UNAVOIDABLE CONNECTED MATROIDS RETAINING A SPECIFIED MINOR

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Abstract. A sufficiently large connected matroid M contains a big circuit or a big cocircuit. Pou-Lin Wu showed that we can ensure that M has a big circuit or a big cocircuit containing any chosen element of M. In this paper, we begin with a fixed connected matroid N and we take M to be a connected matroid that has N as a minor. Our main result establishes that if M is sufficiently large, then, up to duality, either M has a big connected minor in which N is a spanning restriction and the deletion of $E(N)$ is a large connected uniform matroid, or M has as a minor the 2-sum of a big circuit and a connected single-element extension or coextension of N. In addition, we find a set of unavoidable minors for the class of graphs that have a cycle and a bond with a big intersection.

1. Introduction

In this paper, we consider a fixed connected matroid N and look at a sufficiently large connected matroid M that has N as a minor. By a result of Lovász, Schrijver, and Seymour (see [5]), M has a big circuit or a big cocircuit. Hence M itself is a connected matroid that has N as a minor and also has a big uniform minor. But M may have many other elements that are not in either of these minors. Our goal is to pack these two minors compactly into some connected minor of M. Our main theorem proves that this can be done. Two elements, e and f, of M are *clones* if the map that interchanges e and f but fixes every other element of $E(M)$ is an automorphism of M.

Theorem 1.1. *Let N be a connected matroid with n elements and let k be a positive integer. There is a positive integer $f_{1.1}(n, k)$ such that, whenever M is a connected matroid that has at least $f_{1.1}(n, k)$ elements and has N as a minor, one of the following holds for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$.*

(i) *M_0 has a connected minor M_0' with $r(M_0') = r(N_0)$ and $|E(M_0') - E(N_0)| \geq k$ where M_0' has N_0 as a restriction, and all the elements*

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of $E(M_0') - E(N_0)$ are clones in M_0'; in particular, $M_0' \backslash E(N_0)$ is a connected uniform matroid having at least k elements; or

(ii) *for some connected single-element extension or coextension N_0' of N_0 by an element p, the matroid M_0 has, as a minor, the 2-sum with basepoint p of N_0' and a circuit that contains p and has at least k other elements.*

One way to view the last theorem is that if we have a connected matroid M with a fixed connected minor N and a huge connected uniform minor U, then, up to duality. we can find a connected minor M_0 of M such that either $E(M_0)$ has a partition (X, Y) such that $M_0|X = N$ and $M_0|Y$ is a big connected uniform matroid; or M_0 is the 2-sum of a single-element extension or coextension of N and a big circuit. Thus every element of M_0 is in $E(N)$ or the ground set of a big connected uniform minor of M_0. This has some similarity to intertwining two fixed matroids M_1 and M_2 where one seeks a minor-minimal matroid having minors isomorphic to M_1 and M_2. In our theorem, we seek to keep one matroid N as a minor but we allow ourselves to change the other matroid, in this case, so that it remains a big connected uniform matroid.

We continue with the same theme in Section 7 where we suppose that a connected matroid M has both a huge circuit and a huge cocircuit and we try to efficiently pack a big circuit and a big cocircuit into a minor of M. We show that either M has a big set that is the intersection of a circuit and a cocircuit, or M has, as a minor, the 2-sum of a big circuit and a big cocircuit. This leads us to consider what more we can say about M in the former case. In Section 8, we prove the following result for graphs and pose a conjecture about the corresponding result for binary matroids.

Theorem 1.2. *Let n be an integer exceeding two. There is an integer $f_{1.2}(n)$ such that if a graph G has a set Z of at least $f_{1.2}(n)$ edges such that Z is the intersection of a cycle and a bond, then G has, as a minor, one of the graphs $\Gamma_1(n), \Gamma_2(n), \Gamma_3(n)$, or $\Gamma_4(n)$ shown in Figure 1.*

Observe that each of $\Gamma_2(n)$ and $\Gamma_3(n)$ can be formed from n copies of K_4 by a sequence of 2-sums. But, although the cycle matroids of $\Gamma_2(2)$ and $\Gamma_3(2)$ are isomorphic, when $n \geq 3$, the cycle matroids of $\Gamma_2(n)$ and $\Gamma_3(n)$ are not isomorphic. It is also worth noting that $\Gamma_1(n)$ and $\Gamma_4(n)$ are dual graphs.

Sections 2–4 of the paper present some preliminary results that will be used in the proof of Theorem 1.1. In Section 5, we show that if M is a sufficently large connected matroid having a connected matroid N as a minor, then either M has a large connected minor that has N as a spanning restriction, or, up to duality, M has a minor that is the 2-sum of a big circuit and a single-element circuit or cocircuit of N. The proof of Theorem 1.1 is given in Section 6.

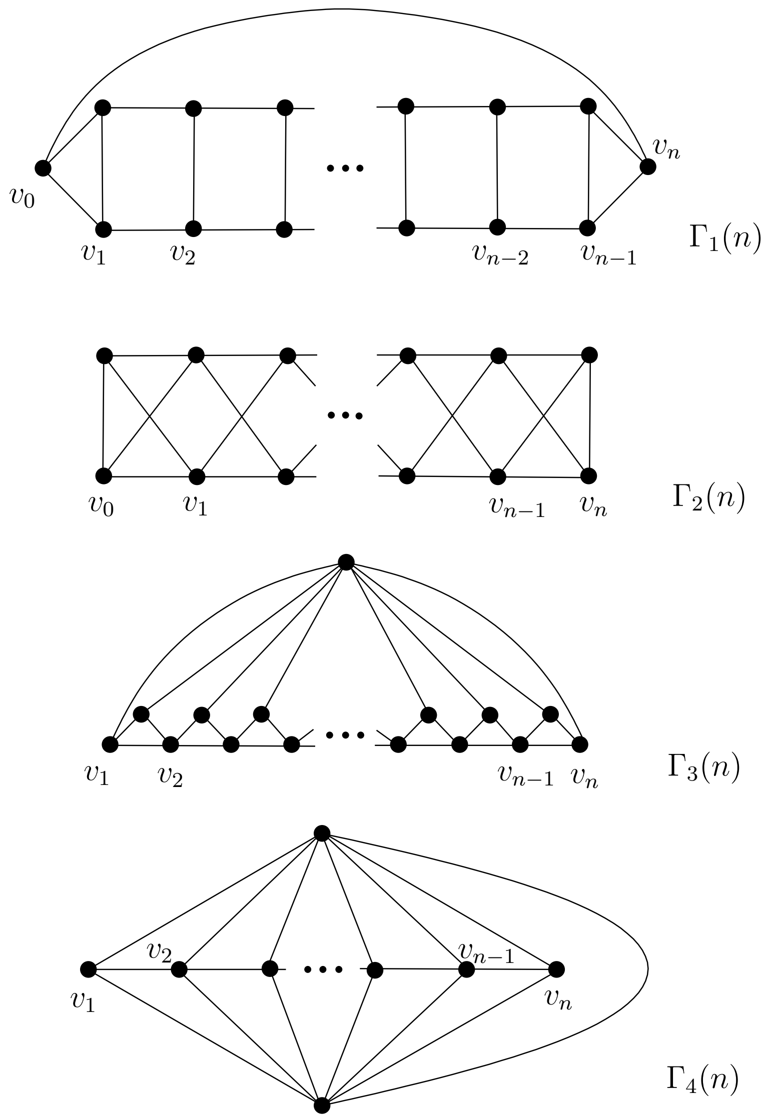

FIGURE 1. The graphs in Theorem 1.2.

2. PRELIMINARIES

The matroid terminology used here will follow Oxley [6]. Let M be a matroid. For subsets X and Y of $E(M)$, the *local connectivity* $\sqcap(X, Y)$ between X and Y is defined by $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$.

The following elementary property of matroids (see, for example, [6, Exercise 1.1.5]) will be used repeatedly throughout the paper.

Lemma 2.1. *In a matroid M, let C be a circuit and e be a non-loop element of $E(M)-C$. If M has a circuit D that contains e and is contained in $C \cup e$,* *then M has a circuit that contains $(C-D)\cup e$ and is contained in $C\cup e$. In particular, if e is in the closure of C, then M has distinct circuits C_1 and C_2 both of which contain e such that $C_1 \cup C_2 = C \cup e$.*

For a set I, let $\mathcal{A}$ be a family $(A_i : i \in I)$ of subsets of a set S. If there is a subset K of S such that $A_j \cap A_k = K$ for all distinct j and k in I, then $\mathcal{A}$ is a *sunflower* with *kernel* K. The following result, which is sometimes called the *sunflower lemma*, was proved by Erdős and Rado [2] .

Lemma 2.2. *Let h and k be positive integers and I be a set. Let $\mathcal{A}$ be a family $(A_i : i \in I)$ of subsets of a set S. If $|I| \geq k!h^{k+1}$ and $|A_i| \leq k$ for all i in I, then I contains a subset J with more than h members such that $(A_i : i \in J)$ is a sunflower.*

Let (X, Y) be a partition of the ground set of a matroid M. An *X-arc* is a minimal non-empty subset A of Y such that M has a circuit C with $C - X = A$ and $C \cap X \neq \emptyset$. This terminology is due to Seymour [8]. It should be noted that Geelen and Whittle [3] call an X-arc a *Y-strand*. Clearly all X-arcs are non-empty independent sets in M. Moreover, no X-arc is a proper subset of another X-arc.

Seymour [8, (3.1), (3.3)] proved the first two parts of the following result. The third part is a straightforward consequence of the second.

Lemma 2.3. *Let X be a set in a matroid M.*
(i) *If C is a circuit of M that meets X, then $C - X$ is expressible as a union of X-arcs.*
(ii) *If x and y are distinct elements of an X-arc A, then $\{x, y\}$ is a cocircuit of $M|(X \cup A)$.*
(iii) *If A is an X-arc, then $M|(X \cup A)$ is the 2-sum, with basepoint p, of an extension M_1 of $M|X$ by p and of a circuit with ground set $A \cup p$. Moreover, $r(M_1) = r(M|X)$.*

Proof. Let A be an X-arc. An immediate consequence of (ii) is that
$$r(X \cup A) = r(X) + |A| - 1.$$
As A is an independent set in M, it follows that $\sqcap(X, A) = 1$. The proof of (iii) is contained in the proof of Theorem 8.3.1 of [6]. □

The next two lemmas establish certain properties of X-arcs that will be used in the proof of Theorem 1.1.

Lemma 2.4. *Let X be a set in a matroid M and C be a circuit of M that meets X. Let A_1 and A_2 be disjoint X-arcs contained in $C - X$. If $a_1 \in A_1$, then A_2 contains an X-arc in $M/(A_1 - a_1)$.*

Proof. We argue by induction on $|A_1|$ noting that the result is immediate if $|A_1| = 1$. Assume the result is true if $|A_1| = k$ and let $|A_1| = k + 1 \geq 2$. Take b_1 in $A_1 - a_1$. Evidently $C - b_1$ is a circuit of M/b_1, and $A_1 - b_1$ is an X-arc in M/b_1. It suffices to show that A_2 contains an X-arc in M/b_1.

Now M has a circuit C_2 meeting X such that $C_2 - X = A_2$. If C_2 is a circuit of M/b_1, then A_2 certainly contains an X-arc in M/b_1 as desired. Thus we may assume that C_2 is not a circuit of M/b_1. By Lemma 2.1, M has distinct circuits D and D' such that each contains b_1 and their union is $C_2 \cup b_1$. We may assume that D meets A_2. If $D \subseteq A_2 \cup b_1$, then D is a circuit of M that is properly contained in C; a contradiction. Hence D meets X and, as desired, we get that A_2 contains an X-arc in M/b_1. The lemma follows immediately by induction. □

Lemma 2.5. *Let n_1 and n_2 be positive integers and X be a rank-n_1 set in a matroid M. Let C be a circuit of M that meets both X and $E(M) - X$ and has at least $n_2! n_1^{n_2}$ elements that are not in X. Then $C - X$ contains an X-arc with at least n_2 elements.*

Proof. Let $\mathcal{A}$ be the set of X-arcs that are contained in $C - X$. Then, by Lemma 2.3(i), every member of $C - X$ is in some member of $\mathcal{A}$. Assume that the lemma fails. Then every member of $\mathcal{A}$ has at most $n_2 - 1$ elements. Thus $\mathcal{A}$ contains at least $(n_2 - 1)! n_1^{n_2}$ members. By Lemma 2.2, there is a subset $\mathcal{A}'$ of $\mathcal{A}$ with at least $n_1 + 1$ members such that $\mathcal{A}'$ is a sunflower. Let K be the kernel of this sunflower. Then $C - K$ is a circuit of M/K and $\mathcal{A}'$ contains a set $\{A_1, A_2, \dots, A_{n_1+1}\}$ of disjoint X-arcs in M/K, each of which is contained in $C - K$. We complete the proof of the lemma by establishing the contradiction that

2.5.1. $r_M(X) \geq n_1 + 1$.

Choose an element a_1 of A_1 and consider $M/K/(A_1 - a_1)$. In this matroid, $\{a_1\}$ is an X-arc and $C - K - (A_1 - a_1)$ is a circuit. Thus a_1 is in the closure of X in $M/K/(A_1 - a_1)$. Also, by Lemma 2.4, for each i in $\{2, 3, \dots, n_1 + 1\}$, the set $A_i - (A_1 - a_1)$ contains an X-arc A'_i in $M/K/(A_1 - a_1)$. Thus A'_2 contains an element a_2 such that $\{a_2\}$ is an X-arc of $M/K/(A_1 - a_1)/(A'_2 - a_2)$. Moreover, for each i in $\{3, 4, \dots, n_1 + 1\}$, the set $A'_i - (A'_2 - a_2)$ contains an X-arc in the last matroid. By repeating this process, we see that $C - X$ contains disjoint subsets $\{a_1, a_2, \dots, a_{n_1+1}\}$ and Z such that, in M/Z, the set $\{a_1, a_2, \dots, a_{n_1+1}\}$ is contained in the closure of X. As $C - X$ is independent in M, it follows that $\{a_1, a_2, \dots, a_{n_1+1}\}$ is independent in M/Z. Since $\{a_1, a_2, \dots, a_{n_1+1}\} \subseteq \mathrm{cl}_{M/Z}(X)$, we deduce that $r_{M/Z}(X) \geq n_1 + 1$. Thus 2.5.1 holds and the lemma follows. □

3. Ramsey's theorem

The main theorem of this paper should be seen in the context of Ramsey theory, which, loosely speaking, asserts that, within a sufficiently large object, some structure must emerge. In this section, we first state Ramsey's original theorem [7] and then give a straightforward consequence of it that we shall need. Let X be a finite set and k be a positive integer. We denote the set of all subsets of X by 2^X and write $\binom{X}{k}$ for the set of all k-element subsets of X. The set of all non-empty subsets of X with at most k elements will be denoted by $\binom{X}{[k]}$. A mapping from a set Y into a set Z is called a *coloring*. The members of Z are called *colors*. When $|Z| = t$, such a mapping is also called a t*-coloring*.

Theorem 3.1. *Let k, t, and $n_1, n_2, \ldots, n_t$ be positive integers. There is a function $f_{3.1}(k; n_1, n_2, \ldots, n_t)$ with the property that, for every set X with at least $f_{3.1}(k; n_1, n_2, \ldots, n_t)$ elements and every coloring of $\binom{X}{k}$ by elements of $\{1, 2, \ldots, t\}$, there is an element j of $\{1, 2, \ldots, t\}$ and an n_j-element subset Y of X such that every member of $\binom{Y}{k}$ receives the color j.*

The following is a well-known corollary of this theorem.

Corollary 3.2. *Let k, t, and n be positive integers. There is a function $f_{3.2}(k, t, n)$ with the property that, for every set X with at least $f_{3.2}(k, t, n)$ elements and every t-coloring of $\binom{X}{[k]}$, there is an n-element subset Y of X such that, for all $i \in \{1, 2, \ldots, k\}$, all members of the set $\binom{Y}{i}$ receive the same color.*

4. Unavoidable minors of big connected matroids

As noted in the introduction, Lovász, Schrijver, and Seymour were the first to show that, in a connected matroid, by bounding the size of a largest circuit and a largest cocircuit, we are also bounding the size of the ground set of the matroid. The following result of Lemos and Oxley [4] gives a best-possible such bound.

Theorem 4.1. *Let M be a connected matroid with at least two elements having largest circuit with c elements and largest cocircuit with c^* elements. Then*

$$|E(M)| \leq \tfrac{1}{2}cc^*.$$

In the proof of our main theorem, we shall use the following result of Pou-Lin Wu [11], which shows that if e is an element of a connected matroid M, and M has a big circuit, then M has a big circuit containing e.

Theorem 4.2. *Let M be a connected matroid with at least two elements having largest circuit with c elements. Then every element of M is in a circuit of size at least $\frac{c}{2} + 1$.*

5. A big spanning restriction or a 2-sum

Given a sufficiently large connected matroid M that has N as a connected minor, one possibility is that M has a large connected minor of which N is a spanning restriction. The task of this section is to show that, up to duality, when this possibility does not arise, M has, as a minor, the 2-sum of a big circuit and a single-element extension or coextension of N.

Theorem 5.1. *Let N be a non-empty connected matroid with n elements and let k be a positive integer. There is a positive integer $f_{5.1}(n, k)$ such* *that, whenever M is a connected matroid that has at least $f_{5.1}(n,k)$ elements and has an N-minor, one of the following holds for some (M_0,N_0) in $\{(M,N),(M^*,N^*)\}$.*

(i) *M_0 has a connected minor M_0' having at least $n+k$ elements such that $r(M_0')=r(N_0)$, and M_0' has N_0 as a restriction; or*

(ii) *for some connected single-element extension or coextension N_0' of N_0 by an element p, the matroid M_0 has, as a minor, the 2-sum with basepoint p of N_0' and a circuit that contains p and has at least k other elements.*

The next lemma contains the core of the proof of this theorem.

Lemma 5.2. *Let N be a non-empty connected matroid with n elements and let k be a positive integer. There is a positive integer $f_{5.2}(n,k)$ such that, whenever M is a connected matroid that has a circuit with at least $f_{5.2}(n,k)$ elements and has an N-minor, one of the following holds.*

(i) *M^* has a connected minor M' having at least $n+k$ elements such that $r(M')=r(N^*)$, and M' has N^* as a restriction; or*

(ii) *for some connected single-element extension or coextension N' of N by an element p, the matroid M has, as a minor, the 2-sum with basepoint p of N' and a circuit that contains p and has at least k other elements.*

Proof. Let M be a connected matroid having a circuit with at least $2(3^k k! n^k+n+k)$ elements and having an N-minor. We shall show that (i) or (ii) holds for M. Let X and Y be subsets of $E(M)$ such that $N=M/X\backslash Y$ and Y is maximal. Then $r^*(M\backslash Y)=r^*(N)$. Moreover, since N is connected and Y is maximal, $M\backslash Y$ is connected. If $|X|\geq k$, then the lemma holds. Thus we may assume that $|X|<k$.

As M has a circuit with at least $2(3^k k! n^k+n+k)$ elements, it follows by Theorem 4.2 that M has a circuit C that meets $E(N)$ and has at least $3^k k! n^k+n+k$ elements. Let $M_1=M\backslash(Y-C)$. Then $N=M_1\backslash(Y\cap C)/X$. Since N is connected, it follows by the maximality of Y that M_1 is connected. Let $M_2=M_1/(X\cap C)$. Again, the maximality of Y implies that M_2 is connected. Moreover, $C-X$ is a circuit C_2 of M_2 meeting $E(N)$ and, since $|X|<k$, it follows that

$$|C_2|>3^k k! n^k+n.$$

Let $X_2=X-C$ and let X_2' be a maximal subset of X_2 such that M_2/X_2' has C_2 as a circuit. Then the maximality of Y means that M_2/X_2' is connected. Let $M_3=M_2/X_2'$ and let $X_3=X_2-X_2'$. Evidently each element of X_3 is in the closure of C_2 in M_3.

Next we prove the following.

5.2.1. *Either*

(i) *M_3/X_3 has a circuit that meets $E(N)$ and has at least $\frac{|C_2|}{3^{|X|}}$ elements; or*

(ii) *for some connected single-element coextension N_1 of N by an element p of X_3, the matroid M has, as a minor, the 2-sum with basepoint p of N_1 and a circuit that contains p and has at least $\frac{2|C_2|}{3^{|X|}}$ other elements.*

We prove this by induction on $|X_3|$. The result is immediate if $|X_3| = 0$. Suppose $|X_3| \geq 1$ and consider x_3 in X_3. As $x_3 \in \mathrm{cl}_{M_3}(C_2)$, Lemma 2.1 implies that M_3 has a circuit that contains x_3, meets $E(N)$, and is contained in $C_2 \cup x_3$. Clearly either

(a) M_3/x_3 has a circuit of size at least $\frac{|C_2|}{3}$ that meets $E(N)$; or
(b) every circuit of M_3/x_3 that meets $E(N)$ has size less than $\frac{|C_2|}{3}$.

In the first case, the result follows by induction. Thus, we may assume that (a) does not hold. Then (b) holds and, by Lemma 2.1, M_3 has a circuit D_3 that contains x_3, that avoids $E(N)$, that is contained in $C_2 \cup x_3$, and that has more than $\frac{2|C_2|}{3} + 1$ elements.

Suppose that M_3/x_3 is connected. Since this matroid has $D_3 - x_3$ as a circuit, it follows by Theorem 4.2 that M_3/x_3 has a circuit that meets $E(N)$ and has at least $\frac{|C_2|}{3}$ elements, which contradicts the fact that (a) does not hold. We deduce that, for every x_3 in X_3, we may assume that M_3/x_3 is disconnected and that

5.2.2. *there is a partition (F_3, G_3) of C_2 such that $F_3 \cup x_3$ and $G_3 \cup x_3$ are circuits of M_3 where $C_2 \cap E(N) \subseteq G_3$ and $|G_3| \leq \frac{|C_2|}{3}$ elements. Moreover, F_3 is a component of M_3/x_3.*

Suppose $X_3 = \{x_3\}$. Then $M_3 \backslash C_2$ is a connected single-element coextension N_1 of N by the element x_3, and $M_3 \backslash (C_2 - F_3)$ is the parallel extension, with basepoint x_3, of N_1 and a circuit with ground set $F_3 \cup x_3$. Thus $M_3 \backslash (C_2 - F_3) \backslash x_3$ is the 2-sum, with basepoint x_3, of N_1 and the circuit $F_3 \cup x_3$. Moreover, $M_3 \backslash (C_2 - F_3) \backslash x_3 / F_3 = N$. Since $|F_3| \geq \frac{2|C_2|}{3}$, the result follows in this case.

We may now assume that $|X_3| \geq 2$. Let x_3 and x_3' be distinct elements of X_3. Let (F_3, G_3) and (F_3', G_3') be corresponding partitions of C_2 given by 5.2.2, where $|F_3|, |F_3'| \geq \frac{2|C_2|}{3}$.

Suppose $F_3' \subseteq F_3$. In M_3/x_3, we have F_3 and G_3 contained in separate components. But $x_3' \in \mathrm{cl}_{M_3}(F_3') \subseteq \mathrm{cl}_{M_3}(F_3)$. Hence we can move x_3' from X into Y contradicting the maximality of the latter. Thus $F_3' \not\subseteq F_3$ and, by symmetry, $F_3 \not\subseteq F_3'$. Hence $F_3' \cap G_3$ and $F_3 \cap G_3'$ are both non-empty. Moreover, $G_3 \cap G_3' \neq \emptyset$ as $G_3 \cap G_3' \supseteq C_2 \cap E(N)$. Finally, as $|G_3|, |G_3'| \leq \frac{|C_2|}{3}$, we see that $|G_3 \cup G_3'| \leq \frac{2|C_2|}{3}$, so $|F_3 \cap F_3'| \geq \frac{|C_2|}{3}$.

Let P_3 be the dual of $M_3|(C_2 \cup x_3 \cup x_3')$. Then $r(P_3) = 3$, and P_3 has $\{x_3, x_3'\}$ as a line. Moreover, P_3/x_3 has $\{x_3'\}, F_3'$, and G_3' as parallel classes, while P_3/x_3' has $\{x_3\}, F_3$, and G_3 as parallel classes. In P_3, there are exactly three lines through x_3. These lines contain $\{x_3, x_3'\}, F_3' \cup x_3$, and $G_3' \cup x_3$.

Likewise, the three lines through x_3' contain $\{x_3, x_3'\}, F_3 \cup x_3'$, and $G_3 \cup x_3'$. Thus $\mathrm{si}(P_3)$ is isomorphic to $M(K_4)$, and P_3 has $F_3 \cap F_3', F_3 \cap G_3', G_3 \cap F_3', G_3 \cap G_3', \{x_3\}$, and $\{x_3'\}$ as parallel classes. Hence P_3 has $\{x_3, x_3'\} \cup (F_3 \cap F_3') \cup (G_3 \cap G_3')$ as a cocircuit that meets $E(N)$. Thus $M_3/x_3, x_3'$ has $(F_3 \cap F_3') \cup (G_3 \cap G_3')$ as a circuit that meets $E(N)$ and that has at least $\frac{|C_2|}{3}$ elements. It now follows, by induction, that 5.2.1 holds.

Now assume that 5.2.1(i) occurs and let C_3 be a circuit of M_3/X_3 that meets $E(N)$ and has at least $\frac{|C_2|}{3^{|X|}}$ elements. Since $|X| < k$ and $|C_2| > 3^k k! n^k + n$, we see that

$$|C_3| > \frac{3^k k! n^k + n}{3^{|X|}} > k! n^k + n.$$

Then $|C_3 - E(N)| > k! n^k$. Since $r(N) < |E(N)| \leq n$, it follows by Lemma 2.5 that C_3 contains an $E(N)$-arc with at least k elements. Hence, by Lemma 2.3(iii), part (ii) of the lemma holds.

Finally, assume that 5.2.1(ii) holds. Then, as $\frac{2|C_2|}{3} \geq k$, we again get that (ii) of the lemma holds. □

Proof of Theorem 5.1. Let M be a connected matroid having at least $[f_{5.2}(n,k)]^2$ elements and having an N-minor. Then, by Theorem 4.1, M has a circuit or a cocircuit with at least $f_{5.2}(n,k)$ elements. By switching to the dual if necessary, we may assume the former. The theorem is now an immediate consequence of Lemma 5.2. □

6. Large matroids of bounded rank

When we begin with a positive integer k and a sufficiently large connected matroid M having some connected minor N as a minor, Theorem 5.1 tells us that one possibility is that M has, as a minor, a connected extension M' of N such that $r(M') = r(N)$ and $|E(M')| - |E(N)| \geq k$. In this section, we show that, when k is sufficiently large, M' has a connected restriction M'' such that all the elements of $E(M'') - E(N)$ are clones. This result gives us the final piece we need to prove the main theorem of the paper, and that proof appears at the end of the section.

Theorem 6.1. *Let N be a matroid with n elements and let k be a positive integer. There is a positive integer $f_{6.1}(n,k)$ such that, whenever M is a matroid with at least $f_{6.1}(n,k)$ elements such that M has N as a spanning restriction, M has a restriction M' with at least $n+k$ elements such that N is a spanning restriction of M' and all the elements of $E(M') - E(N)$ are clones in M'. In particular, $M' \backslash E(N)$ is uniform.*

Proof. Let M be a matroid with at least $f_{3.2}(1 + r(N), 2^{1+2^n}, n)$ elements and suppose that M has N as a spanning restriction. Let $X = E(M) - E(N)$

and let $d = 1 + r(N)$. For every $A \in \binom{X}{[d]}$, let

$$c_1(A) = \begin{cases} 1 & \text{if } A \text{ is a circuit of } M; \\ 0 & \text{if } A \text{ is not a circuit of } M; \end{cases}$$

and let

$$c_2(A) = \{D \in 2^{E(N)} : D \cup A \text{ is a circuit of } M\}.$$

Finally, let $c_0 = c_1 \times c_2$. Then c_0 is a 2^{1+2^n}-coloring of $\binom{X}{[d]}$. By Corollary 3.2, X contains a subset Y such that, for all i in $\{1, 2, \ldots, 1+r(N)\}$, all members of $\binom{Y}{i}$ receive the same color. Let $M' = M|(Y \cup E(N))$. Then all the elements of Y are clones in M' and the theorem holds. $\square$

In the last theorem, there are potentially many different ways for all of the elements of $E(M') - E(N)$ to be clones. For example, all these elements could be parallel, or they could all be added freely to N. In addition, we could take a parallel connection of N and a line with ground set $E(M') - E(N)$ and then truncate this matroid. In general, if $r(M' - E(N)) = t$, then M' can be obtained by extending N by some independent set Z of t elements to give a matroid M'' in which all the elements of Z are clones. We then freely add the elements of $E(M') - E(M'')$ to the flat of M'' that is spanned by Z. It is straightforward to check that, in the resulting matroid, all the elements of $E(M') - E(N)$ are clones.

Applying the last theorem to connected matroids, we immediately obtain the following result.

Corollary 6.2. *Suppose N is a connected matroid with n elements and non-zero rank, and let k be a positive integer. There is a positive integer $f_{6.2}(n, k)$ such that, whenever M is a connected matroid with at least $f_{6.2}(n, k)$ elements such that M has N as a spanning restriction, M has a connected restriction M' with at least $n + k$ elements such that N is a spanning restriction of M' and all the elements of $E(M') - E(N)$ are clones in M'. In particular, $M' \backslash E(N)$ is a connected uniform matroid.*

We are now ready to prove the main theorem.

Proof of Theorem 1.1. Let M be a connected matroid having at least $f_{5.1}(n, f_{6.2}(n, k))$ elements and having an N-minor. It is immediate from Theorem 5.1 and Corollary 6.2 that

$$f_{5.1}(n, k) \geq n + k \text{ and } f_{6.2}(n, k) \geq n + k.$$

Suppose that (M_0, N_0) is a member of $\{(M, N), (M^*, N^*)\}$ for which (i) or (ii) of Theorem 5.1 holds. In the latter case, for some connected single-element extension or coextension N_0' of N_0 by an element p, the matroid M_0 has, as a minor, the 2-sum with basepoint p of N_0' and a circuit that contains p and has at least $f_{6.2}(n, k)$ other elements. Since $f_{6.2}(n, k) \geq n+k$, it follows that part (ii) of the theorem holds.

We may now assume that (i) of Theorem 5.1 holds. Then M_0 has a connected minor M_0' having at least $n+f_{6.2}(n,k)$ elements such that $r(M_0') = r(N_0)$, and M_0' has N_0 as a restriction. Thus, by Corollary 6.2, M_0' has a connected restriction M_0'' with at least $n+k$ elements such that N_0 is a spanning restriction of M_0'', and $M_0''\backslash E(N_0)$ is a connected uniform matroid with at least k elements, so part (i) of the theorem holds. □

7. A big circuit and a big cocircuit

By Theorem 4.1, a sufficiently large connected matroid has a big circuit or a big cocircuit. In this section, we consider what can be said about a connected matroid that has both a big circuit and a big cocircuit. The next lemma is well known (see, for example, [6, Exercise 3.3.11]). It and the subsequent lemma will be used in the proof of the main result of this section.

Lemma 7.1. *Suppose that, in a matroid M, a non-empty set Z is the intersection of a circuit and a cocircuit. Then M has a minor M_0 in which Z is a spanning circuit of both M_0 and M_0^*.*

Lemma 7.2. *Let M be a connected matroid and D be a set of clones in M with $r(D) = t \geq 1$ and $|D| = t + d$. Assume that M has a circuit C with $|C| \geq 2^t(s + 2) - 2$ for some positive integer s. Then M has, as a minor, the parallel connection of a circuit of size at least $s + 1$ and a cocircuit of size at least $d + 1$.*

Proof. Since C is a circuit, it follows by Theorem 4.2 that M has a circuit C_1 of size at least $2^{t-1}(s+2)-1$ that meets D. We argue by induction on t. If $t = 1$, then the result follows immediately. Now assume the result holds when $t < m$ and let $t = m \geq 2$. Take e in $D \cap C_1$. Then $C_1 - e$ is a circuit of M/e and $D - e$ is a set of clones in M/e. Evidently $|D - e| = (t - 1) + d$ and $|C_1 - e| \geq 2^{t-1}(s + 2) - 2$. We may assume that $D - e$ and $C_1 - e$ are in different components of M/e, otherwise the result follows by induction. Thus $D \cap C_1 = \{e\}$. Hence M is the parallel connection, with basepoint e, of a connected matroid that contains D and a connected matroid that contains C_1. It follows that the elements of $D - e$ are all parallel to e in M. We conclude, by induction, that the required result holds. □

Theorem 7.3. *Let n be a positive integer. There is a positive integer $f_{7.3}(n)$ such that, whenever M is a connected matroid having both a circuit and a cocircuit with at least $f_{7.3}(n)$ elements, M has as a minor either*

- (i) *the 2-sum of an $(n + 1)$-element circuit and an $(n + 1)$-element cocircuit; or*
- (ii) *a matroid M_0 that contains a set with at least n elements that is a spanning circuit in both M_0 and M_0^*.*

Proof. It is immediate from Lemma 5.2 and Theorem 6.1 that

$$f_{5.2}(n, k) \geq n + k \text{ and } f_{6.1}(n, k) \geq n + k.$$

Let $m = 2^n(n+2)$ and let M be a connected matroid having both a circuit C and a cocircuit with at least $2f_{5.2}(m, f_{6.1}(m, 2m))$ elements. Then, by Theorem 4.2, M has a cocircuit C^* that meets C and has at least $f_{5.2}(m, f_{6.1}(m, 2m))$ elements.

We may assume that $|C \cap C^*| < n$ otherwise it follows, by Lemma 7.1, that (ii) holds. Let J be a subset of $C^* - C$ such that $|C^* - J| = m$. Let C_1^* be $C^* - J$. Then $M \backslash J$ has a component M_1 in which C_1^* and C are in the same component. Let $N = M_1.C_1^*$. Then N is an m-element cocircuit. Since M_1 has C as a circuit with at least $f_{5.2}(m, f_{6.1}(m, 2m))$ elements, it follows by Lemma 5.2 that either

(i) M_1^* has a connected minor M' with at least $m + f_{6.1}(m, 2m)$ elements such that $r(M') = r(N^*)$ and M' has N^* as a restriction; or
(ii) for some connected single-element extension or coextension N_1 of N by an element p, the matroid M_1 has, as a minor, the 2-sum with basepoint p of N_1 and a circuit that contains p and has at least $f_{6.1}(m, 2m)$ other elements.

Suppose that (ii) holds. If N_1 is an extension of N, then N_1 is also a cocircuit and (i) of the theorem holds. If N_1 is a coextension of N, then p is in a cocircuit of N_1 with at least $\frac{1}{2}f_{6.1}(m, 2m)$ other elements and again (i) of the theorem holds.

We may now assume that (i) holds. Then, since $|E(N)| = m$, it follows, by Corollary 6.2, that M' has a connected restriction M'' with at least $m + 2m$ elements such that N^* is a spanning restriction of M'', and the set D of elements of $E(M'') - E(N^*)$ is a set of clones in M''. Clearly $|D| \geq 2m$. As M'' is spanned by $E(N^*)$, it follows that $r(D) \leq r(N^*) \leq m$. If $r(D) \geq n$, then, since $|D| \geq 2m$, we see that M'', and hence M, has a minor M_0 that contains a set with at least n elements that is a spanning circuit in both M_0 and M_0^*, that is, (ii) of the lemma holds. We may now assume that $r(D) = t < n$. Clearly $|D| \geq t + m$, and M'' has $E(N^*)$ as a cocircuit with m elements. But $m = 2^n(n+2)$. Thus, by Lemma 7.2, M'' has, as a minor, the parallel connection of a circuit of size at least $n + 2$ and a cocircuit of size at least $m + 1$. Since $m \geq n$, we conclude that (i) of the theorem holds in this case, so the proof is complete. □

8. Graphs with a cycle and bond having a big intersection

By Theorem 7.3, one of the possibilities for a connected matroid M that has both a big circuit and a big cocircuit is that M has a minor that contains a big set that is the intersection of a circuit and a cocircuit. The number of matroids like this seems large. We had hoped to be able to identify a family of unavoidable minors for the class of such matroids. But we were unable to solve this problem even when we restrict to the binary case. In the latter case, we have a potential list of unavoidable minors, but we are unsure that this list is complete. The list is given at the end of the section.

The main result of this section solves the problem when we restrict to the class of graphic matroids by proving Theorem 1.2.

Let G be a graph with vertex set V. For any subset X of V, let $G[X] = G - (V - X)$, and let $\delta_G(X)$ denote the set of edges of G that have one end in each of X and $V - X$. When $X \notin \{V, \emptyset\}$ and both $G[X]$ and $G[V - X]$ are connected, $\delta_G(X)$ is a bond in G or, equivalently, it is a cocircuit in $M(G)$.

Our proof of Theorem 1.2 will require some preliminary results.

Lemma 8.1. *Let d and l be non-negative integers. There is an integer $f_{8.1}(d, l)$ such that if G is a connected graph with at least $f_{8.1}(d, l)$ vertices and v is a vertex of G, then either G has a vertex of degree exceeding d, or G has a path that has v as an end and has length exceeding l.*

Proof. Let $n = 1 + d + d(d-1) + d(d-1)^2 + \cdots + d(d-1)^{l-1}$. Suppose G is a connected graph with $|V(G)| \geq \max\{3, 1+n\}$, and let v be a vertex of G. Since $|V(G)| \geq 3$, G has a path of length two. It follows that the lemma holds if $d \leq 1$ or $l \leq 1$. Hence we may assume that $d > 1$ and $l > 1$. We may also assume that every vertex of G has degree at most d. It follows that G has at most n vertices that are distance at most l away from v. Since $|V(G)| \geq 1 + n$, we deduce that G has a path that has v as an end and has length exceeding l. We conclude that the lemma holds if we take $f_{8.1}(d, l) = \max\{3, 1 + n\}$. □

For a positive integer m, we shall denote by mK_2 the disjoint union of m copies of K_2, that is, mK_2 is a matching with m edges.

Lemma 8.2. *Let m, s, and t be positive integers. There exists an integer $f_{8.2}(m, s, t)$ such that every simple graph with at least $f_{8.2}(m, s, t)$ edges has one of mK_2, $K_{1,s}$, or K_t as an induced subgraph.*

Proof. Let G be a simple graph with vertex set $\{1, 2, \ldots, n\}$ having at least $f_{3.1}(2; s, t, t, t, 2s, m)$ edges. For any two distinct edges uv and xy of G, where $u < v$ and $x < y$, we let $Z = \{u, v, x, y\}$ and define

$$c(uv, xy) = \begin{cases} 1 & \text{if } |Z| = 3 \text{ and } |E(G[Z])| = 2; \\ 2 & \text{if } |Z| = 3 \text{ and } |E(G[Z])| = 3; \\ 3 & \text{if } |Z| = 4 \text{ and } vy \in E(G); \\ 4 & \text{if } |Z| = 4 \text{ and } vy \notin E, \text{ but } ux \in E; \\ 5 & \text{if } |Z| = 4 \text{ and } \{vy, ux\} \cap E = \emptyset, \text{ and } |E(G[Z])| \geq 3; \\ 6 & \text{if } |Z| = 4 \text{ and } \{vy, ux\} \cap E = \emptyset, \text{ and } |E(G[Z])| = 2. \end{cases}$$

The six situations arising above are illustrated in Figure 2, where $\{e, f\} = \{uv, xy\}$ and, in the fifth case, vx may be present instead of uy.

Clearly c is a 6-coloring of $\binom{E}{2}$. By Theorem 3.1, for some j in $\{1, 2, 3, 4, 5, 6\}$ and some n_j-element subset F of E, all pairs in $\binom{F}{2}$ have the same color j, where $(n_1, n_2, n_3, n_4, n_5, n_6) = (s, t, t, t, 2s, m)$. Let $F = \{x_1y_1, x_2y_2, \ldots, x_{n_j}y_{n_j}\}$, where $x_i < y_i$ for all i. Let $X = \{x_1, x_2, \ldots, x_{n_j}\}$ and $Y = \{y_1, y_2, \ldots, y_{n_j}\}$.

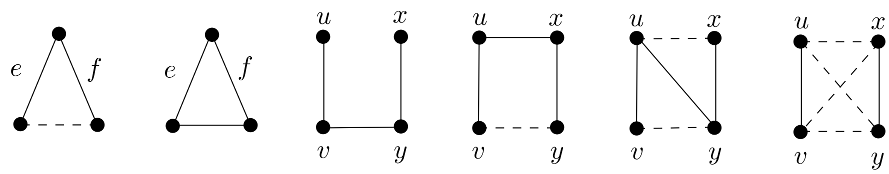

FIGURE 2. Dashed lines represent non-edges of G; unlinked vertices may or may not be adjacent in G.

If j is 1 or 6, then F forms an induced subgraph of G isomorphic to $K_{1,s}$ or mK_2, respectively. If $j = 2$, then $G[X \cup Y]$ is K_{t+1}, unless $t = 3$ when $G[X \cup Y] \in \{K_3, K_4\}$. If j is 3 or 4, then $G[Y]$ or $G[X]$, respectively, is K_t. Finally, if $j = 5$, then, for all i in $\{2, 3, \dots, n_5\}$, at least one of x_1y_i and y_1x_i is an edge of G. It follows that at least one of $G[\{x_1\} \cup Y]$ and $G[\{y_1\} \cup X]$ contains an induced subgraph isomorphic to $K_{1,s}$. We conclude that the lemma holds if we take $f_{8.2}(m, s, t) = f_{3.1}(2; s, t, t, t, 2s, m)$. □

Lemma 8.3. *Let t_1, t_2, t_3, and t_4 be positive integers. There is an integer $f_{8.3}(t_1, t_2, t_3, t_4)$ such that if G is a simple graph with vertex set $\{1, 2, \dots, n\}$ and G has at least $f_{8.3}(t_1, t_2, t_3, t_4)$ edges, then G has distinct edges $x_1y_1, x_2y_2, \dots, x_ty_t$ such that at least one of the following holds:*

(i) $t = t_1$ *and* $x_1 = x_2 = \cdots, = x_t$;
(ii) $t = t_2$ *and* $x_1 < y_1 < x_2 < y_2 < \cdots < x_t < y_t$;
(iii) $t = t_3$ *and* $x_1 < x_2 < \cdots < x_t < y_1 < y_2 < \cdots < y_t$;
(iv) $t = t_4$ *and* $x_1 < x_2 < \cdots < x_t < y_t < y_{t-1} < \cdots < y_1$.

Proof. Let G be a simple graph with vertex set $\{1, 2, \dots, n\}$ and suppose that G has at least $f_{3.1}(2; t_1t_4, 2t_2 - 1, t_3)$ edges. Let uv and xy be an arbitrary pair of distinct edges of G with $u < v$ and $x < y$, where $u \le x$. Define

$$c(uv, xy) = \begin{cases} 1 & \text{if } u = x \text{ or } y \le v; \\ 2 & \text{if } v \le x; \\ 3 & \text{otherwise, that is, if } u < x < v < y. \end{cases}$$

Then c is a 3-coloring of $\binom{E}{2}$. By Theorem 3.1, there is an element j in $\{1, 2, 3\}$ and an n_j-element subset F of E such that all pairs in $\binom{F}{2}$ have the same color j, where $(n_1, n_2, n_3) = (t_1t_4, 2t_2 - 1, t_3)$. Let $F = \{x_1y_1, x_2y_2, \dots, x_{n_j}y_{n_j}\}$. If $j = 1$, then we may assume $x_1 \le x_2 \le \cdots \le x_{n_j} \le y_{n_j} \le y_{n_j-1} \le \cdots \le y_1$. In this case, either a subset of F satisfies (i), or $\{x_{1+it_1}y_{1+it_1} : i \in \{0, 1, \dots, t_4 - 1\}\}$ satisfies (iv). If $j = 2$, then we may assume $x_1 < y_1 \le x_2 < y_2 \le \cdots \le x_{n_j} < y_{n_j}$. In this case, $\{x_{2i-1}y_{2i-1} : i \in \{1, 2, \dots, t_2\}\}$ satisfies (ii). Finally, if $j = 3$, then F satisfies (iii). □

From now on, we will call a vertex in a graph *universal* if it is adjacent to all other vertices of the graph. In addition, when x and y are vertices of a path P, we denote by $P[x, y]$ the subpath of P between x and y.

Lemma 8.4. *Let n be a positive integer. There exists an integer $f_{8.4}(n)$ with the following property. If a graph G has a bond $\delta_G(V_1)$ that is contained in a cycle C such that $G[V_1]$ has a universal vertex, and $|\delta_G(V_1)| \geq f_{8.4}(n)$, then G has $\Gamma_1(n)$, $\Gamma_3(n)$, or $\Gamma_4(n)$ as a minor.*

Proof. Let $l = f_{8.3}(3, n+2, n+1, n+1)$. We shall show that the lemma holds with $f_{8.4}(n) = 2f_{8.1}(2n-1, l-1)$. Let G be a graph satisfying the specified conditions, and let $D = \delta_G(V_1)$ and u_1 be a universal vertex of $G[V_1]$. Let $V_2 = V(G) - V_1$. By contracting edges of C in $G[V_2]$, we may assume that no edge of C is contained in $G[V_2]$. Also, by repeatedly contracting edges of $G[V_2]$ that have at most one end on C, we may assume that $V_2 \subseteq V(C)$. After these reductions, every vertex of V_2 meets exactly two edges of C and so meets exactly two edges of D.

Since $|V_2| = |D|/2 \geq f_{8.1}(2n-1, l-1)$, it follows by Lemma 8.1 that $G[V_2]$ has subgraph T such that T is a star on $2n+1$ vertices or a path on $l+1$ vertices. Let us delete all the edges of $G[V_2]$ that are not in T. Then each vertex v in $V_2 - V(T)$ has degree two as it meets two edges in D. For all such v, contract exactly one of the two edges of D that are incident with v. Let G' be the resulting graph, and let $C' = E(G') \cap C$. Then C' is a cycle of G'. Let $V_2' = V(T)$. Then (V_1, V_2') is a partition of $V(G')$ and $G'[V_2'] = T$. Moreover, u_1 is a universal vertex of $G'[V_1]$, and $\delta_{G'}(V_1)$ is a bond D' of G' that is contained in the cycle C'. Now we apply the same reductions used in the first paragraph to $G'[V_1]$. First we contract edges if necessary to get that no edge of C' is contained in $G'[V_1]$. Then further contractions, this time of edges with at most one end in $V(C')$, mean that we may assume that $V_1 \subseteq V(C')$. Note that u_1 remains a universal vertex of $G'[V_1]$ where we follow the convention that if an edge e that is incident with u_1 is contracted, the composite vertex that results from identifying the ends of e will retain the label u_1.

Suppose that T is a star with center vertex u_2. Let P be a u_1u_2-path in C' with $|V(P)| \geq (|V(C')| + 2)/2$. Then $|V(P)| \geq |V_2'| + 1 = 2n + 2$. Now the vertices of P alternate between V_1 and V_2'. By contracting all of the even-numbered edges in P, we get a path with at least n edges in which every interior vertex is adjacent to both u_1 and u_2. Thus the union of P, T, and the edges between u_1 and P contains $\Gamma_4(n)$ as a minor.

We may now suppose that T is an l-edge path P and that its vertices are $1, 2, \ldots, l+1$, listed as they appear on the path. Note that C' is divided by these vertices into $l+1$ two-edge paths $P_1, P_2, \ldots, P_{l+1}$, each having both ends in P and each vertex in P is an end of exactly two such paths. Apply Lemma 8.3 to the graph with vertex set $V(P)$ and edge set $\{x_iy_i : x_i, y_i$ are the ends of P_i and $i \in \{1, 2, \ldots, l\}\}$, where we assume that P_{l+1} contains the universal vertex u_1. Recall that $l = f_{8.3}(3, n+2, n+1, n+1)$.

Without loss of generality, we may assume that $x_1y_1, x_2y_2, \dots, x_ty_t$ satisfy one of the outcomes (ii)–(iv) in Lemma 8.3. If (ii) holds, then, as $t = t_2 = n+2$, the union of P, $P_1, P_2, \dots, P_t$, and edges from u_1 to each P_i with i in $\{1, 2, \dots, t\}$ has $\Gamma_3(n)$ as a minor. If (iii) or (iv) of Lemma 8.3 holds, then, as $t_3 = t_4 = n + 1$, the union of $P[x_1, x_t]$, $P[y_1, y_t]$, $P_1, P_2, \dots, P_t$, and an edge from u_1 to each of P_1 and P_t has $\Gamma_1(n)$ as a minor. □

We are now ready to prove the main result of this section.

Proof of Theorem 1.2. Let $d = f_{8.4}(n)/2$ and $m = f_{8.3}(2, n+2, 2, 3n+3)$. In addition, let $l_2 = f_{8.2}(m, 2n, n+1)$ and $l_1 = f_{8.1}(d-2, l_2+2)$. We shall prove that the theorem holds when we take $f_{1.2}(n) = 2f_{8.1}(d-2, l_1)$. Specifically, we show that, for a graph G having a set Z with at least $2f_{8.1}(d-2, l_1)$ edges such that Z is the intersection of a cycle and a bond, G has, as a minor, one of $\Gamma_1(n), \Gamma_2(n), \Gamma_3(n)$, or $\Gamma_4(n)$.

Suppose Z is the intersection of a cycle C and a bond C^*. By deleting the edges of $C^* - C$, we obtain a minor of G is which C is a cycle and $C \cap C^*$ is a bond D. For notational convenience, we shall relabel this minor of G as G. Let (V_1, V_2) be a partition of $V(G)$ such that $D = \delta_G(V_1)$. As in the proof of the preceding lemma, we also assume that $V_1 \subseteq V(C)$ and that no edge of C is contained in $G[V_1]$. It follows that every vertex of V_1 meets exactly two edges of C and so meets exactly two edges of D. Thus $|V_1| = |D|/2 \geq f_{8.1}(d-2, l_1)$.

Suppose $G[V_1]$ has a vertex u of degree at least $d-1$. Delete all the edges of $G[V_1]$ that are not incident with u letting G' be the resulting graph. Let V_0 be u together with its neighbors in V_1 and let $D_0 = \delta_{G'}(G'[V_0])$. Then $|V_0| \geq d$, so $|D_0| \geq 2d$. We can now apply Lemma 8.4 to the bond D_0 of G'. As $G'[V_0]$ has u as a universal vertex and $|D_0| \geq f_{8.4}(n)$, the theorem follows in this case. We may now assume that the maximum degree of $G[V_1]$ is at most $d-2$. Thus, by Lemma 8.1, $G[V_1]$ has a path P_1 of length l_1. By deleting all the edges of $G[V_1]$ that are not in P_1, we obtain a subgraph G_0 of G in which $\delta_{G_0}(V(P_1))$ is a bond D' contained in C. Note that $V(P_1) \subseteq V(C)$, that $E(C) \cap E(P_1) = \emptyset$, and that P_1 is an induced path in G_0. By contracting edges in $G_0 - V(P_1)$ as in the first paragraph of the last proof, we can obtain a minor G' such that D' forms a cycle C' that spans $M(G')$ and if (V_1', V_2') is the partition of G' with $V_1' = V(P_1)$, then $V_2' \subseteq V(C')$ and $E(C') \cap G'[V_2'] = \emptyset$. As every vertex of each of V_1' and V_2' meets exactly two edges of D', it follows that $|V_1'| = |V_2'|$. Let u_1, v_1 be the ends of P_1.

If $G'[V_2']$ has a vertex of degree at least $d-1$, then, as in the previous paragraph, the theorem holds by Lemma 8.4. Thus we may assume that the maximum degree of $G'[V_2']$ is at most $d-2$. Let p label u_1u_2, one of the two edges in D' that are incident with u_1. Since $|V_2'| = |V_1'| = l_1 + 1 > f_{8.1}(d-2, l_2+2)$, we deduce from Lemma 8.1 that $G'[V_2']$ has a path P_2 of length $l_2 + 3$ having u_2 as an end. Let v_2 be the other end of this path. Let G'' be obtained by deleting all the edges of $G'[V_2']$ that are not in P_2, and, for each w in $V_2' - V(P_2)$, contracting exactly one of the two edges of D' that are incident with w. Note that, after these contractions, the remaining edges of C' form a cycle C'' that spans $M(G'')$. Let P be the union of P_1, P_2, and p. Then P is a path with vertex set $V(G'')$ and ends v_1 and v_2. Moreover, $(E(C''\backslash p), E(P))$ is a partition of $E(G'')$. We also observe that no edge of C'' has both ends in P_2 but C'' may have some edges with both ends in P_1.

We shall say that two edges x_1x_2 and y_1y_2 of $C''\backslash p$ *cross* if $P[x_1, x_2]$ and $P[y_1, y_2]$ have at least one common edge yet neither is a subpath of the other.

8.6.1. *If w_1w_2 is an edge e having one end in $V(P_1) - \{u_1, v_1\}$ and the other in $V(P_2) - \{u_2, v_2\}$, then e crosses some edge of $C''\backslash p$.*

Let Q be the component of $C''\backslash\{e, p\}$ that contains v_1. Then Q is a path with ends u_i and w_j for some i and j in $\{1, 2\}$. Note that $Q[u_i, v_1]$ is a path from the interior of $P[w_1, w_2]$ to v_1, and it contains neither w_1 nor w_2. Thus its last edge leaving $P[w_1, w_2]$ crosses e. Hence 8.6.1 holds.

Let F be the set of edges joining a vertex in $V(P_1) - \{u_1, v_1\}$ to a vertex in $V(P_2) - \{u_2, v_2\}$, and let F' be the set of edges of $C'' - p$ that cross at least one edge in F. We now construct an auxiliary simple graph H that has $F \cup F'$ as its vertex set. Two vertices in H are adjacent if and only if the corresponding edges of G'' cross. By 8.6.1, H has no isolated vertices. Now each vertex of $V(P_2) - \{u_2, v_2\}$ meets exactly two edges of $C''\backslash p$. Since each of these edges has its other end in $V(P_1)$ and at most four of these edges have u_1 or v_1 as an end, $|F| \geq 2(|V(P_2)| - 2) - 4 = 2|V(P_2)| - 8$. Thus $|E(H)| \geq |V(H)|/2 = |F \cup F'|/2 \geq |F|/2 \geq |V(P_2)| - 4 = l_2 = f_{8.2}(m, 2n, n+1)$. By Lemma 8.2, $F \cup F'$ has a subset F_0 such that $H[F_0]$ is mK_2, $K_{1,2n}$, or K_{n+1}. In the third case, F_0 corresponds to a set of $n+1$ edges of $C''\backslash p$ every two of which cross. Beginning at v_1, order the vertices of P as they occur along the path. Then the edges of F_0 can be ordered $x_1y_1, x_2y_2, \ldots, x_{n+1}y_{n+1}$ so that $x_1 < x_2 < \cdots < x_{n+1} < y_1 < y_2 < \cdots < y_{n+1}$. Then the union of $P[x_1, x_{n+1}], P[y_{n+1}, y_1]$, and the edges of F_0 form a subdivided ladder. Since the path $P[x_{n+1}, y_1]$ is internally disjoint from this ladder, when we add this path to the ladder and suppress the degree-two vertices, we get $\Gamma_1(n)$.

Next suppose that $H[F_0]$ is $K_{1,2n}$. Then, in G'', the edges of F_0 consist of a $2n$-edge matching together with a single edge e that crosses all of the edges in the matching. If xy and $x'y'$ are in the matching, then $\{x, y\} \cap \{x', y'\} = \emptyset$. Moreover, either $P[x, y]$ and $P[x', y']$ are disjoint, or one of $P[x, y]$ and $P[x', y']$ is a subpath of the other. As e crosses all the edges in the matching, it follows that the matching contains n edges, $x_1y_1, x_2y_2, \ldots, x_ny_n$, such that, after interchanging x_i and y_i where necessary, $x_1 < x_2 < \cdots < x_n < y_n < y_{n-1} < \cdots < y_1$. Moreover, we may assume that $e = x_0y_0$ where $x_0 < x_1$ and $x_n < y_0 < y_n$. Then the union of $P[x_1, x_n], P[y_1, y_n]$, and $\{x_1y_1, x_2y_2, \ldots, x_ny_n\}$ is a subdivided ladder. Taking the union of it with

$x_0y_0, P[x_n, y_0], P[y_0, y_n]$, and $P[x_0, x_1]$ gives a graph that has $\Gamma_1(n)$ as a minor.

It remains to consider the case when $H[F_0]$ is mK_2. In that case, the m edges of $H[F_0]$ correspond to m disjoint pairs $\{e_i, e'_i\}$ of edges of $C''\backslash p$ in G'' where the edges in each pair cross each other but edges from different pairs do not cross. Now we apply Lemma 8.3 to the subgraph of G'' with edge set $\{e_1, e_2, \dots, e_m\}$, where their end vertices are ordered as they occur on P. We know that the edges in $\{e_1, e_2, \dots, e_m\}$ form a matching and no two of them cross. Since $m = f_{8.3}(2, n+2, 2, 3n+3)$, it follows by Lemma 8.3 that either (ii) of the lemma holds with $t = n+2$, or (iv) of the lemma holds with $t = 3n+3$.

Consider the first possibility. Then at most one e_i has one end in P_1 and the other end in P_2. Thus, since no e_i has both ends in P_2, it follows that $n+1$ of the edges, say $e_1, e_2, \dots, e_{n+1}$, have both ends in P_1. Then each of $e_1, e_2, \dots, e_{n+1}$ is in F'. The definition of F' guarantees that, for each i in $\{1, 2, \dots, n+1\}$, there is an edge g_i of F that crosses e_i. Then, by contracting P_2 to a single vertex, we see that the union of P and $\{e_1, e_2, \dots, e_{n+1}, g_1, g_2, \dots, g_{n+1}\}$ contains $\Gamma_3(n)$ as a minor.

Now suppose (iv) of Lemma 8.3 occurs. Let $e_i = x_iy_i$ for all i in $\{1, 2, \dots, 3n+3\}$ and suppose that $x_1 < x_2 < \cdots < x_{3n+3} < y_{3n+3} < \cdots < y_2 < y_1$. For each i in $\{2, 3, \dots, 3n+2\}$, there are three possibilities for the position of e'_i:

(a) both ends of e'_i are in $P[x_{i-1}, x_{i+1}]$;
(b) both ends of e'_i are in $P[y_{i+1}, y_{i-1}]$;
(c) e'_i joins a vertex in $P[x_{i-1}, x_{i+1}]$ to a vertex in $P[y_{i+1}, y_{i-1}]$.

Put e'_i into I_a, I_b, or I_c depending on which of (a), (b), or (c) holds. Then I_a, I_b, and I_c are disjoint sets whose union is $\{2, 3, \dots, 3n+2\}$. Hence one of I_a, I_b, or I_c has at least $n+1$ members. If $|I_a| \geq n+1$, then the union of P and the edges $e_ie'_i$ with i in I_a contains $\Gamma_3(n)$ as a minor. By symmetry, when $|I_b| \geq n+1$, we also get $\Gamma_3(n)$ as a minor. Thus we may assume that $|I_c| \geq n+1$. By relabelling, we may assume that $\{2, 3, \dots, n+2\} \subseteq I_c$. Then, by taking the union of $P[x_1, x_{n+3}]$, $P[y_{n+3}, y_1]$, and $\{e_i, e'_i : i \in \{2, 3, \dots, n+2\}\}$, we get a graph that contains $\Gamma_2(n)$ as a minor. □

A list of unavoidable minors for the class of binary matroids that contain a big set that is the intersection of a circuit and a cocircuit must include the cycle matroids of $\Gamma_1(n), \Gamma_2(n), \Gamma_3(n)$, and $\Gamma_4(n)$. In addition, the list should include the tipless binary spike of rank n, that is, the vector matroid of the binary matrix $[I_n | J_n - I_n]$, where J_n is the $n \times n$ matrix of all ones. We were unable to prove that this list is complete, nor could we find a counterexample.

Conjecture 8.7. *Let n be an integer exceeding two. There is an integer $f_{8.7}(n)$ such that if a binary matroid M contains a set that is the intersection of a circuit and a cocircuit and has at least $f_{8.7}(n)$ elements, then M has,* *as a minor, one of* $M(\Gamma_1(n)), M(\Gamma_2(n)), M(\Gamma_3(n)), M(\Gamma_4(n))$, *or the vector matroid of the binary matrix* $[I_n|J_n - I_n]$.

The next result shows that if the list in the conjecture is incomplete, any other matroids on the list can be assumed to be 3-connected.

Theorem 8.8. *Let* M *be a binary matroid that contains a set* Z *that is the intersection of a circuit and a cocircuit. For each integer* n *exceeding two, there is a positive integer* $f_{8.8}(n)$ *such that if* $|Z| \geq f_{8.8}(n)$, *then*

(i) M *has a* 3*-connected minor* M' *that contains a set* C' *that has at least* n *elements and is both a spanning circuit and a cospanning cocircuit; or*

(ii) M *has* $M(\Gamma_2(n))$ *or* $M(\Gamma_3(n))$ *as a minor.*

Proof. Assume that $|Z| \geq nf_{8.1}(2^n, n^2)$. By Lemma 7.1, M has a minor M_0 such that Z is a spanning circuit of both M_0 and M_0^*. For notational convenience, we relabel M_0 as M. We may assume that M is not 3-connected, otherwise (i) certainly holds. We now apply a result of Cunningham and Edmonds (see Cunningham [1]) following the treatment given in [6, Section 8.3]. By that result, M has a *canonical tree decomposition.* This consists of a tree T whose vertex set is labelled by a set $\{M_1, M_2, \dots, M_k\}$ of matroids and whose edge set is $\{e_1, e_2, \dots, e_{k-1}\}$, say, such that

(i) each M_i is a circuit, a cocircuit, or a 3-connected matroid; and no two adjacent vertices of T are both labelled by circuits, or are both labelled by cocircuits;

(ii) if M_{j_1} and M_{j_2} are joined by an edge e_i of T, then $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$, and $\{e_i\}$ is not a component of M_{j_1} or M_{j_2};

(iii) if M_{j_1} and M_{j_2} are non-adjacent, then $E(M_{j_1}) \cap E(M_{j_2}) = \emptyset$;

(iv) $E(M) = (E(M_1) \cup E(M_2) \cup \cdots \cup E(M_k)) \cup \{e_1, e_2, \dots, e_{k-1}\}$;

(v) $|E(M_i)| \geq 3$ for all i; and

(vi) M is the matroid that labels the single vertex of $T/e_1, e_2, \dots, e_{k-1}$ where, when an edge is contracted, the resulting composite vertex is labelled by the 2-sum of the two vertices that had labelled its ends.

As M has Z as a spanning circuit, whenever M is written as a 2-sum of N_1 and N_2 with respect to the basepoint p, either

(a) $Z = (C_1 - p) \cup (C_2 - p)$ where C_i is a spanning circuit of N_i containing p; or

(b) Z is a spanning circuit of one of N_1 and N_2 avoiding p while the other N_i has rank one.

Because Z is also a spanning circuit of M^*, the latter cannot occur. Thus (a) holds. Hence no vertex of the tree T labels a cocircuit otherwise M has a circuit that is properly contained in Z. Because T is also the canonical tree decomposition for M^*, where each vertex label M_i is replaced by M_i^*, we deduce that no vertex of T labels a circuit. Hence every vertex of T labels a 3-connected binary matroid, which must have at least six elements. If some M_i has rank at least $n-1$, then the theorem holds. Thus we may assume that $r(M_i) < n-1$ for all i. But $r(M) = \sum_{i=1}^{k} r(M_i) - (k-1)$. Hence $r(M) < k(n-1)$.

We also know that $r(M) \geq |Z|-1 \geq nf_{8.1}(2^n, n^2)-1 \geq (n-1)f_{8.1}(2^n, n^2)$. Thus

$$k(n-1) > r(M) > (n-1)f_{8.1}(2^n, n^2).$$

Hence $k > f_{8.1}(2^n, n^2)$, that is, T has at least $f_{8.1}(2^n, n^2)$ vertices. Therefore, by Lemma 8.1, either T has a vertex of degree exceeding 2^n, or T contains a path with at least n^2 vertices. The first possibility does not arise since a simple binary matroid with more than 2^n elements has rank more than n, and we know $r(M_i) < n-1$ for all i. Thus we may assume that T contains a path P with vertex set $\{N_1, N_2, \dots, N_{n^2}\}$ where $E(N_i) \cap E(N_{i+1}) = \{f_i\}$ for all i in $\{1, 2, \dots, n^2-1\}$. Let f_0 be an element of $E(N_1) - f_1$, and let f_{n^2} be an element of $E(N_{n^2}) - f_{n^2-1}$.

Now P is the canonical tree decomposition for a minor N of M. For each i in $\{1, 2, \dots, n^2\}$, since N_i is 3-connected and binary, it follows by Tutte's Wheels-and-Whirls Theorem [10] that N_i has an $M(K_4)$-minor. Moreover, by a result of Seymour [9], N_i has an $M(K_4)$-minor N_i' whose ground set contains $\{f_{i-1}, f_i\}$. Thus N has a minor N' for which P is the canonical tree decomposition where we replace the vertex label N_i by N_i'. We color N_i' black if it has a triangle containing $\{f_{i-1}, f_i\}$ and color N_i' white otherwise. Since P has at least n^2 vertices, it certainly has at least n black vertices or at least n white vertices. We can eliminate a vertex N_i' of P by contracting elements of it until f_{i-1} and f_i are parallel. We then delete the remaining elements of N_i' except f_i and relabel f_{i-1} in N_{i-1}' by f_i. By this process, we get a minor of M for which the canonical tree decomposition is an n-vertex path in which all vertices are the same color. We conclude that M has $M(\Gamma_2(n))$ or $M(\Gamma_3(n))$ as a minor. $\square$

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