements of $F_N - x_4$. Therefore they intersect in at least three elements, so we can apply Lemma 5.8. This shows that the elements common to both fans form contiguous subsequences in both. This implies $\{g_1, g_2, g_3\} = \{x_2, e, x_4\}$ is a triad contained in $(f_1, \ldots, f_{s-2}, x_2)$. Therefore Proposition 5.6 implies $\{x_2, e, x_4\} = \{x_2, f_{s-2}, f_{s-3}\} = \{x_2, x_1, f_{s-3}\}$. This is impossible as e, x_1,

and x_4 are distinct elements. We have a contradiction that completes the proof of Lemma 6.15. □

If $M_0 \backslash e$ is 3-connected with N as a minor, where M_0 is isomorphic to M and is not a fan-extension, then $M_0 \backslash e$ must be a fan-extension, by the minimality of M. This implies $M_0 \backslash e$ has a covering family. In the next result, we examine this covering family, and discover that the elements in $E(M_0 \backslash e) - E(N)$ are concentrated at the ends of fans.

Lemma 6.16. *Let M_0 be isomorphic to M. Assume that N is a minor of M_0, but that M_0 is not a fan-extension of N. Assume also that $M_0 \backslash e$ is 3-connected and has N as a minor. Let $\mathcal{F}_e$ be a covering family of $M_0 \backslash e$, and let $(e_1, \ldots, e_m)$ be a fan in $\mathcal{F}_e$. If $2 < i < m - 1$, then e_i is in $E(N)$.*

Proof. Assume that the lemma fails, so that $(e'_1, \ldots, e'_m)$ is a fan in a covering family of $M_0 \backslash e$ and e'_j is in $E(M_0 \backslash e) - E(N)$, for some $j \in \{3, \ldots, m - 2\}$. Note that $m \geq 5$.

We claim that either e'_{j-1} or e'_{j+1} is in $E(M_0 \backslash e) - E(N)$. Assume for a contradiction that $\{e'_{j-1}, e'_{j+1}\} \subseteq E(N)$. If e'_j is a rim element in $(e'_1, \ldots, e'_m)$ then let M' be $M_0 \backslash e$ and let N' be N. Otherwise let M' be $(M_0 \backslash e)^*$ and let N' be N^*. Thus e'_j is a rim element in the fan $(e'_1, \ldots, e'_m)$ in M'. Now N' is a minor of $M' \backslash e'_j$, for if N' is a minor of M'/e'_j, then N' contains the parallel pair $\{e'_{j-1}, e'_{j+1}\}$. Therefore $m = 5$ and $j = 3$, because otherwise $\{e'_{j-3}, e'_{j-2}, e'_{j-1}\}$ or $\{e'_{j+1}, e'_{j+2}, e'_{j+3}\}$ is a codependent triangle in $M' \backslash e'_j$. From $m = 5$ and $j = 3$, it follows that $\{e'_1, e'_2, e'_4, e'_5\}$ is a union of series pairs in $M' \backslash e'_j$, and contains at least three elements of $E(N')$. This leads to a contradiction to the 3-connectivity of N'. Therefore e'_{j-1} or e'_{j+1} is in $E(M_0 \backslash e) - E(N)$, as claimed.

By reversing $(e'_1, \ldots, e'_m)$ as necessary, we can assume that e'_i and e'_{i+1} are in $E(M_0 \backslash e) - E(N)$, where $1 < i < m$, and e'_i is a rim element. Proposition 6.5 tells us that N is a minor of $M_0 \backslash e / e'_i \backslash e'_{i+1}$.

By applying Lemma 6.15 to the fan $(e'_{i-1}, e'_i, e'_{i+1}, e'_{i+2})$, we see that we can swap labels on e'_{i-1} and e'_{i+1} without any loss of generality. If $i - 1 > 1$, we would then be able to apply Lemma 6.15 to the fan $(e'_{i-2}, e'_{i+1}, e'_i, e'_{i-1})$, and swap labels on e'_{i-2} and e'_i. Deleting e from the resulting matroid reveals the following fan:

$$(e_1, \ldots, e'_{i-3}, e'_i, e'_{i+1}, e'_{i-2}, e'_{i-1}, e'_{i+2}, \ldots, e_m).$$

In fact, by starting from $(e'_1, \ldots, e'_i, e'_{i+1}, \ldots, e'_m)$, and repeatedly applying Lemma 6.15, it is possible for us to assume that any sequence of the form

$$(e'_1, \ldots, e'_j, x, y, e'_{j+1}, \ldots, e'_{i-1}, e'_{i+2}, \ldots, e'_m) \quad \text{or}$$
$$(e'_1, \ldots, e'_{i-1}, e'_{i+2}, \ldots, e'_j, x, y, e'_{j+1}, \ldots, e'_m)$$

is a fan of $M_0 \backslash e$. If we have applied Lemma 6.15 an even number of times in this process, then (x, y) is (e'_i, e'_{i+1}), otherwise it is (e'_{i+1}, e'_i). Note that this collection of sequences includes $(x, y, e'_1, \ldots, e'_{i-1}, e'_{i+2}, \ldots, e'_m)$ and

$(e'_1, \ldots, e'_{i-1}, e'_{i+2}, \ldots, e'_m, x, y)$. We remark that if i had been equal to 1 or $m-1$, then Lemma 6.15 would not have been applicable. This is why it was necessary to show that i lies between 2 and $m-1$.

Since there is a fan of $\mathcal{F}_N$ that is consistent with $(e'_1, \ldots, e'_m)$, by repeatedly applying Lemma 6.15, we can assume that we have the following situation: $(e_1, \ldots, e_{n+2})$ is a contiguous subsequence of $(e'_1, \ldots, e'_m)$, and is therefore a fan of $M_0\backslash e$. Some fan $F \in \mathcal{F}_N$ is consistent with $(e_1, \ldots, e_n)$ and contains e_1 and e_n. Also, e_{n+1} and e_{n+2} are not in $E(N)$. Let $\{x, y\} = \{e_{n+1}, e_{n+2}\}$, where x is a rim element of $(e_1, \ldots, e_{n+2})$. Then N is a minor of $M_0\backslash e/x\backslash y$ by Proposition 6.5.

6.16.1. *Let e_i be a rim element of $(e_1, \ldots, e_{n+2})$, where $1 \leq i \leq n$. Then $\{e_i, e_{i+1}, e_{i+2}\}$ is not a triad in M_0.*

Proof. Assume this is false. By starting from $(e'_1, \ldots, e'_m)$, and using Lemma 6.15 to shift x and y, we can, without losing any generality, assume that $(e_1, \ldots, e_{i-1}, x, y, e_i, \ldots, e_n)$ is a fan of $M_0\backslash e$. Here x is a rim element and $\{x, y, e_i\}$ is a triad of M_0, while N is a minor of $M_0\backslash e/x\backslash y$. By Proposition 6.4, $M_0\backslash e/x\backslash y$ is 3-connected. If $M_0/x\backslash y$ is not 3-connected, there is a triangle of M_0 that contains x and e but not y. Orthogonality with $\{x, y, e_i\}$ shows that $\{e, x, e_i\}$ is a triangle of M_0, so $\{e, x, e_i\}$ is a codependent triangle in $M_0\backslash y$. This contradiction shows that $M_0/x\backslash y$ is 3-connected, and is therefore a fan-extension of N.

Let F' be the fan in a covering family of $M_0/x\backslash y$ such that F is consistent with F'. Now $(e_1, \ldots, e_n)$ is a fan of $M_0/x\backslash y\backslash e$, by Proposition 5.2, and F is consistent with $(e_1, \ldots, e_n)$ and contains e_1 and e_n. Note that if e is in F', then it is a terminal spoke element, as $M_0/x\backslash y\backslash e$ is 3-connected. Hence $F' - e$ is a fan of $M_0/x\backslash y\backslash e$. Because $F' - e$ and $(e_1, \ldots, e_n)$ have at least three elements in common, and these elements include e_1 and e_n, it follows from Lemma 5.8 that the elements in $\{e_1, \ldots, e_n\}$ form a contiguous subsequence of $F' - e$.

Assume that the elements $\{e_1, \ldots, e_n\}$ do not appear in the order $(e_1, \ldots, e_n)$ in $F' - e$. Then it follows easily from Proposition 5.6 that $n < 5$. Note that F is consistent with $(e_1, \ldots, e_n)$ and with F', so the elements of F appear in F' in the same order as they do in $(e_1, \ldots, e_n)$. Therefore our assumption means that F is not equal to $(e_1, \ldots, e_n)$ or its reversal, although it contains e_1 and e_n. If $n = 3$, then F is equal to (e_1, e_2, e_3) or its reversal, contrary to our conclusion. Therefore $n = 4$, and F is either (e_1, e_2, e_4) or (e_1, e_3, e_4), or one of their reversals. Now $\{e_1, e_2, e_3, e_4\}$ is contained in $E(N)$, or else N contains a series or parallel pair. Therefore (e_1, e_2, e_3, e_4) is a fan of N. As (e_1, e_2, e_4) or (e_1, e_3, e_4) is also a fan of N, we have contradicted Proposition 5.6. Thus $(e_1, \ldots, e_n)$ is a contiguous subsequence of $F' - e$, and hence of F'.

Let F' be $(f_1, \ldots, f_{j-1}, e_1, \ldots, e_n, f_j, \ldots, f_p)$, so this sequence is a fan in $M_0/x\backslash y$. We will show that

$$(f_1, \ldots, f_{j-1}, e_1, \ldots, e_{i-1}, x, y, e_i, \ldots, e_n, f_j, \ldots, f_p)$$

is a fan in M_0. Let this sequence be labeled $(g_1, \ldots, g_{k-1}, x, y, g_k, \ldots, g_r)$, so that $(g_1, \ldots, g_r)$ is a fan in $M_0/x\backslash y$. Note that $i \leq n$, so there is at least one element in $\{g_k, \ldots, g_r\}$. By assumption, $\{x, y, e_i\} = \{x, y, g_k\}$ is a triad of M_0.

Assume that $r \geq k+1$. If $g_{k+1} = e_{i+1}$, then $\{y, e_i, e_{i+1}\} = \{y, g_k, g_{k+1}\}$ is a triangle of M_0, since $(e_1, \ldots, e_{i-1}, x, y, e_i, \ldots, e_n)$ is a fan in $M_0\backslash e$. On the other hand, if $g_{k+1} = f_j$, then $i = n$, and $\{e_{n-1}, e_n, f_j\} = \{g_{k-1}, g_k, g_{k+1}\}$ is a triangle in $M_0/x\backslash y$. It cannot be a triangle in M_0 by orthogonality with the triad $\{x, y, g_k\}$. Therefore $\{g_{k-1}, x, g_k, g_{k+1}\}$ is a circuit in M_0. Now $\{g_{k-1}, x, y\}$ is a triangle of $M_0\backslash e$, and therefore of M_0. Strong circuit-exchange between $\{g_{k-1}, x, g_k, g_{k+1}\}$ and $\{g_{k-1}, x, y\}$ shows that there is a circuit in $\{g_{k-1}, y, g_k, g_{k+1}\}$ that contains y. Orthogonality shows that this circuit cannot contain g_{k-1}, because $n \geq 3$, and $\{e_{n-2}, e_{n-1}, x\} = \{g_{k-2}, g_{k-1}, x\}$ is a triad of $M_0\backslash e$. Therefore $\{y, g_k, g_{k+1}\}$ is a triangle in M_0 in any case.

Next we assume that $r \geq k+2$. Then $\{g_k, g_{k+1}, g_{k+2}\}$ is a triad in $M_0/x\backslash y$. Assume that it is not a triad in M_0, so that $\{y, g_k, g_{k+1}, g_{k+2}\}$ is a cocircuit. If $n \geq i+2$, then $\{e_i, e_{i+1}, e_{i+2}\} = \{g_k, g_{k+1}, g_{k+2}\}$ is a triad in $M_0\backslash e$, and $\{y, g_k, g_{k+1}, g_{k+2}\}$ is a union of cocircuits. Therefore $\{y, g_k, g_{k+1}, g_{k+2}\}$ is a $U_{2,4}$-corestriction of $M_0\backslash e$. As $\{y, g_k, g_{k+1}\}$ is a triangle this is a contradiction to Corollary 6.3, so $n < i+2$, meaning that $i > 1$. Now $\{e_{i-1}, x, y\} = \{g_{k-1}, x, y\}$ is a triangle of $M_0\backslash e$ that violates orthogonality with $\{y, g_k, g_{k+1}, g_{k+2}\}$. Hence $\{g_k, g_{k+1}, g_{k+2}\}$ is a triad in M_0.

Now we consider all the sets comprising three consecutive elements in $(g_{k+1}, \ldots, g_r)$. If such a set is a triangle in $M_0/x\backslash y$, then it is a triangle in M_0, by orthogonality with the triad $\{x, y, g_k\}$. If such a set is a triad in $M_0/x\backslash y$, then it is a triad in M_0 by orthogonality with the triangle $\{y, g_k, g_{k+1}\}$. We have shown that $(x, y, g_k, \ldots, g_r)$ is a fan in M_0.

Assume that $k > 1$. If $g_{k-1} = e_{i-1}$, then $\{g_{k-1}, x, y\}$ is a triangle in M_0, since $(e_1, \ldots, e_{i-1}, x, y, e_i, \ldots, e_n)$ is a fan of $M_0\backslash e$. Therefore we assume that $g_{k-1} = f_{j-1}$. This means that $i = 1$, which implies that $\{g_k, \ldots, g_r\}$ contains at least three elements. Note $\{f_{j-1}, e_1, e_2\} = \{g_{k-1}, g_k, g_{k+1}\}$ is a triangle in $M_0/x\backslash y$, but not in M_0, by orthogonality with the triad $\{x, y, g_k\}$. Therefore $\{g_{k-1}, x, g_k, g_{k+1}\}$ is a circuit in M_0. We perform strong circuit-exchange on $\{g_{k-1}, x, g_k, g_{k+1}\}$ and the triangle $\{y, g_k, g_{k+1}\}$ to obtain a circuit of M_0 contained in $\{g_{k-1}, x, y, g_{k+1}\}$ that contains y. Orthogonality with the triad $\{g_k, g_{k+1}, g_{k+2}\}$ shows that this circuit cannot contain g_{k+1}. Therefore $\{g_{k-1}, x, y\}$ is a triangle of M_0 in any case.

Next we assume that $k > 2$. We wish to show that $\{g_{k-2}, g_{k-1}, x\}$ is a triad in M_0. Now $\{g_{k-2}, g_{k-1}, g_k\}$ is a triad in $M_0/x\backslash y$. It is not a triad in M_0, by orthogonality with the triangle $\{g_{k-1}, x, y\}$, so $\{g_{k-2}, g_{k-1}, y, g_k\}$ is a cocircuit of M_0. By strong cocircuit-exchange with $\{x, y, g_k\}$, we find a cocircuit, C^*, of M_0 contained in $\{g_{k-2}, g_{k-1}, x, g_k\}$ that contains x. If

C^* does not contain g_k, then $\{g_{k-2}, g_{k-1}, x\}$ is a triad, as desired. Therefore we assume that $g_k \in C^*$. Orthogonality with the triangle $\{g_{k-1}, x, y\}$ shows that $g_{k-1} \in C^*$. Because $\{x, y, g_k\}$ is a triad of M_0, it follows that if $C^* = \{g_{k-1}, x, g_k\}$, then $C^* \cup y$ is a $U_{2,4}$-corestriction of M_0. As $\{g_{k-1}, x, y\}$ is a triangle, this contradicts Corollary 6.3. Hence $C^* = \{g_{k-2}, g_{k-1}, x, g_k\}$. It follows that $r = k$, for otherwise C^* and $\{y, g_k, g_{k+1}\}$ violate orthogonality. From $r = k$, we deduce that $i = n$, and as $n \geq 3$, it follows that $(g_{k-2}, g_{k-1}) = (e_{n-2}, e_{n-1})$, so $\{g_{k-2}, g_{k-1}, x\}$ is a triad in $M_0 \backslash e$. Now C^* is a union of cocircuits in $M_0 \backslash e$ that contains the triad $\{g_{k-2}, g_{k-1}, x\}$, so C^* is a $U_{2,4}$-corestriction in $M_0 \backslash e$. Since $\{g_{k-1}, x, y\}$ is a triangle, this is impossible. Therefore $\{g_{k-2}, g_{k-1}, x\}$ is a triad in M_0, as we claimed.

Finally, we observe that any set of three consecutive elements from $(g_1, \ldots, g_{k-1})$ that is a triangle in $M_0 / x \backslash y$ is a triangle in M_0, since $\{x, y, g_k\}$ is a triad. Any such set that is a triad in $M_0 / x \backslash y$ is a triad in M_0, as $\{g_{k-1}, x, y\}$ is a triangle.

We have shown that

$$(g_1, \ldots, g_{k-1}, x, y, g_k, \ldots, g_r) =$$
$$(f_1, \ldots, f_{j-1}, e_1, \ldots, e_{i-1}, x, y, e_i, \ldots, e_n, f_j, \ldots, f_p)$$

is a fan of M_0. Now $M_0 / x \backslash y$ has a covering family containing $F' = (f_1, \ldots, f_{j-1}, e_1, \ldots, e_n, f_j, \ldots, f_p)$, and M_0 is obtained from $M_0 / x \backslash y$ by a fan-lengthening move on this fan. Hence M_0 is a fan-extension of N. This contradiction completes the proof of 6.16.1. $\qquad\square$

6.16.2. *If e_i is a rim element of $(e_1, \ldots, e_{n+2})$, and there is a triad of $M_0 \backslash e$ contained in $\{e_1, \ldots, e_{n+2}\}$ that does not contain e_i, then there is a triangle of M_0 containing $\{e, e_i\}$.*

Proof. Assume that the claim fails. We will start from $(e'_1, \ldots, e'_m)$, and then use Lemma 6.15 to shift x and y appropriately. By then possibly reversing, we can assume that $(e_1, \ldots, e_{i-1}, x, y, e_i, \ldots, e_n)$ is a fan of $M_0 \backslash e$, where x is a rim element and $M_0 \backslash e / x \backslash y$ has N as a minor; moreover, we assume there is no triangle of M_0 containing $\{e, x\}$, and $\{e_j, e_{j+1}, e_{j+2}\}$ is a triad of $M_0 \backslash e$, for some $j \in \{i, \ldots, n - 2\}$. Proposition 6.4 implies that $M_0 \backslash e / x \backslash y$ is 3-connected. Because there is no triangle containing e and x in M_0, it follows that $M_0 / x \backslash y$ is 3-connected, and hence a fan-extension of N. Let F' be the fan in a covering family of $M_0 / x \backslash y$ such that F is consistent with F'. If F' contains e, then e is a terminal element because $M_0 / x \backslash y \backslash e$ is 3-connected, so $F' - e$ is a fan of $M_0 / x \backslash y \backslash e$. Now F is also consistent with $(e_1, \ldots, e_n)$, which is a fan of $M_0 / x \backslash y \backslash e$. Since $(e_1, \ldots, e_n)$ and $F' - e$ have at least three elements in common, including e_1 and e_n, it follows from Lemma 5.8 that the elements in $\{e_1, \ldots, e_n\}$ form a contiguous subsequence of $F' - e$. As $\{e_j, e_{j+1}, e_{j+2}\}$ is a triad in $M_0 / x \backslash y \backslash e$, it is a set of three consecutive elements in F' by the dual of Proposition 5.6. It cannot be a triangle in $M_0 / x \backslash y$, for then it would be a triad and a triangle in $M_0 / x \backslash y \backslash e$. Therefore $\{e_j, e_{j+1}, e_{j+2}\}$ is a triad in $M_0 / x \backslash y$, but not in M_0

by 6.16.1. This shows that y is in the coclosure of $\{e_i, \ldots, e_n\}$ in M_0, and hence in $M_0\backslash e$. It is in the closure of the same set because $\{y, e_i, e_{i+1}\}$ is a triangle. Thus $\{y, e_i, \ldots, e_n\}$ is 2-separating in $M_0\backslash e$, so the complement of $\{y, e_i, \ldots, e_n\}$ contains at most one element. Proposition 5.4 implies $M_0\backslash e$ is a wheel or a whirl. This contradiction completes the proof of 6.16.2. $\square$

6.16.3. $n \leq 5$, and if $n = 5$, then e_1 is a spoke element of $(e_1, \ldots, e_{n+2})$.

Proof. If this statement is false, then, by starting from $(e'_1, \ldots, e'_m)$, and applying Lemma 6.15 to shift x and y as necessary, we can let $(x_1, \ldots, x_5, y, x)$ be either $(e_1, \ldots, e_5, y, x)$ or $(e_2, \ldots, e_6, y, x)$, and assume that $(x_1, \ldots, x_5, y, x)$ is a fan of $M_0\backslash e$, where N is a minor of $M_0\backslash e/x\backslash y$, and x_1 is a rim element of $(x_1, \ldots, x_5, y, x)$. Since $\{x_1, x_2, x_3\}$, $\{x_3, x_4, x_5\}$, and $\{x_5, y, x\}$ are triads in $M_0\backslash e$, we apply 6.16.1 and 6.16.2 and see that $\{e, x_1, x_2, x_3\}$, $\{e, x_3, x_4, x_5\}$, and $\{e, x_5, y, x\}$ are cocircuits in M_0, and that $\{e, x_1\}$, $\{e, x_3\}$, $\{e, x_5\}$, and $\{e, x\}$ are all contained in triangles of M_0. Now it is easy to see, using orthogonality, that $\{e, x_1, x_5\}$ and $\{e, x_3, x\}$ are triangles of M_0. This implies that $x \in \mathrm{cl}_{M_0\backslash e}(\{x_1, \ldots, x_5, y\})$. Since x is also in $\mathrm{cl}^*_{M_0\backslash e}(\{x_1, \ldots, x_5, y\})$, we see that $\{x_1, \ldots, x_5, y, x\}$ is 2-separating in $M_0\backslash e$, and it follows that $M_0\backslash e$ is a wheel or a whirl. This contradiction completes the proof. $\square$

6.16.4. $n \leq 4$.

Proof. If $n > 4$, then by 6.16.3, $n = 5$, and e_1 is a spoke element. Therefore $(e_1, \ldots, e_5, x, y)$ is a fan of $M_0\backslash e$, where N is a minor of $M_0\backslash e/x\backslash y$. As $\{e_2, e_3, e_4\}$ and $\{e_4, e_5, x\}$ are triads of $M_0\backslash e$, 6.16.1 and 6.16.2 imply that $\{e, e_2, e_3, e_4\}$ and $\{e, e_4, e_5, x\}$ are cocircuits of M_0, and $\{e, e_2\}$ and $\{e, x\}$ are contained in triangles of M_0. Assume $\{e, e_2, x\}$ is not a triangle. Then orthogonality requires that $\{e, e_2\}$ is in a triangle with e_4 or e_5, and that $\{e, x\}$ is in a triangle with e_3 or e_4. This implies that x is in the closure and the coclosure of $\{e_1, \ldots, e_5\}$ in $M_0\backslash e$, and we arrive at the contradiction that $M_0\backslash e$ is a wheel or a whirl. Therefore $\{e, e_2, x\}$ is a triangle.

Assume $M_0\backslash y$ is not 3-connected. Then there is a triad of M_0 that contains y. By orthogonality with $\{e_5, x, y\}$, $\{e_3, e_4, e_5\}$, and $\{e, e_2, x\}$, we see that y is in the coclosure and closure of $\{e_1, \ldots, e_5, x\}$ in M_0, and also in $M_0\backslash e$. This leads to a contradiction as before, so $M_0\backslash y$ is 3-connected, and therefore a fan-extension of N. This implies there is a fan, F', of $M_0\backslash y$ such that F is consistent with F' and e_1 and e_5 are terminal elements of F'.

Proposition 6.4 implies that $M_0\backslash y\backslash e/x$ is 3-connected. If neither e nor x is an internal element of F', then $F' - \{e, x\}$ is a fan of $M_0\backslash y\backslash e/x$ by the fact that $M_0\backslash y\backslash e/x$ is 3-connected. Assume either e or x is an internal element in F'. By applying Proposition 6.6 to $M' = M_0\backslash y$, we see that e and x are consecutive in F', and x is a rim element. Then Proposition 5.2 implies $F' - \{e, x\}$ is a fan of $M_0\backslash y\backslash e/x$, so this is true in any case. We note that $(e_1, \ldots, e_5)$ is also a fan in $M_0\backslash y\backslash e/x$. As $F' - \{e, x\}$ and $(e_1, \ldots, e_5)$ have at least three elements in common, including e_1 and e_5, Lemma 5.8

shows that $\{e_1, \ldots, e_5\}$ is contained in $F' - \{e, x\}$. Applying Proposition 5.6
three times shows that $(e_1, \ldots, e_5)$ is a contiguous subsequence of $F' - \{e, x\}$.
Assume (e_2, e_3, e_4) is a contiguous subsequence of F'. Since $\{e_2, e_3, e_4\}$ is a
triad in $M_0 \backslash y \backslash e / x$, it cannot be a triangle in $M_0 \backslash y$. Hence it is a triad in
$M_0 \backslash y$, but not in M_0. This means y is in the coclosure and the closure of
$\{e_1, \ldots, e_5, x\}$ in $M_0 \backslash e$. This leads to a contradiction, so (e_2, e_3, e_4) is not a
contiguous subsequence of F'. This means that the elements $\{e, x, e_3\}$ are
consecutive in F', so $\{e, x, e_3\}$ is a triad or a triangle in $M_0 \backslash y$. The former
case is impossible, by orthogonality with $\{e_3, e_4, e_5\}$. Therefore $\{e, x, e_3\}$ is
a triangle of $M_0 \backslash y$. Since $\{e, x, e_2\}$ is also a triangle, we see that $\{e, x, e_2, e_3\}$
is a $U_{2,4}$-restriction of $M_0 \backslash y$ that is contained in the fan F'. Therefore it
intersects a triad, and this contradiction to Corollary 6.3 shows that $n \leq$
4. $\square$

6.16.5. $n = 3$.

Proof. If $n > 3$, then by 6.16.4, $n = 4$. If e_1 is a rim element, we
let $(x_1, x_2, x_3, x_4, x, y)$ be $(e_1, \ldots, e_6)$. Otherwise, we shift x and y using
Lemma 6.15 so that $(y, x, e_1, e_2, e_3, e_4)$ is a fan of $M_0 \backslash e$, and now we let
$(x_1, x_2, x_3, x_4, x, y)$ be $(e_4, e_3, e_2, e_1, x, y)$. In either case $(x_1, x_2, x_3, x_4, x, y)$
is a fan of $M_0 \backslash e$ with x_1 as a rim element, and N is a minor of $M_0 / x \backslash y$.
Since $\{x_1, x_2, x_3\}$ and $\{x_3, x_4, x\}$ are triads of $M_0 \backslash e$, we deduce from 6.16.1
and 6.16.2 that $\{e, x_1, x_2, x_3\}$ and $\{e, x_3, x_4, x\}$ are cocircuits of M_0, and
that $\{e, x_1\}$ and $\{e, x\}$ are contained in triangles. If $\{e, x_1, x\}$ is not a trian-
gle, then x is contained in the closure and the coclosure of $\{x_1, x_2, x_3, x_4\}$ in
$M_0 \backslash e$, exactly as in the proof of 6.16.4. Because this leads to a contradiction,
$\{e, x_1, x\}$ is a triangle of M_0.

If $M_0 \backslash y$ is not 3-connected, then there is a triad of M_0 containing y, and
y is in the coclosure and the closure of $\{x_1, x_2, x_3, x_4, x\}$ in $M_0 \backslash e$, as in the
proof of 6.16.4. This gives a contradiction, so $M_0 \backslash y$ is 3-connected, and
hence a fan-extension of N. There is a fan, F', in $M_0 \backslash y$ such that F is
consistent with F', and x_1 and x_4 are terminal elements of F'. If neither e
nor x is an internal element of F', then $F' - \{e, x\}$ is a fan of $M_0 \backslash y \backslash e / x$.
If either e or x is an internal element, then Propositions 5.2 and 6.6 show
$F' - \{e, x\}$ is a fan of $M_0 \backslash y \backslash e / x$. As (x_1, x_2, x_3, x_4) is also a fan in $M_0 \backslash y \backslash e / x$,
the elements $\{x_1, x_2, x_3, x_4\}$ form a contiguous subsequence of $F' - \{e, x\}$.

Assume (x_1, x_2, x_3, x_4) is not a contiguous subsequence of $F' - \{e, x\}$. As
F is consistent with $F' - \{e, x\}$, it follows that, up to reversing, F is either
(x_1, x_2, x_4) or (x_1, x_3, x_4). In this case $\{x_1, x_2, x_3, x_4\} \subseteq E(N)$, for otherwise
N contains a series or parallel pair. But then (x_1, x_2, x_3, x_4) is a fan of N,
and so is either (x_1, x_2, x_4) or (x_1, x_3, x_4), contradicting Proposition 5.6.
Therefore (x_1, x_2, x_3, x_4) is a contiguous subsequence of $F' - \{e, x\}$. Assume
(x_1, x_2, x_3) is a contiguous subsequence of F'. Then $\{x_1, x_2, x_3\}$ is a triad of
$M_0 \backslash y$ but not of M_0, so y is in the coclosure and closure of $\{x_1, x_2, x_3, x_4, x\}$
in $M_0 \backslash e$. This leads to a contradiction, so (x_1, x_2, x_3) is not a contiguous
subsequence, and thus e, x, and x_2 are consecutive in F'. By orthogonality

with $\{x_2, x_3, x_4\}$, $\{e, x, x_2\}$ is not a triad in $M_0 \backslash y$, so $\{e, x, x_2\}$ is a triangle of $M_0 \backslash y$. Thus $\{e, x_1, x_2, x\}$ is a $U_{2,4}$-restriction of $M_0 \backslash y$ that intersects a triad contained in F'. This contradiction completes the proof. $\qquad \square$

6.16.6. e_1 *is a spoke element.*

Proof. Assume that e_1 is a rim element. Then (e_1, e_2, e_3, y, x) is a fan of $M_0 \backslash e$, and F is equal to (e_1, e_2, e_3) or its reversal. As before, we see that $\{e, e_1, e_2, e_3\}$ and $\{e, e_3, y, x\}$ are cocircuits in M_0, and $\{e, e_1, x\}$ is a triangle.

Assume $M_0 \backslash y$ is not 3-connected, so that y is contained in a triad, T^*, of M_0. Note T^* is a triad of $M_0 \backslash e$ but $T^* \neq \{e_3, y, x\}$. If $T^* \cap \{e_3, x\} \neq \emptyset$, then $T^* \cup \{e_3, x\}$ is a $U_{2,4}$-corestriction in $M_0 \backslash e$, and $\{e_2, e_3, y\}$ is a triangle. This contradiction to Corollary 6.3 shows that $T^* \cap \{e_3, x\} = \emptyset$. Orthogonality with $\{e_2, e_3, y\}$ now shows that e_2 is in T^*. Assume that we swap labels on e_3 and x in M_0, and then contract x. The resulting matroid has N as a minor, as $\{e_3, x\}$ is a series pair in $M_0 \backslash e \backslash y$. It also has the triad T^*, which contains the parallel pair $\{e_2, y\}$. This contradiction shows that $M_0 \backslash y$ is 3-connected, and therefore a fan-extension of N.

Let F' be a fan of $M_0 \backslash y$ such that (e_1, e_2, e_3) is consistent with F', and e_1 and e_3 are terminal elements. Now $M_0 \backslash y \backslash e / x$ is 3-connected, and contains the triad $\{e_1, e_2, e_3\}$. As before, $F' - \{e, x\}$ is a fan of $M_0 \backslash y \backslash e / x$. It follows that (e_1, e_2, e_3) is a contiguous subsequence of $F' - \{e, x\}$. If (e_1, e_2, e_3) is a contiguous subsequence of F', then $\{e_1, e_2, e_3\}$ is a triad of $M_0 \backslash y$, since it cannot be a triangle and a triad in $M_0 \backslash y \backslash e / x$. Therefore $\{e_1, e_2, e_3\}$ is a triad in $M_0 \backslash y$ but not in M_0. This means $\{e_1, e_2, e_3, y\}$ is 2-separating in $M_0 \backslash e$, leading to the contradiction that $M_0 \backslash e$ is a wheel or a whirl. Therefore (e_1, e_2, e_3) is not a contiguous subsequence of F', so $\{e, x, e_2\}$ is a set of consecutive elements in F'. If $\{e, x, e_2\}$ is a triad in $M_0 \backslash y$, then $\{e, x, e_2, y\}$ is a cocircuit in M_0, by orthogonality with $\{e_2, e_3, y\}$. Therefore $\{e_2, y, x\}$ and $\{e_3, y, x\}$ are both triads in $M_0 \backslash e$, so $M_0 \backslash e$ has a $U_{2,4}$-corestriction that intersects a triangle, a contradiction. This shows that $\{e, x, e_2\}$ is a triangle in $M_0 \backslash y$, so $\{e, x, e_1, e_2\}$ is a $U_{2,4}$-restriction of $M_0 \backslash y$ that intersects a triad contained in F', a contradiction. $\qquad \square$

Now we know that e_1 is a spoke element and (e_1, e_2, e_3) or its reversal is in $\mathcal{F}_N$. By shifting x and y, we can assume that (e_1, e_2, y, x, e_3) is a fan in $M_0 \backslash e$ with e_1 as a spoke element, and N is a minor of $M_0 \backslash e / x \backslash y$. It follows from 6.16.1 that $\{e, e_2, y, x\}$ is a cocircuit in M_0. Assume that y is in a triad, T^*, of M_0. Then $T^* \neq \{e_2, y, x\}$, and T^* is a triad of $M_0 \backslash e$. If $T^* \cap \{e_2, x\} \neq \emptyset$, then $T^* \cup \{e_2, x\}$ is a $U_{2,4}$-corestriction of $M_0 \backslash e$ that intersects a triangle, which is impossible. Therefore $T^* \cap \{e_2, x\} = \emptyset$. Orthogonality shows that $T^* = \{e_1, y, e_3\}$. This means M_0 / x contains a dependent triad, so we have a contradiction. We conclude that y is in no triad in M_0, so $M_0 \backslash y$ is 3-connected, and is therefore a fan-extension of N.

Let F' be a fan of $M_0 \backslash y$ such that (e_1, e_2, e_3) is a subsequence of F' and e_1 and e_3 are terminal elements. As $\{e_1, e_2, e_3\}$ is a triangle in $M_0 \backslash y \backslash e / x$, we conclude as before that $F' - \{e, x\} = (e_1, e_2, e_3)$. If $F' = (e_1, e_2, e_3)$, then

$\{e_1, e_2, e_3\}$ is a triangle in $M_0 \backslash y$. As $\{e_1, e_2, y\}$ is a triangle in $M_0 \backslash e$, we see that $\{e_1, e_2, e_3, y\}$ is a $U_{2,4}$-restriction in $M_0 \backslash e$, and $\{e_2, y, x\}$ is a triad, so we have a contradiction. Now it follows that $F' \neq (e_1, e_2, e_3)$, so e and x are internal elements of F'. By Proposition 6.6, x is a rim element of F', and x and e are consecutive. As $\{e_1, e_2, e_3\}$ is a triangle in $M_0 \backslash y \backslash e / x$, it follows from Proposition 5.2 that e_1 is a spoke element of F'. It follows that F' is either (e_1, e_2, e, x, e_3) or (e_1, x, e, e_2, e_3). In the first case, $\{e_1, e_2, e, y\}$ and $\{e_3, e, x, y\}$ are $U_{2,4}$-restrictions of M_0. If $r_{M_0}(\{e, x, y, e_1, e_2, e_3\}) > 2$, then submodularity implies that $r_{M_0}(\{e, y\}) \leq 1$, which contradicts 3-connectivity. Therefore $r_{M_0 \backslash e}(\{x, y, e_1, e_2, e_3\}) = 2$, which quickly leads to a contradiction. We conclude that $F' = (e_1, x, e, e_2, e_3)$, so $\{e_1, x, e\}$ and $\{e, e_2, e_3\}$ are triangles of M_0.

Because $\{e, e_1\}$ and $\{e_3, y\}$ are parallel pairs in M_0 / x, we will swap the labels on e and e_1, and on e_3 and y. Let M' be the resulting matroid, so N is a minor of $M' \backslash e / x \backslash y$, and M' has triangles $\{e, e_2, e_3\}$, $\{e_3, x, y\}$, $\{e, e_1, x\}$, and $\{e_1, e_2, y\}$, as well as the cocircuit $\{e_1, e_2, e_3, x\}$. Assume that e is in a triad, T^*, of M'. Orthogonality implies that T^* contains an element from each of $\{e_2, e_3\}$ and $\{e_1, x\}$. By orthogonality with $\{e_1, e_2, y\}$ and $\{e_3, x, y\}$, and the fact that $\{e_1, e_2\}$ is not a series pair in N (and therefore not in $M' \backslash e$), we see that $T^* = \{e, e_3, x\}$. Therefore $\{e_3, x, y\}$ is a codependent triangle in $M' \backslash e$. This contradiction shows that $M' \backslash e$ is 3-connected, and is therefore a fan-extension of N.

Let F' be a fan in $M' \backslash e$ that contains (e_1, e_2, e_3) as a subsequence, and that has e_1 and e_3 as terminal elements. By the same arguments as before, we see that $F' - \{x, y\}$ is a fan in $M' \backslash e / x \backslash y$. However, $\{e_1, e_2, e_3\}$ is a triangle in $M' \backslash e / x \backslash y$, so $F' - \{x, y\} = (e_1, e_2, e_3)$. Assume $F' = (e_1, e_2, e_3)$. Because $\{e_3, x, y\}$ is a triangle in $M' \backslash e$, orthogonality shows that $\{e_1, e_2, e_3\}$ is also a triangle. Because $\{e, e_2, e_3\}$, $\{e_1, e_2, y\}$, and $\{e, e_1, x\}$ are also triangles in M', it follows that $r_{M'}(\{e_1, e_2, e_3, e, x, y\}) = 2$. This means that $\{e_1, e_2, e_3\}$ is a parallel class in M' / x, so we have a contradiction. Therefore F' is not (e_1, e_2, e_3), so $\{x, y, e_2\}$ is a consecutive set in F'. If $\{x, y, e_2\}$ is a triangle in $M' \backslash e$, then exactly as before, we see that $r_{M'}(\{e_1, e_2, e_3, e, x, y\}) = 2$. Therefore $\{x, y, e_2\}$ is a triad of $M' \backslash e$, and $\{e, e_2, x, y\}$ is a cocircuit of M', by orthogonality with $\{e, e_2, e_3\}$. Because $\{e_1, e_2, e_3, x\}$ is also a cocircuit, we see that $\{e, e_2, e_3, x\}$ cospans $\{e_1, e_2, e_3, e, x, y\}$ in M'. As $\{e, e_3, y\}$ spans the same set, it follows that $\lambda_{M'}(\{e_1, e_2, e_3, e, x, y\}) \leq 1$. This means that $|E(M' \backslash e)| \leq 6$, and as $|F'| = 5$, it follows from Proposition 5.4 that $M' \backslash e$ is a wheel or a whirl. This contradiction completes the proof of Lemma 6.16. $\square$

In fact, we can strengthen Lemma 6.16 further, and show that elements in $E(M_0 \backslash e) - E(N)$ are terminal elements of fans in a covering family.

Lemma 6.17. *Let M_0 be isomorphic to M. Assume that N is a minor of M_0, but that M_0 is not a fan-extension of N. Assume also that $M_0 \backslash e$ is 3-connected and has N as a minor. Let $\mathcal{F}_e$ be a covering family of $M_0 \backslash e$, and let $(e_1, \ldots, e_m)$ be a fan in $\mathcal{F}_e$. If $1 < i < m$, then e_i is in $E(N)$.*

Proof. Assume that the lemma fails. By applying Lemma 6.16 and reversing as required, we can assume that $(e_1, \ldots, e_m)$ is in $\mathcal{F}_e$, and that e_2 is in $E(M_0 \backslash e) - E(N)$.

6.17.1. e_1 *is in* $E(M_0 \backslash e) - E(N)$.

Proof. If e_2 is a rim element in $(e_1, \ldots, e_m)$, then we let M' be $M_0 \backslash e$ and we let N' be N. Otherwise we let M' be $(M_0 \backslash e)^*$ and we let N' be N^*. Thus M' is a fan-extension of N' and $\mathcal{F}_e$ is a covering family, while $e_2 \in E(M') - E(N')$ is a rim element in the fan $(e_1, \ldots, e_m) \in \mathcal{F}_e$ of M'.

Assume for a contradiction that e_1 is in $E(N')$. If N' is a minor of $M' \backslash e_2$, then $m = 4$, for otherwise $\{e_3, e_4, e_5\}$ is a codependent triangle in $M' \backslash e_2$, violating Proposition 6.2. But if $m = 4$, then $\{e_1, e_3, e_4\}$ are all contained in a fan in $\mathcal{F}_N$, and $\{e_3, e_4\}$ is a series pair in $M' \backslash e_2$, which leads to a contradiction to the 3-connectivity of N'. Therefore N' is a minor of M'/e_2. If $m = 4$, then $\{e_1, e_3, e_4\} \subseteq E(N')$, and $\{e_1, e_3\}$ is a parallel pair in M'/x_2. This leads to a contradiction, so $m \geq 5$, meaning that Lemma 6.16 implies $e_3 \in E(N')$. Now $\{e_1, e_3\}$ is a parallel pair in M'/e_2, and we again obtain a contradiction to the 3-connectivity of N'. $\square$

It follows from 6.17.1 that $m \geq 5$. If e_1 is a rim element of $(e_1, \ldots, e_m)$, then it is easily checked that N is a minor of $M_0 \backslash e/e_1 \backslash e_2$. If e_1 is a spoke element, then N is a minor of $M_0 \backslash e \backslash e_1/e_2$. Let M' be the matroid obtained from M_0 by swapping the labels on e_1 and e_3. If e_1 is a rim element, then $\{e_1, e_3\}$ is a series pair in $M_0 \backslash e \backslash e_2$, so $M' \backslash e/e_1 \backslash e_2 = M_0 \backslash e/e_1 \backslash e_2$. If e_1 is a spoke element, then we can similarly show that $M' \backslash e \backslash e_1/e_2 = M_0 \backslash e \backslash e_1/e_2$. Therefore N is a minor of $M' \backslash e/e_1 \backslash e_2$ or $M' \backslash e \backslash e_1/e_2$.

Assume that M' is not a fan-extension of N. Since e_1 and e_2 are not in $E(N)$, it is easy to see that

$$(\mathcal{F}_e - \{(e_1, \ldots, e_m)\}) \cup \{(e_3, e_2, e_1, e_4, \ldots, e_m)\}$$

is a covering family of $M' \backslash e$. But this contradicts Lemma 6.16, as e_1 is not in $E(N)$. Therefore M' is a fan-extension of N. Note that this does not contradict Lemma 6.15, which would apply if N were a minor of $M_0 \backslash e/e_2 \backslash e_3$ or $M_0 \backslash e \backslash e_2/e_3$.

Let F_N be the fan in $\mathcal{F}_N$ that is consistent with $(e_1, \ldots, e_m)$.

6.17.2. e_3 *is in* $E(N)$, *but not in* F_N, *and therefore* e_3 *is contained in no fan in* $\mathcal{F}_N$.

Proof. It follows immediately from Lemma 6.16 that e_3 is in $E(N)$. Assume that 6.17.2 fails, so that e_3 is in F_N. Because M' is a fan-extension of N, there is a covering family, $\mathcal{F}'$, of M'. Let F' be the fan in $\mathcal{F}'$ such that F_N is consistent with F'. If e is in F', then it is a terminal spoke element, as $M' \backslash e$ is isomorphic to $M \backslash e$, which is 3-connected. Therefore $F' - e$ is a fan in $M' \backslash e$, and so is $(e_3, e_2, e_1, e_4, \ldots, e_m)$. As these fans both contain the elements of F_N, it follows from Lemma 5.8 that the elements they have in common form a contiguous subsequence in both fans. In particular, this means that

$\{e_3, e_2, e_1\}$ is contained in $F' - e$. As this set is either a triad or a triangle in $M'\backslash e$, it forms a set of three consecutive elements in $F' - e$, and hence in F'. Let F'' be the sequence obtained from F' by swapping the positions of e_1 and e_3. As e_1 and e_2 are not in $E(N)$, it follows that $(\mathcal{F}' - \{F'\}) \cup \{F''\}$ is a covering family of M_0. This contradiction to Lemma 6.11 completes the proof of 6.17.2. $\qquad\square$

6.17.3. *Let $\mathcal{F}'$ be a covering family of M'. Then no fan in $\mathcal{F}'$ contains e_3.*

Proof. Assume for a contradiction that e_3 is contained in a fan in $\mathcal{F}'$. Let $\mathcal{F}''$ be the family of fans in M_0 obtained from $\mathcal{F}'$ by swapping the labels on e_1 and e_3. Obviously there are $|\mathcal{F}_N|$ pairwise disjoint fans in $\mathcal{F}''$. Since e_3 is not contained in a fan in $\mathcal{F}_N$ (by 6.17.2), and the same statement applies to e_1, it follows that every fan in $\mathcal{F}_N$ is consistent with a fan in $\mathcal{F}''$. Moreover, every element in $E(M') - E(N)$ is contained in a fan in $\mathcal{F}'$, and since e_3 is contained in a fan in $\mathcal{F}'$, and $e_3 \in E(N)$, we conclude that every element in $E(M_0) - E(N)$ is contained in a fan in $\mathcal{F}''$. We have just shown that $\mathcal{F}''$ is a covering family of M_0, contradicting Lemma 6.11. $\qquad\square$

Let $\{x, y\} = \{e_1, e_2\}$, where N is a minor of $M\backslash e/x\backslash y$.

6.17.4. *Let $\mathcal{F}'$ be a covering family of M'. Then there is a fan in $\mathcal{F}'$ that contains e_1 and e_2.*

Proof. Let $\mathcal{F}'$ be a covering family of M'. Assume that 6.17.4 fails, so that there are distinct fans, F_1 and F_2, in $\mathcal{F}'$ that (respectively) contain e_1 and e_2. Note that F_1 and F_2 contain at least four elements each. If e is contained in either F_1 or F_2, then it is a terminal spoke element, as $M'\backslash e$ is 3-connected. Therefore $F_1 - e$ and $F_2 - e$ are fans in $M'\backslash e$, and these fans must each contain at least four elements, as they contain a fan in $\mathcal{F}_N$, and an element in $\{e_1, e_2\}$. Now $\{e_1, e_2, e_3\}$ is a triangle or a triad in $M'\backslash e$, and $F_1 - e$ and $F_2 - e$ are fans. Since 6.17.3 asserts that e_3 is in neither $F_1 - e$ nor $F_2 - e$, orthogonality requires that either e_1 and e_2 are both terminal spoke elements of $F_1 - e$ and $F_2 - e$, or they are both terminal rim elements. In this former case, $M'\backslash e/x$ contains a triad that contains a parallel pair, and in the latter case, $M'\backslash e\backslash y$ contains a triangle that contains a series pair. In either case we have a contradiction, so 6.17.4 must hold. $\qquad\square$

Let $\mathcal{F}'$ be a covering family of M', and let F' be a fan in $\mathcal{F}'$ that contains e_1 and e_2. Then F' contains at least five elements. If e is in F', then it is a terminal spoke element, as $M'\backslash e$ is 3-connected, so $F' - e$ is a fan in $M'\backslash e$. Let $F' - e$ be $(f_1, \ldots, f_n)$. Note that $n \geq 5$. Now $\{e_1, e_2, e_3\}$ is a triangle or a triad in $M'\backslash e$ that intersects in $F' - e$ in exactly the elements e_1 and e_2 (by 6.17.3). We apply Proposition 5.3 or its dual. Assume that statement (iii), (iv), or (v) holds. Then either $\{e_1, e_2\} = \{f_1, f_n\}$, or $n = 5$ and $\{e_1, e_2\} = \{f_2, f_4\}$, and in any case e_1 and e_2 are both spoke elements in $F' - e$, or both are rim elements. If both are spoke elements, $M'\backslash e/x$ contains a triad that contains a parallel pair, and if both are rim

elements, then $M'\backslash e\backslash y$ contains a triangle that contains a series pair. In either case we have a contradiction, so statements (iii), (iv), and (v) in Proposition 5.3 cannot hold. Now, by reversing as necessary, we can assume that $\{e_1, e_2\} = \{f_1, f_2\}$.

6.17.5. F' *is either* $(f_1, \ldots, f_n)$ *or* $(f_1, \ldots, f_n, e)$.

Proof. If e is not contained in F', then $F' = F' - e = (f_1, \ldots, f_n)$, and we are done. Therefore assume that e is a terminal spoke element of F'. The only way 6.17.5 can fail to be true is if $F' = (e, f_1, \ldots, f_n)$, so let us assume this is the case. Then $\{e, e_1, e_2\} = \{e, f_1, f_2\}$ is a triangle of M'. If $\{e_1, e_2, e_3\}$ is a triangle of M', then $\{e, e_1, e_2, e_3\}$ is a $U_{2,4}$-restriction, which is a contradiction, as $\{f_1, f_2, f_3\} = \{e_1, e_2, e_3\}$ is a triad. Similarly, if $\{e_1, e_2, e_3\}$ is a triad in M', then $\{e_2, e_1, e_4\}$ is a triangle, so $\{e, e_2, e_1, e_4\}$ is a $U_{2,4}$-restriction. This again leads to a contradiction. Therefore 6.17.5 holds. $\square$

6.17.6. $\{e, e_1, e_2, e_3\}$ *is a cocircuit in* M'.

Proof. Assume that $\{e_1, e_2, e_3\}$ is a triangle in $M'\backslash e$, and hence in M'. If $\{f_1, f_2, f_3\}$ is a triangle in M', then $\{e_3, f_1, f_2, f_3\}$ is a $U_{2,4}$-restriction in M', and $\{f_2, f_3, f_4\}$ is a triad. As this is a contradiction, it follows that $\{f_1, f_2, f_3\}$ is a triad in M'. Let $F' + e_3$ be the sequence obtained from F' by appending e_3 to the beginning. Then $F' + e_3$ is a fan in M', and it follows from 6.17.2 and 6.17.3 that $(\mathcal{F}' - \{F'\}) \cup \{F' + e_3\}$ is a covering family of M'. But this contradicts 6.17.3, so we conclude that $\{e_1, e_2, e_3\}$ is a triad in $M'\backslash e$.

To conclude the proof of 6.17.6, assume that $\{e_1, e_2, e_3\}$ is a triad in M'. If $\{f_1, f_2, f_3\}$ is a triad in M', then $\{e_3, f_1, f_2, f_3\}$ is a $U_{2,4}$-corestriction in M', which is a contradiction as $\{f_2, f_3, f_4\}$ is a triangle. Therefore $\{f_1, f_2, f_3\}$ is a triangle in M'. We again let $F' + e_3$ be obtained from F' by appending e_3 to the beginning. This leads to a contradictory covering family, just as in the previous paragraph. Therefore $\{e, e_1, e_2, e_3\}$ is a cocircuit in M. $\square$

As e is in $E(M') - E(N)$, there is a fan, F_e, in $\mathcal{F}'$, that contains e. Then F_e contains at least four elements. As $M'\backslash e$ is 3-connected, it follows that e is a terminal spoke element in F_e. Let $\{e, u, v\}$ be the triangle contained in F_e that contains e. Orthogonality between the triangle $\{e, u, v\}$ and the cocircuit $\{e, e_1, e_2, e_3\}$ shows that either e_1, e_2, or e_3 is in $\{u, v\}$. The last case is impossible by 6.17.3, so e_1 or e_2 is in $\{u, v\}$. This means that F_e and F' cannot be disjoint fans, so $F' = (f_1, \ldots, f_n, e)$, by 6.17.5. However it now follows that $\{e_1, e_2\} = \{f_1, f_2\}$, and $\{u, v\} = \{f_{n-1}, f_n\}$. Thus F' can contain at most four elements, which is impossible, as it contains e, e_1, e_2, and a fan in $\mathcal{F}_N$. This completes the proof of Lemma 6.17. $\square$

Lemma 6.14 supplies us with a matroid M_0, which we now relabel as M, such that M has N as a minor, but is not a fan-extension of N. By replacing M, N, and $\mathcal{M}$ with their duals as necessary, we can assume that $M\backslash e$ is

3-connected and has N as a minor. The minimality of M means that $M\backslash e$ is a fan-extension of N.

Lemma 6.18. *Let $\mathcal{F}_e$ be a covering family of $M\backslash e$, and assume that $(e_1, \ldots, e_m)$ is a fan in $\mathcal{F}_e$. Assume also that e_1 is not in $E(N)$. If e_1 is a spoke element of $(e_1, \ldots, e_m)$, then $M\backslash e\backslash e_1$ is 3-connected and has N as a minor. If e_1 is a rim element, then $M\backslash e/e_1$ is 3-connected and has N as a minor.*

Proof. If e_1 is a rim element, then let $M' = M\backslash e$ and let $N' = N$. Otherwise, let $M' = (M\backslash e)^*$ and let $N' = N^*$. Thus, in either case, we assume that $\mathcal{F}_e$ is a covering family of M' relative to N' and $\mathcal{F}_N$, that $(e_1, \ldots, e_m)$ is a fan in $\mathcal{F}_e$, and e_1 is a rim element that is not in $E(N')$. Certainly M'/e_1 has N' as a minor, or else $M'\backslash e_1$ has N' as a minor and $\{e_2, e_3, e_4\}$ as a codependent triangle, contradicting Proposition 6.2. Assume M'/e_1 is not 3-connected. Then e_1 is contained in a triangle, T, of M'. Orthogonality with the triad $\{e_1, e_2, e_3\}$ shows that T contains e_2 or e_3. Note that $T \not\subseteq \{e_1, \ldots, e_m\}$, for otherwise Proposition 5.6 implies $T = \{e_1, e_2, e_3\}$, and thus T is a triad and a triangle in M'. Let x be the element in $T - \{e_1, e_2, e_3\}$.

Assume that e_2 is in T. Since e_2 is in $E(N')$ by Lemma 6.17, and $\{e_2, x\}$ is a parallel pair in M'/e_1, it follows that N' is a minor of $M'/e_1\backslash x$. The definition of a covering family means there is a fan in $\mathcal{F}_e$ that contains x. Let F_x be this fan. Then F_x contains at least four elements. Orthogonality with T shows that x is a terminal spoke element of F_x. Since x is not in $E(N')$, it now follows that $(\mathcal{F}_e - \{(e_1, \ldots, e_m), F_x\}) \cup \{(x, e_1, \ldots, e_m), F_x - x\}$ is a covering family in M', and now we have a contradiction to Lemma 6.17 as $x, e_1 \notin E(N')$. Therefore $T = \{e_1, e_3, x\}$.

Because x is not in $(e_1, \ldots, e_m)$, we see that $m = 4$, as otherwise we violate orthogonality between T and $\{e_3, e_4, e_5\}$. Hence (e_2, e_3, e_4) or its reversal is in $\mathcal{F}_N$. Let M'' be obtained from M' by swapping labels on e_3 and x. Thus (e_3, e_1, x, e_2, e_4) is a fan of M''. As $\{e_3, x\}$ is a parallel pair in M'/e_1, we see that M'' has N' as a minor. Since $|E(M'')| = |E(M')| < |E(M)|$, it follows that M'' is a fan-extension of N'. Therefore there is a fan of M'' that contains (e_2, e_3, e_4) as a subsequence. Let this fan be F. By comparing F with (e_3, e_1, x, e_2, e_4) in M'' using Lemma 5.8, we see that there is a contiguous subsequence of F using the elements $\{e_3, e_1, x, e_2, e_4\}$. We apply Proposition 5.6 and its dual to $\{e_3, e_1, x\}$, $\{e_1, x, e_2\}$, and $\{x, e_2, e_4\}$ and see that F contains (e_3, e_1, x, e_2, e_4) as a contiguous subsequence, which contradicts the definition of F. Therefore M'/e_1 is 3-connected, and this completes the proof. $\square$

Lemma 6.19. *Let $\mathcal{F}_e$ be a covering family of $M\backslash e$, and assume that $(e_1, \ldots, e_m)$ is a fan in $\mathcal{F}_e$. Assume also that e_1 is not in $E(N)$. If e_1 is a rim element of $(e_1, \ldots, e_m)$, then assume that there is no triangle of M that contains $\{e_1, e\}$. Let $(e_s, \ldots, e_t)$ be a minimal contiguous subsequence of $(e_1, \ldots, e_m)$ such that a fan, $F_N \in \mathcal{F}_N$ is consistent with $(e_s, \ldots, e_t)$. Then $2 \le s < s + 2 \le t \le m$ and F_N contains e_s and e_t. Moreover, F_N is equal to*

$(e_s, \ldots, e_t)$ or its reversal, and is a fan of M. However, $(e_1, \ldots, e_t)$ is not a fan in M.

Proof. Since $(e_1, \ldots, e_m)$ is a fan in a covering family, there is a fan in $\mathcal{F}_N$ that is consistent with $(e_1, \ldots, e_m)$. Let e_s be the first element in this fan, and let e_t be the last. Then $2 \leq s < s+2 \leq t \leq m$ because each fan contains at least three elements, and $e_1 \notin E(N)$. By Lemma 6.17 it follows that every element in $\{e_s, \ldots, e_t\}$ is in $E(N)$. Since N is 3-connected, $(e_s, \ldots, e_t)$ is a fan of N. As F_N is also a fan of N, we apply Lemma 5.8 and deduce that F_N and $(e_s, \ldots, e_t)$ use the same set of elements. Since F_N is consistent with $(e_s, \ldots, e_t)$, this means that F_N is equal to $(e_s, \ldots, e_t)$ or its reversal.

If e_1 is a spoke element of $(e_1, \ldots, e_m)$, let M' be $M\backslash e_1$. In this case, M' is 3-connected, since $M'\backslash e$ is 3-connected by Lemma 6.18. If e_1 is a rim element, then let M' be M/e_1. In this case also, $M'\backslash e$ is 3-connected, and the hypotheses imply that e is not in a parallel pair in M'. Therefore M' is 3-connected in either case, and M' has N as a minor. As $|E(M')| < |E(M)|$, it follows that M' is a fan-extension of N.

Let F be a fan of M' such that $(e_s, \ldots, e_t)$ is consistent with F. If e is in F, then it is a terminal spoke element, as $M'\backslash e$ is 3-connected, so in this case $F - e$ is a fan in $M'\backslash e$. In fact, $F - e$ is a fan of $M'\backslash e$ whether or not e is in F. Since $(e_s, \ldots, e_t)$ is also a fan in $M'\backslash e$, by Lemma 5.8, there is a contiguous subsequence of $F - e$ using the elements $\{e_s, \ldots, e_t\}$. Since $(e_s, \ldots, e_t)$ is consistent with $F - e$, this means $(e_s, \ldots, e_t)$ is a contiguous subsequence of $F - e$. Hence $(e_s, \ldots, e_t)$ is a fan in M'.

Assume $(e_s, \ldots, e_t)$ is not a fan in M. If e_1 is a spoke element in $(e_1, \ldots, e_m)$, then $M' = M\backslash e_1$, so there is a rim element, e_i, of $(e_s, \ldots, e_t)$, such that $s \leq i \leq t - 2$, and $\{e_1, e_i, e_{i+1}, e_{i+2}\}$ is a cocircuit of M. But in this case e_1 is in the closure and coclosure of $(e_2, \ldots, e_m)$ in $M\backslash e$. Hence $(e_1, \ldots, e_m)$ is 2-separating in $M\backslash e$, so Proposition 5.4 implies $M\backslash e$ is a wheel or a whirl, a contradiction. The argument when e_1 is a rim element in $(e_1, \ldots, e_m)$ is similar: In this case $M' = M/e_1$, so there is a spoke element, e_i, in $(e_s, \ldots, e_t)$, such that $\{e_1, e_i, e_{i+1}, e_{i+2}\}$ is a circuit of M. Thus e_1 is in the closure and coclosure of $(e_2, \ldots, e_m)$ in $M\backslash e$, so we can deduce that $M\backslash e$ is a wheel or a whirl, contradicting Proposition 6.1. From this contradiction we see that $(e_s, \ldots, e_t)$ is a fan of M.

To complete the proof, we will assume that $(e_1, \ldots, e_t)$ is a fan of M, and deduce a contradiction. This assumption means that $(e_2, \ldots, e_t)$ is a fan in M'.

6.19.1. *There is a covering family of M' containing a fan that has $(e_2, \ldots, e_t)$ as a contiguous subsequence.*

Proof. Let $\mathcal{F}$ be a covering family of M'. Then $\mathcal{F}$ contains a fan, F, such that $(e_s, \ldots, e_t)$ is consistent with F. As $(e_s, \ldots, e_t)$ is a fan in M', we can use Lemma 5.8 to show that $(e_s, \ldots, e_t)$ is a contiguous subsequence of F. Assume that F contains $(e_i, \ldots, e_t)$ as a contiguous subsequence, where $2 \leq i \leq s$, and $\mathcal{F}$ and F have been chosen so that i is as small as

possible. If $i = 2$, then 6.19.1 is already proved, so we assume $2 < i$. Let $F = (f_1, \ldots, f_n)$, where $(e_i, \ldots, e_t) = (f_j, \ldots, f_{j+t-i})$.

Assume $j > 1$. Then $f_{j-1} \neq e_{i-1}$, for otherwise $(e_{i-1}, \ldots, e_t)$ is a contiguous subsequence of F, and the minimality of i is contradicted. Since $\{f_j, f_{j+1}, f_{j+2}\} = \{e_i, e_{i+1}, e_{i+2}\}$ is not a triad and a triangle, $f_j = e_i$ is a spoke element in both F and $(e_2, \ldots, e_m)$, or a rim element in both. Now $\{f_{j-1}, f_j, f_{j+1}\}$ and $\{e_{i-1}, e_i, e_{i+1}\}$ are distinct triads or distinct triangles that intersect in two elements, and their union is a $U_{2,4}$-corestriction or restriction of M' that intersects a triangle or triad. This contradiction shows that $j = 1$.

Now $(e_{i-1}, f_1, \ldots, f_n)$ is a fan of M'. Let F' be this fan. Note that because $i - 1 < s$, the element e_{i-1} is not contained in the fan $(e_s, \ldots, e_t)$, and since it is contained in $(e_1, \ldots, e_m) \in \mathcal{F}_e$, the definition of a covering family tells us that e_{i-1} is contained in no fan in $\mathcal{F}_N$. If e_{i-1} is in no fan in $\mathcal{F}$, then $(\mathcal{F} - \{F\}) \cup \{F'\}$ is a covering family in M'. If e_{i-1} is in a fan, $F'' \in \mathcal{F}$, then it is a terminal element of F'' by orthogonality with $\{e_{i-1}, e_i, e_{i+1}\}$, so $(\mathcal{F} - \{F, F''\}) \cup \{F', F'' - e_{i-1}\}$ is a covering family of M'. In either case we have constructed a covering family of M' that contains the fan F', so the minimality of i is contradicted. This completes the proof of 6.19.1. $\square$

Now we assume $\mathcal{F}$ is a covering family of M', and $\mathcal{F}$ contains a fan, F, that has $(e_2, \ldots, e_t)$ as a contiguous subsequence. Let $F = (f_1, \ldots, f_n)$, where $(e_2, \ldots, e_t) = (f_j, \ldots, f_{j+t-2})$. Assume $j > 1$. If e_1 is a rim element of $(e_1, \ldots, e_m)$, then $\{e_2, e_3, e_4\}$ is a triangle in M, and hence in $M' = M/e_1$. Therefore e_2 is a spoke element in $(f_j, \ldots, f_{j+t-2})$, so $\{f_{j-1}, f_j, f_{j+1}\}$ is a triad of M' and therefore in M. Now $\{e_1, f_{j-1}, f_j, f_{j+1}\}$ is a $U_{2,4}$-corestriction of M that intersects a triangle. If e_1 is a spoke element, we reach a similar contradiction. Therefore $j = 1$. But this means $(e_1, f_1, \ldots, f_n)$ is a fan of M, and M is obtained from M' by a fan-lengthening move on F. This implies M is a fan-extension of N, so we have a contradiction that completes the proof of Lemma 6.19. $\square$

Now we fix $\mathcal{F}_e$ to be a covering family of $M \backslash e$. Since we have assumed $|E(M)| - |E(N)| > 2$, it follows that $M \backslash e \neq N$. Any element in $E(M \backslash e) - E(N)$ is a terminal element of a fan in $\mathcal{F}_e$, by Lemma 6.17.

Lemma 6.20. *If $(e_1, \ldots, e_m)$ is a fan in $\mathcal{F}_e$ with the property that $e_1 \notin E(N)$, then e_1 is a rim element, and $\{e_1, e\}$ is contained in a triangle of M.*

Proof. If the lemma fails, then there is a fan $(e_1, \ldots, e_m)$ in $\mathcal{F}_e$ such that e_1 is not in $E(N)$, and either e_1 is a spoke element, or e_1 is a rim element of $(e_1, \ldots, e_m)$ and there is no triangle of M containing $\{e_1, e\}$.

There are indices p and q such that $2 \leq p < p + 2 \leq q \leq m$, and there is a fan in $\mathcal{F}_N$ consistent with $(e_p, \ldots, e_q)$ that contains e_p and e_q. Now Lemma 6.19 applies, so $(e_p, \ldots, e_q)$ or its reversal is in $\mathcal{F}_N$, and is a fan of M. However, $(e_1, \ldots, e_q)$ is not a fan of M. Since it is a fan in $M \backslash e$, there is some $j \in \{1, \ldots, p-1\}$ such that $\{e, e_j, e_{j+1}, e_{j+2}\}$ is a cocircuit of M.

Let M' be $M \backslash e_1$ if e_1 is a spoke element, and let it be M/e_1 otherwise. Lemma 6.18 implies that $M' \backslash e$ is 3-connected with N as a minor. Because the hypotheses imply M' has no parallel pair containing e, we now see that M' is 3-connected. Therefore M' is a fan-extension of N.

As $|E(M)| - |E(N)| > 2$, we can let u be an element in $E(M' \backslash e) - E(N)$. Either u belongs to a fan in $\mathcal{F}_e$ that is distinct, and hence disjoint, from $(e_1, \ldots, e_m)$, or, by Lemma 6.17, $u = e_m$. The analyses in the two cases are similar, so we combine them. If necessary we can reverse the fan that contains u, and assume that we are in one of the following situations.

(I) $(x_1, \ldots, x_n)$ is a fan in $\mathcal{F}_e$ that is disjoint from $(e_1, \ldots, e_m)$, and x_1 is not in $E(N)$.

(II) e_m is not in $E(N)$. In this case we let $(x_1, \ldots, x_n)$ be $(e_m, \ldots, e_2)$.

Note that in either of these cases, $(x_1, \ldots, x_n)$ is a fan in $M \backslash e$ and in $M' \backslash e$, and x_1 is not in $E(N)$. By taking a minimal contiguous subsequence of $(x_1, \ldots, x_n)$ such that a fan in $\mathcal{F}_N$ is consistent with the subsequence, we see there are integers s and t such that $2 \le s < s + 2 \le t \le n$ and such that some fan in $\mathcal{F}_N$ contains x_s and x_t and is consistent with $(x_s, \ldots, x_t)$.

6.20.1. *Either $(x_s, \ldots, x_t)$ or its reversal is in $\mathcal{F}_N$. Moreover, $(x_s, \ldots, x_t)$ is a fan in M, but $(x_1, \ldots, x_t)$ is not.*

Proof. We first show that $(x_s, \ldots, x_t)$ is a fan in M. In Case (II), $(x_s, \ldots, x_t)$ is equal to $(e_q, \ldots, e_p)$, and we have already noted this is a fan in M. Therefore we assume Case (I) holds. We can apply Lemma 6.19 to $(x_1, \ldots, x_n)$, unless x_1 is a rim element and $\{x_1, e\}$ is contained in a triangle of M. If Lemma 6.19 does apply, then it tells us that $(x_s, \ldots, x_t)$ is a fan of M. Therefore we assume that x_1 is a rim element, and $\{x_1, e\}$ is contained in a triangle. Orthogonality with the cocircuit $\{e, e_j, e_{j+1}, e_{j+2}\}$ shows that $\{x_1, e, e_k\}$ is a triangle of M for some $k \in \{j, j+1, j+2\}$. Because $(x_s, \ldots, x_t)$ is a fan in $M \backslash e$, it follows that if it is not a fan of M, then there is a cocircuit of M that contains e and three consecutive elements from $(x_s, \ldots, x_t)$. This violates orthogonality with $\{x_1, e, e_k\}$ as $s \ge 2$. Hence $(x_s, \ldots, x_t)$ is a fan of M in any case.

Next we show that $(x_s, \ldots, x_t)$ or its reversal is in $\mathcal{F}_N$. Since x_s and x_t are in $E(N)$, Lemma 6.17 implies $\{x_s, \ldots, x_t\} \subseteq E(N)$. It follows that $(x_s, \ldots, x_t)$ is a fan of N. Some fan in $\mathcal{F}_N$ is consistent with $(x_s, \ldots, x_t)$ and contains x_s and x_t. It follows easily from Lemma 5.8 that this fan must be either $(x_s, \ldots, x_t)$ or its reversal.

Finally we show that $(x_1, \ldots, x_t)$ is not a fan of M. We can apply Lemma 6.19, unless x_1 is a rim element of $(x_1, \ldots, x_n)$ and $\{x_1, e\}$ is contained in a triangle of M. If we can apply Lemma 6.19, then it tells us that $(x_1, \ldots, x_t)$ is not a fan in M, as desired. Therefore we assume x_1 is a rim element, and $\{x_1, e\}$ is contained in a triangle, T, of M. If $\{x_1, x_2, x_3\}$ is not a triad of M, then $(x_1, \ldots, x_t)$ is not a fan of M, so we are done. Therefore we assume $\{x_1, x_2, x_3\}$ is a triad of M. Orthogonality shows that T is $\{e, x_1, x_2\}$ or $\{e, x_1, x_3\}$.

If Case (I) holds, then we have a violation of orthogonality between T and the cocircuit $\{e, e_j, e_{j+1}, e_{j+2}\}$. Therefore Case (II) holds, so $(x_1, x_2, x_3) = (e_m, e_{m-1}, e_{m-2})$. Orthogonality between $\{e, e_j, e_{j+1}, e_{j+2}\}$ and T requires that $j + 2 \geq m - 2$. Recall that $j \leq p - 1$. Certainly $p + 2 \leq q$, as $(e_p, \ldots, e_q)$ or its reversal is in $\mathcal{F}_N$. Because e_q is in $E(N)$, but e_m is not, it follows that $q \leq m - 1$. Putting these together, we see that $j + 2 \leq p + 1 \leq q - 1 \leq m - 2$, so $j + 2 = m - 2$, and equality holds throughout this expression. Again applying orthogonality between $\{e, e_j, e_{j+1}, e_{j+2}\}$ and T, we see that $T = \{e, x_1, x_3\} = \{e, e_m, e_{m-2}\}$. As $p = m - 3$ and $q = m - 1$, this shows that $(x_s, \ldots, x_t) = (e_{m-1}, e_{m-2}, e_{m-3})$ and $(x_2, x_3, x_4) = (e_{m-1}, e_{m-2}, e_{m-3})$ or its reversal is in $\mathcal{F}_N$. If $(x_1, \ldots, x_t) = (e_m, e_{m-1}, e_{m-2}, e_{m-3})$ is not a fan in M, then we are done, so we assume it is a fan in M. Therefore (x_1, x_2, x_3, x_4) is a fan in M' also.

Because x_1 is not in $E(N)$, it follows easily that N is a minor of $M' \backslash e / x_1$. Let M'' be the matroid obtained from M' by swapping labels on e and x_3. Since $\{e, x_3\}$ is a parallel pair in M' / x_1, M'' has N as a minor. Moreover M'' is 3-connected since M' is, and $|E(M'')| = |E(M')| < |E(M)|$, so M'' is a fan-extension of N. Let F be a fan of M'' such that (x_2, x_3, x_4) is consistent with F. Because $M'' \backslash x_3$ is isomorphic to $M' \backslash e$, and is therefore 3-connected, x_3 is contained in no triad in F. If F contains at least four elements, then this means that x_3 is a terminal element of F, which is impossible as (x_2, x_3, x_4) is consistent with F. Therefore F contains exactly three elements, so $\{x_2, x_3, x_4\}$ is a triangle of M''. As $\{x_1, x_2, e\}$ is a triad of M'', we have a contradiction to orthogonality. $\qquad\square$

Applying 6.20.1, we let C_x^* be a cocircuit of M of the form $\{e, x_k, x_{k+1}, x_{k+2}\}$, where x_k is a rim element of $(x_1, \ldots, x_t)$ in $M \backslash e$, and $1 \leq k \leq s - 1$.

6.20.2. C_x^* *is a cocircuit of* M'.

Proof. Assume Case (I) holds. If $M' = M / e_1$, then C_x^* is a cocircuit in M', as e_1 is not in C_x^*. On the other hand, if $M' = M \backslash e_1$, then $\{e_1, e_2, e_3\}$ is a triangle disjoint from $\{x_1, \ldots, x_n, e\}$, so C_x^* is a cocircuit of M'. Now assume Case (II) holds. Note $C_x^* - e \subseteq \{x_1, \ldots, x_{s+1}\} \subseteq \{e_m, \ldots, e_3\}$, so $e_1 \notin C_x^*$. Therefore, if $M' = M / e_1$, then C_x^* is a cocircuit of M'. If $M' = M \backslash e_1$, then C_x^* is a cocircuit of M', for otherwise e_1 is in the coclosure and closure of $\{e_2, \ldots, e_m\}$ in $M \backslash e$. This implies $\{e_1, \ldots, e_m\}$ is 2-separating in $M \backslash e$, so $M \backslash e$ is a wheel or whirl, a contradiction. $\qquad\square$

Recall that M' is a fan-extension of N. Let F_e be the fan containing e in a covering family of M'. As $M' \backslash e$ is 3-connected, e is a terminal spoke element of F_e. Let (e, x, y, z) be the initial four elements of F_e, so $\{e, x, y\}$ is a triangle of M' and $\{x, y, z\}$ is a triad.

6.20.3. $\{x, y, z\} \subseteq E(N)$.

Proof. This follows from Lemma 6.17 if $|F_e| > 4$, and from the fact that F_e contains e and a fan in $\mathcal{F}_N$ otherwise. $\qquad\square$

Now $\{e, x, y\}$ is a triangle in M', and C_x^* is a cocircuit, so orthogonality requires that $(C_x^* - e) \cap \{x, y\} \neq \emptyset$. Thus $\{x, y, z\}$ is a triad of $M' \backslash e$ that intersects the fan $(x_1, \ldots, x_n)$.

As $\{x, y, z\}$ is contained in $E(N)$ and x_1 is not, we see that $x_1 \notin \{x, y, z\}$. Furthermore, any element in $C_x^* \cap \{x, y\}$ is not equal to x_n, since $C_x^* - e$ is contained in $\{x_1, \ldots, x_{s+1}\} \subseteq \{x_1, \ldots, x_{n-1}\}$. This means that if $\{x, y, z\}$ intersects $\{x_1, \ldots, x_n\}$ in exactly one element, then it is a triad of $M' \backslash e$ that intersects the fan $(x_1, \ldots, x_n)$ in a single, internal, element, violating orthogonality. Therefore $|\{x, y, z\} \cap \{x_1, \ldots, x_n\}| \geq 2$.

6.20.4. $|\{x, y, z\} \cap \{x_1, \ldots, x_n\}| \neq 2$.

Proof. Assume $|\{x, y, z\} \cap \{x_1, \ldots, x_n\}| = 2$. We apply the dual of Proposition 5.3 in $M' \backslash e$ to $\{x, y, z\}$ and the fan $(x_1, \ldots, x_n)$. Since x_1 is not in $\{x, y, z\}$, statement (i) or (iii) in Proposition 5.3 cannot apply. By the same reasoning, if statement (v) holds, then $n = 5$. In summary, either $\{x, y, z\}$ intersects $\{x_1, \ldots, x_n\}$ in $\{x_{n-1}, x_n\}$, or $n \leq 5$, x_1 is a rim element, and $\{x, y, z\}$ intersects $\{x_1, \ldots, x_n\}$ in $\{x_2, x_4\}$.

Assume $\{x, y, z\} \cap \{x_1, \ldots, x_n\} = \{x_{n-1}, x_n\}$. Then Proposition 5.3 implies x_n is a spoke element in $M' \backslash e$. Since $C_x^* - e = \{x_k, x_{k+1}, x_{k+2}\}$ contains x or y, and x_{k+2} is a rim element, it follows that $C_x^* - e = \{x_{n-3}, x_{n-2}, x_{n-1}\}$. This implies $s = n - 2$, so (x_{n-2}, x_{n-1}, x_n) or its reversal is in $\mathcal{F}_N$. Since F_e contains (e, x, y, z) as a contiguous subsequence, it therefore contains an element of (x_{n-2}, x_{n-1}, x_n). As F_e is in a covering family of M', this means that (x_{n-2}, x_{n-1}, x_n) is consistent with F_e. However, $x_{n-2} \notin \{x, y, z\}$, and $x_{n-1} \in \{x, y\}$ as $(C_x^* - e) \cap \{x, y\} \neq \emptyset$. Since (x_{n-2}, x_{n-1}, x_n) is consistent with F_e, we are forced to the conclusion that $x = x_n$ and $y = x_{n-1}$. Now $\{e, x_n, x_{n-1}\}$ and $\{x_n, x_{n-1}, x_{n-2}\}$ are triangles in M', and $\{x, y, z\} = \{x_n, x_{n-1}, z\}$ is a triad, which leads to a contradiction to Corollary 6.3.

Next we consider the case that $n \leq 5$, and $\{x, y, z\} \cap \{x_1, \ldots, x_n\} = \{x_2, x_4\}$. If $s = 2$, then (x_2, x_3, x_4) must be consistent with F_e, but (e, x, y, z) contains x_2 and x_4 and not x_3. If $s \neq 2$, then $s = 3$, $n = 5$, and (x_3, x_4, x_5) or its reversal is in $\mathcal{F}_N$. However, (e, x, y, z) contains x_4 and neither x_3 nor x_5, so it is impossible for (x_3, x_4, x_5) to be consistent with F_e. $\square$

Now we know that $\{x, y, z\} \subseteq \{x_1, \ldots, x_n\}$. Applying the dual of Proposition 5.6 to $\{x, y, z\}$ and $(x_1, \ldots, x_n)$ in $M' \backslash e$, we see that $\{x, y, z\} = \{x_i, x_{i+1}, x_{i+2}\}$, where x_i is a rim element of $(x_1, \ldots, x_n)$, and $i \leq n - 2$.

6.20.5. $1 < i < n - 2$.

Proof. Since $x_1 \notin E(N)$ and $\{x, y, z\} \subseteq E(N)$ it follows that $x_1 \notin \{x, y, z\}$, so $i > 1$. Assume that $i + 2 = n$. Then $\{x_{n-2}, x_{n-1}, x_n\}$ is a triad in M'. As C_x^* is a cocircuit in M' by 6.20.2, it follows that $C_x^* - e \neq \{x_{n-2}, x_{n-1}, x_n\}$. As C_x^* contains at least one of x or y, we see that $C_x^* - e = \{x_{n-4}, x_{n-3}, x_{n-2}\}$. As $(x_s, \ldots, x_t)$ is a fan in M, it follows that $s \geq n - 3$, so some three- or four-element contiguous subsequence of $(x_{n-3}, x_{n-2}, x_{n-1}, x_n)$ is in $\mathcal{F}_N$ (up

to reversing). If $\{x, y\} = \{x_{n-1}, x_n\}$, then the triangle $\{e, x_{n-1}, x_n\}$ violates orthogonality with C_x^* in M'. Assume that $\{x, y\} = \{x_{n-2}, x_{n-1}\}$. Now $\{x_{n-3}, x_{n-2}, x_{n-1}\}$ is a triangle in $M \backslash e$, and hence in M'. As $\{e, x, y\}$ is also a triangle in M', it follows that $\{e, x_{n-3}, x_{n-2}, x_{n-1}\}$ is a $U_{2,4}$-restriction in M' that intersects the triad $\{x, y, z\}$. This is a contradiction, so $\{x, y\} = \{x_{n-2}, x_n\}$. Therefore (e, x, y, z) is $(e, x_{n-2}, x_n, x_{n-1})$ or $(e, x_n, x_{n-2}, x_{n-1})$. This means any three- or four-element contiguous subsequence of $(x_{n-3}, x_{n-2}, x_{n-1}, x_n)$ contains elements from (e, x, y, z), but cannot be consistent with F_e, contradicting the fact that F_e is in a covering family of M'. $\qquad\square$

6.20.6. $\{x, y\} = \{x_i, x_{i+2}\}$, *and therefore $z = x_{i+1}$*.

Proof. Assume that $\{x, y\} = \{x_i, x_{i+1}\}$. Then $\{e, x_{i-1}, x_i, x_{i+1}\}$ is a $U_{2,4}$-restriction in M' that intersects the triad $\{x, y, z\}$, a contradiction. Next assume that $\{x, y\} = \{x_{i+1}, x_{i+2}\}$. Then $\{e, x_{i+1}, x_{i+2}\}$ and $\{x_{i+1}, x_{i+2}, x_{i+3}\}$ are triangles of M'. We again find that M' has a $U_{2,4}$-restriction that intersects a triad. $\qquad\square$

Note that F_e contains e and a fan in $\mathcal{F}_e$, so it contains at least four elements.

6.20.7. $|F_e| > 4$.

Proof. If F_e contains exactly four elements, then (x, y, z) or its reversal is in $\mathcal{F}_N$. However, (x, y, z) is (x_i, x_{i+2}, x_{i+1}) or (x_{i+2}, x_i, x_{i+1}), implying that (x, y, z) is not consistent with $(x_1, \ldots, x_n)$, although $\{x, y, z\} \subseteq \{x_1, \ldots, x_n\}$. This contradicts the fact that either $(x_1, \ldots, x_n)$ or $(x_1, \ldots, x_n, e_1) = (e_m, \ldots, e_1)$ is in the covering family $\mathcal{F}_e$ (up to reversing). $\qquad\square$

Let (e, x, y, z, v) be the first five elements of F_e. If $(y, z) = (x_i, x_{i+1})$, then $v = x_{i-1}$, for otherwise $\{v, x_{i-1}, x_i, x_{i+1}\}$ is a $U_{2,4}$-restriction in M', and $\{x_i, x_{i+1}, x_{i+2}\}$ is a triad. A similar argument shows that if $(y, z) = (x_{i+2}, x_{i+1})$, then $v = x_{i+3}$, since otherwise $\{v, x_{i+1}, x_{i+2}, x_{i+3}\}$ is a $U_{2,4}$-restriction in M'.

Assume that F_e contains more than five elements. Let (e, x, y, z, v, w) be the first six elements of F_e. Then $\{z, v, w\}$ is a triad of $M' \backslash e$. The previous paragraph shows that $\{z, v, w\}$ cannot be a set of three consecutive elements in $(x_1, \ldots, x_n)$, so the dual of Lemma 5.8 shows that w is not in $\{x_1, \ldots, x_n\}$. Because z and v are not consecutive in $(x_1, \ldots, x_n)$, it follows from Proposition 5.3 that $n \le 5$. Now 6.20.5 implies that $n = 5$, and $i = 2$, so $z = x_3$. But now the triad $\{z, v, w\}$ and the triangle $\{x_1, x_2, x_3\}$ violate orthogonality. Hence $F_e = (e, x, y, z, v)$.

Either (e, x, y, z, v) is $(e, x_i, x_{i+2}, x_{i+1}, x_{i+3})$, or it is $(e, x_{i+2}, x_i, x_{i+1}, x_{i-1})$. Now some three- or four-element subsequence of $(x_i, x_{i+2}, x_{i+1}, x_{i+3})$ or $(x_{i+2}, x_i, x_{i+1}, x_{i-1})$ is in $\mathcal{F}_N$ (up to reversing). But this subsequence must be equal to $(x_s, \ldots, x_t)$ for some values of s and

t. It is clear that no such subsequence exists, so this completes the proof of Lemma 6.20. $\qquad\square$

By reversing as necessary, we can assume that $(e_1, \ldots, e_m)$ is a fan in $\mathcal{F}_e$ and $e_1 \notin E(N)$. Lemma 6.20 tells us that e_1 is a rim element, and $\{e, e_1\}$ is contained in a triangle, T_e, of M. By Lemma 6.18, $M\backslash e/e_1$ is 3-connected and has N as a minor. As $|E(M)| - |E(N)| > 2$, we let x be an element in $E(M\backslash e/e_1) - E(N)$. First we assume that the fan in $\mathcal{F}_e$ that contains x is disjoint from $(e_1, \ldots, e_m)$. Let $(x_1, \ldots, x_n)$ be this fan, where we can reverse as necessary and assume that $x = x_1$. Then x_1 is a rim element of $(x_1, \ldots, x_n)$ and $\{e, x_1\}$ is contained in a triangle, T_x. Note that N is a minor of $M\backslash e/e_1/x_1$, for $\{x_2, x_3, x_4\}$ is a codependent triangle in $M\backslash e/e_1\backslash x_1$. This means $T_e \neq T_x$, for otherwise e is a loop in $M/e_1/x_1$, and this matroid has N as a minor, so is connected. Let C be a circuit of $M\backslash e$ contained in $(T_e \cup T_x) - e$ that contains e_1. If $|C| = 3$, then $M\backslash e/e_1$ contains a parallel pair, contradicting its 3-connectivity, so $C = (T_e \cup T_x) - e$, and therefore $x_1 \in C$. By orthogonality with $\{e_1, e_2, e_3\}$ we deduce that C contains an internal element from $(e_1, \ldots, e_m)$ (note $m \geq 4$ as $(e_1, \ldots, e_m)$ contains e_1 and a fan from $\mathcal{F}_N$), and this element is in $E(N)$ by Lemma 6.17. By orthogonality with $\{x_1, x_2, x_3\}$, we see that C also contains an element from $(x_2, \ldots, x_n)$ that is in $E(N)$. Thus $C - \{e_1, x_1\}$ is a parallel pair in $M\backslash e/e_1/x_1$, and in N, a contradiction.

Now the only elements in $E(M\backslash e/e_1) - E(N)$ belong to the same fan of $\mathcal{F}_e$ as e_1. By Lemmas 6.17 and 6.20, we can assume $(e_1, \ldots, e_m)$ is a fan in $\mathcal{F}_e$ such that e_1 and e_m are rim elements not in $E(N)$, and that $N = M\backslash e/e_1/e_m$. Let T_1 and T_m be triangles of M such that $\{e, e_1\} \subseteq T_1$ and $\{e, e_m\} \subseteq T_m$. Note that $T_1 \neq T_m$, as $M/e_1/e_m$ has N as a minor, and is therefore connected. Let C be a circuit of $M\backslash e$ contained in $(T_1 \cup T_m) - e$ that contains e_1. Then $|C| = 4$, or else $M\backslash e/e_1$ contains a parallel pair. Hence $e_m \in C$. Therefore $C - \{e_1, e_m\}$ is a parallel pair in $M\backslash e/e_1/e_m = N$, a contradiction. Now we have completed the proof of Theorem 6.10.

7. Acknowledgements

References

[1] C. Chun, D. Chun, B. Clark, D. Mayhew, G. Whittle, and S. H. M. van Zwam. Computer-verification of the structure of some classes of fragile matroids. arXiv:1312.5175 [math.CO]

[2] J. Geelen, B. Gerards, and G. Whittle. Towards a matroid-minor structure theory. In *Combinatorics, complexity, and chance*, volume 34 of *Oxford Lecture Ser. Math. Appl.*, pp. 72–82. Oxford Univ. Press, Oxford (2007).

[3] J. Geelen, J. Oxley, D. Vertigan, and G. Whittle. Weak maps and stabilizers of classes of matroids. *Adv. in Appl. Math.* **21** (1998), no. 2, 305–341.

[4] J. F. Geelen, A. M. H. Gerards, and A. Kapoor. The excluded minors for GF(4)-representable matroids. *J. Combin. Theory Ser. B* **79** (2000), no. 2, 247–299.

[5] R. Hall, D. Mayhew, and S. H. M. van Zwam. The excluded minors for near-regular matroids. *European J. Combin.* **32** (2011), no. 6, 802–830.

[6] S. R. Kingan and M. Lemos. Almost-graphic matroids. *Adv. in Appl. Math.* **28** (2002), no. 3-4, 438–477. Special issue in memory of Rodica Simion.

[7] S. R. Kingan and M. Lemos. Strong splitter theorem. arXiv:1201.4427 [math.CO]

[8] D. Mayhew, G. Whittle, and S. H. M. van Zwam. Stability, fragility, and Rota's Conjecture. *J. Combin. Theory Ser. B* **102** (2012), no. 3, 760–783.

[9] J. Oxley. *Matroid theory*. Oxford University Press, New York, second edition (2011).

[10] J. Oxley and H. Wu. On the structure of 3-connected matroids and graphs. *European J. Combin.* **21** (2000), no. 5, 667–688.

[11] J. G. Oxley. A characterization of a class of nonbinary matroids. *J. Combin. Theory Ser. B* **49** (1990), no. 2, 181–189.

[12] R. A. Pendavingh and S. H. M. van Zwam. Confinement of matroid representations to subsets of partial fields. *J. Combin. Theory Ser. B* **100** (2010), no. 6, 510–545.

[13] K. Truemper. A decomposition theory for matroids. VI. Almost regular matroids. *J. Combin. Theory Ser. B* **55** (1992), no. 2, 253–301.

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