A CHAIN THEOREM FOR INTERNALLY 4-CONNECTED BINARY MATROIDS

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ABSTRACT. Let M be a matroid. When M is 3-connected, Tutte's Wheels-and-Whirls Theorem proves that M has a 3-connected proper minor N with $|E(M) - E(N)| = 1$ unless M is a wheel or a whirl. This paper establishes a corresponding result for internally 4-connected binary matroids. In particular, we prove that if M is such a matroid, then M has an internally 4-connected proper minor N with $|E(M) - E(N)| \leq 3$ unless M or its dual is the cycle matroid of a planar or Möbius quartic ladder, or a 16-element variant of such a planar ladder.

1. INTRODUCTION

When dealing with matroid connectivity, it is often useful in inductive arguments to be able to remove a small set of elements from a matroid M to obtain a minor N that maintains the connectivity of M. Results that guarantee the existence of such removal sets are referred to as *chain theorems*. Tutte [16] proved that, when M is 2-connected, if $e \in E(M)$, then $M\backslash e$ or M/e is 2-connected. More significantly, when M is 3-connected, Tutte [16] proved the following result, his Wheels-and-Whirls Theorem.

Theorem 1.1. *Let M be a 3-connected matroid. Then M has a proper 3-connected minor N such that $|E(M)| - |E(N)| = 1$ unless $r(M) \geq 3$ and M is a wheel or a whirl.*

This result has proved to be such a useful tool for 3-connected matroids that it is natural to seek a corresponding result for 4-connected matroids. Since higher connectivity for matroids may be unfamiliar, we now define it. Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. For a positive integer k, a subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation* if $|X|, |E - X| \geq k$. A matroid having no k-separations for all $k < n$ is *n-connected*.

In an $n \times n$ grid graph with n large, identify the top and bottom sides and the left and right sides of the grid to get a 4-connected graph embedded on the torus in which every face is a 4-cycle and every vertex has degree 4. Using this graph, it is not difficult to see that, for all positive integers m,

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there is a 4-connected matroid M having no proper 4-connected minor N with $|E(M)| - |E(N)| \leq m$. Nevertheless, chain theorems have been proved for certain classes of 3-connected matroids which are partially 4-connected. More precisely, instead of eliminating all 3-separations as one does in a 4-connected matroid, one can allow certain constrained 3-separations. Let k be an integer exceeding one. A matroid M is $(4,k)$-*connected* if M is 3-connected and, whenever (X,Y) is a 3-separating partition of $E(M)$, either $|X| \leq k$ or $|Y| \leq k$. Hall [6] called such a matroid 4-*connected up to separators of size* k. She proved a chain theorem for $(4,5)$-connected matroids. Matroids that are $(4,4)$-connected have also been called *weakly* 4-*connected*. The next result is a chain theorem for such matroids proved by Geelen and Zhou [4]. It identifies certain exceptional matroids. For $n \geq 3$, a *planar cubic ladder* is a graph with vertex set $\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$ that consists of two disjoint cycles, $\{u_1u_2, u_2u_3, \ldots, u_nu_1\}$ and $\{v_1v_2, v_2v_3, \ldots, v_nv_1\}$, and a matching $\{u_1v_1, u_2v_2, \ldots, u_nv_n\}$; a *Möbius cubic ladder* has the same vertex set and consists of a Hamiltonian cycle $\{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_nv_1, v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nu_1\}$ and a matching $\{u_1v_1, u_2v_2, \ldots, u_nv_n\}$. In particular, the planar cubic ladder with $n = 4$ coincides with the graph of the *cube*. Note that the planar dual of the cube is the *octahedron*, $K_{2,2,2}$. A *trident* is a 12-element rank-6 matroid whose ground set is the union of three disjoint 4-element 3-separating sets of rank 3. We remark that this is quite different from what is defined as a 'trident' in Oxley, Semple, and Whittle [13].

Theorem 1.2. *Let M be a weakly 4-connected matroid with $|E(M)| \geq 7$. Then M has a weakly 4-connected proper minor N with $|E(M)|-|E(N)| \leq 2$ unless M is the cycle matroid of a planar or Möbius cubic ladder, or M is a trident.*

An *internally* 4-*connected matroid* is one that is $(4,3)$-connected. Geelen and Zhou [4, p.539] observed that:"For binary matroids, internal 4-connectivity is certainly the most natural variant of 4-connectivity and it would be particularly useful to have an inductive construction for this class." In a sequel to this paper, we give such an inductive construction. Most of the work towards obtaining such a construction appears in the current paper. Indeed, with a view to this sequel, several results in this paper derive more structural details than are needed to prove the main results of the current paper. These results involve quartic ladders. For $n \geq 3$, a *planar quartic ladder* is obtained from a planar cubic ladder by adding another matching $\{u_1v_n, u_2v_1, \ldots, u_nv_{n-1}\}$; a *Möbius quartic ladder* consists of a Hamiltonian cycle $\{v_1v_2, v_2v_3, \ldots, v_{2n-2}v_{2n-1}, v_{2n-1}v_1\}$ along with the set of edges $\{v_iv_{i+n-1}, v_iv_{i+n} : 1 \leq i \leq n\}$ where all subscripts are interpreted modulo $2n-1$. In particular, for $n = 3$, the planar and Möbius quartic ladders coincide with the octahedron and K_5, respectively. A *terrahawk* is the graph $\mathcal{T}$ that is obtained from the cube by adjoining one new vertex and adding edges from this vertex to each of the four vertices that bound some fixed

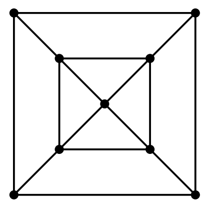

FIGURE 1. A terrahawk.

face of the cube (see Figure 1). Clearly $M^*(\mathcal{T}) \cong M(\mathcal{T})$ and $\mathcal{T}$ has both the cube and the octahedron as minors.

Theorem 1.3. *Let M be an internally 4-connected binary matroid with $|E(M)| \geq 7$. Then M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 3$ unless M or its dual is the cycle matroid of a planar or Möbius quartic ladder, or a terrahawk. In the exceptional cases, either M has an internally 4-connected minor N with $|E(M)| - |E(N)| = 4$; or M is the cycle matroid of an octahedron or a cube.*

We observe that, when M is the cycle matroid of an octahedron or a cube, M has an $M(K_4)$-minor but has no internally 4-connected proper minor N with $|E(M)| - |E(N)| < 6$. Because the proof of Theorem 1.3 is long, we outline its main steps in Section 3. In the next section, we give some basic definitions and results that will be needed in this proof.

Although our concern here is for a chain theorem, we should note significant related work involving splitter theorems. The aim of the latter results is to remove a small set of elements while retaining not only the connectivity but also a copy of some fixed minor. For 3-connected matroids, Seymour [15] generalized the Wheels-and-Whirls Theorem by proving that if N is a 3-connected minor of a 3-connected matroid M such that if N is a wheel or a whirl, it is the largest wheel or whirl minor of M, then there is a sequence $M_0, M_1, \dots, M_k$ of 3-connected matroids such that $M_0 = M$ and $M_k \cong N$, while each M_{i+1} is a minor of M_i with $|E(M_i) - E(M_{i+1})| = 1$. Johnson and Thomas [7] considered the problem of trying to find a splitter theorem for internally 4-connected graphs. There are immediate difficulties with this since, for example, the cubic planar ladder with $2n$ vertices is a minor of the quartic planar ladder with $2n$ vertices. Both these graphs are internally 4-connected, but there is no internally 4-connected graph that lies strictly between them in the minor order. Hence, even for graphic matroids, we can be forced to remove arbitrarily many elements to recover internal 4-connectivity while maintaining a copy of a specified minor. By controlling the presence of ladders and double wheels, Johnson and Thomas [7] were able to prove a theorem of this type for internally 4-connected graphs. In their result, each intermediate graph is obtained from its predecessor by removing, via deletion or contraction, at most two edges, and each such intermediate graph is $(4, 4)$-connected satisfying some additional constraints.

Geelen and Zhou [3, 5] proved two analogues of this theorem for internally 4-connected binary matroids, the second strengthening the first.

The specialization of our main theorem to graphs is also new and we end this section by stating this corollary. For $n \geq 2$, a *cubic planar biwheel* is a planar graph with vertex set $\{v_1, v_2, \ldots, v_{2n}, u, w\}$ and edge set $\{v_1v_2, v_2, v_3, \ldots, v_{2n}v_1\} \cup \{uv_{2i-1}, wv_{2i} : 1 \leq i \leq n\}$. Its dual is a quartic planar ladder.

Corollary 1.4. *Let G be an internally 4-connected graph with $|E(G)| \geq 7$. Then G has a proper internally 4-connected minor H with $|E(G)|-|E(H)| \leq 3$ unless G is K_5, a terrahawk, a planar or Möbius quartic ladder, or a cubic planar biwheel. In the exceptional cases, either G has an internally 4-connected minor H with $|E(G)| - |E(H)| = 4$; or G is an octahedron or a cube.*

2. Preliminaries

The matroid terminology used here will follow Oxley [10] except that the simplification and cosimplification of a matroid M will be denoted by $\mathrm{si}(M)$ and $\mathrm{co}(M)$, respectively. A *quad* in a matroid is a 4-element set that is both a circuit and a cocircuit. The property that a circuit and a cocircuit in a matroid cannot have exactly one common element will be referred to as *orthogonality*.

In a matroid M, a k-separating set X, or a k-separating partition $(X, E-X)$, or a k-separation $(X, E-X)$ is *exact* if $\lambda_M(X) = k-1$. A k-separation $(X, E-X)$ is *minimal* if $|X| = k$ or $|E-X| = k$. It is well known (see, for example, [10, Corollary 8.1.11]) that if M is k-connected having $(X, E-X)$ as a k-separation with $|X| = k$, then X is a circuit or a cocircuit of M.

A set X in a matroid M is *fully closed* if it is closed in both M and M^*, that is, $\mathrm{cl}(X) = X$ and $\mathrm{cl}^*(X) = X$. Thus the full closure of X is the intersection of all fully closed sets that contain X. One way to obtain $\mathrm{fcl}(X)$ is to take $\mathrm{cl}(X)$, and then $\mathrm{cl}^*(\mathrm{cl}(X))$ and so on until neither the closure nor coclosure operator adds any new elements of M. The full closure operator enables one to define a natural equivalence on exactly 3-separating partitions as follows. Two exactly 3-separating partitions (A_1, B_1) and (A_2, B_2) of a 3-connected matroid M are *equivalent*, written $(A_1, B_1) \cong (A_2, B_2)$, if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$. If $\mathrm{fcl}(A_1) = E(M)$, then B_1 has an ordering $(b_1, b_2, \ldots, b_n)$ such that $\{b_1, b_2, \ldots, b_k\}$ is 3-separating for all k in $\{1, 2, \ldots, n\}$. We call such an ordering a *sequential ordering* of B_1 and say that the set B_1 is *sequential*. Similarly, A_1 is sequential if $\mathrm{fcl}(B_1) = E(M)$. We say (A_1, B_1) is *sequential* if A_1 or B_1 is sequential. A *sequentially 4-connected* matroid is a 3-connected matroid in which every 3-separation is sequential. A 3-connected matroid M is $(4, k, S)$-*connected* if M is both $(4, k)$-connected and sequentially 4-connected.

The following elementary lemma [12, Lemma 3.1] will be in repeated use throughout the paper.

Lemma 2.1. *For a positive integer k, let (A, B) be an exactly k-separating partition in a matroid M.*

(i) *For e in $E(M)$, the partition $(A \cup e, B - e)$ is k-separating if and only if $e \in \mathrm{cl}(A)$ or $e \in \mathrm{cl}^*(A)$.*
(ii) *For e in B, the partition $(A \cup e, B - e)$ is exactly k-separating if and only if e is in exactly one of $\mathrm{cl}(A) \cap \mathrm{cl}(B - e)$ and $\mathrm{cl}^*(A) \cap \mathrm{cl}^*(B - e)$.*
(iii) *The elements of $\mathrm{fcl}(A) - A$ can be ordered $b_1, b_2, \dots, b_n$ so that $A \cup \{b_1, b_2, \dots, b_i\}$ is k-separating for all i in $\{1, 2, \dots, n\}$.*

Next we state a well-known lemma that specifies precisely when a single element z of a matroid M blocks a k-separating partition of $M \backslash z$ from extending to a k-separating partition of M. This result and its dual underlie numerous arguments in this paper.

Lemma 2.2. *In a matroid M with an element z, let (A, B) be an exactly k-separating partition of $M \backslash z$. Then both $\lambda_M(A \cup z)$ and $\lambda_M(B \cup z)$ exceed $k - 1$ if and only if $z \in \mathrm{cl}^*(A) \cap \mathrm{cl}^*(B)$.*

A subset S of a 3-connected matroid M is a *fan* in M if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \dots, s_n)$ of S such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \dots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads beginning with either. We call $(s_1, s_2, \dots, s_n)$ a *fan ordering* of S. If $n \geq 4$, then s_1 and s_n, which are the only elements of S that are not in both a triangle and a triad contained in S, are the *ends* of the fan.

The following basic property of maximal fans [11, Theorem 1.6] will be used frequently in the paper without explicit reference.

Lemma 2.3. *Let M be a 3-connected matroid other than a wheel or a whirl. Let F be a maximal fan in M having at least four elements. If x is an end of F that is in a triangle of M, then $M \backslash x$ is 3-connected.*

The connectivity function λ_M of a matroid M has a number of attractive properties. For example, $\lambda_M(X) = \lambda_M(E - X)$. Moreover, the connectivity functions of M and its dual M^* are equal. To see this, it suffices to note the easily verified fact that

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

We shall often abbreviate λ_M as λ.

One of the most useful features of the connectivity function of M is that it is submodular, that is, for all $X, Y \subseteq E(M)$,

$$\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y).$$

This means that if X and Y are k-separating, and one of $X \cap Y$ or $X \cup Y$ is not $(k-1)$-separating, then the other must be k-separating. The next lemma specializes this fact.

Lemma 2.4. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If* $|X \cap Y| \geq 2$*, then* $X \cup Y$ *is* 3*-separating.*
(ii) *If* $|E(M) - (X \cup Y)| \geq 2$*, then* $X \cap Y$ *is* 3*-separating.*

The following elementary property of internally 4-connected matroids will be used repeatedly. The first two parts are in [13, Lemma 6.1], while the third part follows immediately from the first two.

Lemma 2.5. *Let* M *be an internally* 4*-connected matroid with* $|E(M)| \geq 8$*.*
(i) *If* e *is an element of* M *that is not in a triad, then* $M\backslash e$ *is* 3*-connected.*
(ii) *Every triad of* M *avoids every triangle of* M*.*
(iii) *If* e *is an element of* M *that is in a triangle, then* $M\backslash e$ *is* 3*-connected.*

Geelen and Whittle [2, Theorem 1.2] proved a chain theorem for sequentially 4-connected matroids. They used the following result [2, Theorem 5.1] in the proof of that theorem. We shall use it here to prove our main theorem when M is 4-connected.

Theorem 2.6. *Let* M *be a* 4*-connected matroid. Then* M *has an element* x *such that* $M\backslash x$ *or* M/x *is sequentially* 4*-connected.*

A 3-separation (X, Y) of a 3-connected matroid M is a $(4, 3)$*-violator* if $|X|, |Y| \geq 4$. Evidently M is internally 4-connected if and only if it has no $(4, 3)$-violators.

Theorem 2.7. *Let* M *be a binary* 4*-connected matroid. Then* M *has an element* x *such that* $M\backslash x$ *or* M/x *is internally* 4*-connected.*

Proof. By Theorem 2.6 and duality, we may assume that M has an element x such that $M\backslash x$ is sequentially 4-connected. If $M\backslash x$ is not internally 4-connected, then it has a 3-separation (X, Y) with $|X|, |Y| \geq 4$. Thus $|E(M)| \geq 9$. Without loss of generality, X has a sequential ordering $(x_1, x_2, \ldots, x_n)$. Since M is 4-connected, it has no triangles. Thus $\{x_1, x_2, x_3\}$ is a triad of $M\backslash x$. Since M is binary, $\{x_1, x_2, x_3, x_4\}$ is not a circuit of $M\backslash x$ and $x_4 \notin \mathrm{cl}^*{}_{M\backslash x}(\{x_1, x_2, x_3\})$. Thus x_4 is in neither the closure nor the coclosure of $\{x_1, x_2, x_3\}$ in $M\backslash x$; a contradiction. □

The next two lemmas establish some basic properties of binary 3-connected matroids. One well-known such property is that every two distinct elements in such a matroid are contained in at most one triangle and at most one triad.

Lemma 2.8. *Let* z *be an element of a binary internally* 4*-connected matroid* M *such that* $M\backslash z$ *is* 3*-connected. Let* (X, Y) *be a* $(4, 3)$*-violator of* $M\backslash z$*. If* Y *is sequential, then* Y *contains a* 4*-element fan of* $M\backslash z$ *that is the union of a triangle* T *and a triad* T^**. Moreover,* T *is a triangle of* M *and* $T^* \cup z$ *is a cocircuit of* M*.*

Proof. Let $(y_1, y_2, \ldots, y_n)$ be a sequential ordering of Y. Then (y_1, y_2, y_3, y_4) is a sequential ordering of a 3-separating set of $M\backslash z$, so $\{y_1, y_2, y_3\}$ is a triangle or a triad of $M\backslash z$. Since M is binary, we deduce that $\{y_1, y_2, y_3, y_4\}$ is a fan of $M\backslash z$. As M is internally 4-connected, the lemma follows. □

Lemma 2.9. *Let (X, Y) be a 3-separation of a binary 3-connected matroid M. If X is sequential and $|X| \leq 5$, then X is a fan.*

Proof. We have $r(X) + r^*(X) - |X| = 2$, so $r(X) + r^*(X) \leq 7$. Hence, by duality, we may assume that $r(X) \leq 3$. Thus $M|X$ is a restriction of the Fano matroid. Since M is sequential, it follows easily that M is a fan. □

In [4], Geelen and Zhou introduced a structure in a matroid that they call a rotor. In [5], they introduced a slight variant on this structure that is defined as follows. Let M be an internally 4-connected matroid. A *quasi rotor* with *central triangle* $\{a, b, c\}$ is an 8-tuple $(a, b, c, d, e, T_a, T_c, Z)$ such that the following hold:

(i) $E(M) = \{a, b, c, d, e\} \cup T_a \cup T_c \cup Z$;
(ii) a, b, c, d, and e are distinct, and T_a, T_c, and $\{a, b, c\}$ are disjoint triangles with d in T_a and e in T_c;
(iii) $T_a \cup \{b, e\}$ and $T_c \cup \{b, d\}$ are 3-separating in $M\backslash a$ and $M\backslash c$, respectively; and
(iv) T_a and T_c are 2-separating in $M\backslash a, b$ and $M\backslash b, c$, respectively.

If, in addition to (i)–(iv), there is a proper non-empty subset A of Z such that $T_a \cup a \cup A$ is 3-separating in $M\backslash b$, then $(a, b, c, d, e, T_a, T_c, A, Z - A)$ is a *rotor* with *central triangle* $\{a, b, c\}$.

Because we are concerned here exclusively with internally 4-connected binary matroids, when quasi rotors arise in such matroids, we can be more explicit about their structure. The next lemma is obtained by a straightforward specialization of [4, 3.7.1, 3.7.2]

Lemma 2.10. *Let $(a, b, c, d, e, T_a, T_c, Z)$ be a quasi rotor with central triangle $\{a, b, c\}$ in an internally 4-connected binary matroid M. Then*

(i) *$\{b, d, e\}$ is a triangle of M;*
(ii) *M has a 4-cocircuit containing $\{a, b, d\}$ and one element of $T_a - d$; and*
(iii) *M has a 4-cocircuit containing $\{b, c, e\}$ and one element of $T_c - e$.*

Proof. As a is in a triangle of M, Lemma 2.5(iii) implies that $M\backslash a$ is 3-connected. Since T_a is 2-separating in $M\backslash a, b$, the set $T_a \cup b$ is 3-separating in $M\backslash a$; so too is $T_a \cup \{b, e\}$. Thus $e \in \mathrm{cl}_{M\backslash a}(T_a \cup b)$ or $e \in \mathrm{cl}^*{}_{M\backslash a}(T_a \cup b)$. But e is in the triangle T_c, so $e \notin \mathrm{cl}^*{}_{M\backslash a}(T_a \cup b)$. Hence $e \in \mathrm{cl}_{M\backslash a}(T_a \cup b)$. As M is binary, $e \notin \mathrm{cl}_{M\backslash a}(T_a)$. Thus $b \in \mathrm{cl}_{M\backslash a}(T_a \cup e)$. By symmetry, $b \in \mathrm{cl}_{M\backslash a}(T_c \cup d)$. Thus $T_a \cup \{b, e\}$ and $T_c \cup \{b, d\}$ both have rank 3. Their union has rank at least four, so their intersection, $\{b, d, e\}$, has rank at most 2. Thus $\{b, d, e\}$ is a triangle of M.

As M is binary and T_a is 2-separating in $M\backslash a, b$, there is a 2-cocircuit C^* of $M\backslash a, b$ contained in T_a. Since M is internally 4-connected, $C^* \cup \{a, b\}$ is a cocircuit of M. By orthogonality with the triangle $\{b, d, e\}$ of M, we deduce that $d \in C^*$. Hence M has a 4-cocircuit containing $\{a, b, d\}$ and one element of $T_a - d$. By symmetry, M has a 4-cocircuit containing $\{b, c, e\}$ and one element of $T_c - e$. □

The last lemma established that we can associate an additional triangle and two 4-cocircuits with a quasi rotor. The next lemma shows that, if we have four triangles and two 4-cocircuits as in a quasi rotor, then we do indeed have a quasi rotor.

Lemma 2.11. *Let M be an internally 4-connected binary matroid. Let $\{1, 2, \ldots, 9\}$ be a set of distinct elements of M such that $\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}$, and $\{3, 5, 7\}$ are triangles, while $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ are cocircuits. Then $(4, 5, 6, 3, 7, \{1, 2, 3\}, \{7, 8, 9\}, E(M) - \{1, 2, \ldots, 9\})$ is a quasi rotor in M.*

Proof. In $M\backslash 4$, the set $\{1, 2, 3, 5, 7\}$ is a 5-element fan, so it is 3-separating. By symmetry, $\{7, 8, 9, 5, 3\}$ is 3-separating in $M\backslash 6$. Also $M\backslash 4, 5$ has $\{2, 3\}$ as a cocircuit and so has $\{1, 2, 3\}$ as a 2-separating set. By symmetry, $M\backslash 5, 6$ has $\{7, 8, 9\}$ as a 2-separating set. Thus $(4, 5, 6, 3, 7, \{1, 2, 3\}, \{7, 8, 9\}, E(M) - \{1, 2, \ldots, 9\})$ is, indeed, a quasi rotor in M. □

In view of the last two lemmas, we shall modify Geelen and Zhou's terminology slightly. From now on, in an internally 4-connected binary matroid M, we shall call $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ a *quasi rotor* with *central triangle* $\{4, 5, 6\}$ if $\{1, 2, 3\}, \{4, 5, 6\}$, and $\{7, 8, 9\}$ are disjoint triangles in M such that $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ are cocircuits and $\{3, 5, 7\}$ is a triangle.

3. Outline

The proof of Theorem 1.3 is long, occupying the rest of the paper. In this section, we outline the strategy of the proof. Let M be an internally 4-connected binary matroid. In Section 4, we prove the theorem in the case when $|E(M)| \leq 12$. Hence we may assume that $|E(M)| \geq 13$. In Theorem 2.7, we proved that, when M is 4-connected, it has an internally 4-connected minor N with $|E(M)| - |E(N)| = 1$. Thus we may assume that M is not 4-connected. Then, by switching to the dual if necessary, we may assume that M has a triangle T. Theorem 5.1 proves that either T is the central triangle of a rotor, or T contains an element e such that $M\backslash e$ is $(4, 4, S)$-connected.

In Section 6, we prove that, when M contains a quasi rotor, and hence when M contains a rotor, M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 3$. This enables us to assume that every triangle of M contains an element e for which $M\backslash e$ is $(4, 4, S)$-connected.

In Section 7, we show that if M has a restriction isomorphic to $M(K_4)$, then M has an internally 4-connected proper minor that is obtained by removing at most two elements from M. This means that, in addition to

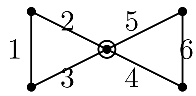

FIGURE 2. A bowtie.

assuming that M contains no quasi rotor, we may also assume that M has no $M(K_4)$-restriction.

Next we consider a triangle T in M and an element e of T for which $M\backslash e$ is $(4, 4, S)$-connected. Then $M\backslash e$ has a 4-element fan, $\{a, b, c, d\}$ say, where $\{a, b, c\}$ is a triangle and $\{b, c, d\}$ is a triad. As M has no 4-element fans, $\{b, c, d, e\}$ is a cocircuit of M. Moreover, by orthogonality, $T - e$ contains an element of $\{b, c, d\}$. By symmetry, there are two possibilities:

(i) T contains d;
(ii) T contains b.

In the first case, M contains a structure consisting of two disjoint triangles, T_1 and T_2, and a 4-cocircuit D^* that is contained in their union. We call such a structure a *bowtie* in M. By orthogonality, $|T_1 \cap D^*| = 2 = |T_2 \cap D^*|$. Although the matroid M we are dealing with need not be graphic, it will be convenient to use modified graph diagrams to keep track of some of the circuits and cocircuits in M. For example, we represent the bowtie $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ as in Figure 2. By convention, the cycles in the graph correspond to circuits of the matroid while a circled vertex indicates a known cocircuit of M.

In Section 8, we prove some general lemmas that enable us to build up additional structure surrounding a bowtie or other similar submatroid of M. In Section 9, we prove the main theorem in case (ii) above by showing that, in that case, if M has no bowties and has no $M(K_4)$-restriction, then M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 2$.

The results described above put us into the position where we may assume that M has a bowtie but M contains no quasi rotor and no $M(K_4)$-restriction. What we attempt to do next is to build up interlocking bowties. Formally, a *string of bowties* is a sequence $T_1, D_1^*, T_2, D_2^*, \dots, D_{n-1}^*, T_n$ such that $T_1, T_2, \dots, T_n$ are pairwise disjoint triangles, each D_i^* is a 4-cocircuit contained in $T_i \cup T_{i+1}$, and $|D_j^* \cap D_{j+1}^*| = 1$ for all j with $1 \leq j \leq n - 2$.

In Section 10, we prove that if M has a bowtie (T_1, T_2, D_1^*) that cannot be extended to a string $T_0, D_0^*, T_1, D_1^*, T_2$ of bowties, then M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 3$ unless M is isomorphic to the cycle matroid of a terrahawk.

It remains for us to consider the case when M has a bowtie and, from every bowtie, we can build a string of bowties where we can specify the direction of this building. In Section 11, we prove, in this case, that M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 3$ unless M is the cycle matroid of a planar or Möbius quartic ladder. That result essentially completes the proof of the main theorem. All that is left to show is that, when $|E(M)| \geq 13$ and M is the cycle matroid of a terrahawk or a planar or Möbius quartic ladder, M has an internally 4-connected minor N with $|E(M)| - |E(N)| = 4$.

Our initial hope had been to determine all internally 4-connected matroids M having no internally 4-connected minor N with $1 \leq |E(M) - E(N)| \leq 2$. To solve this problem, we would need to add to the list of exceptional matroids in Theorem 1.3, the cycle and bond matroids of the planar and Möbius cubic ladders. In addition, as the next lemma shows, we would need to add the cycle and bond matroids of the line graphs of these ladders.

Lemma 3.1. *Let G be an internally 4-connected cubic graph. Then the line graph, $L(G)$, of G is internally 4-connected. Moreover, there is no internally 4-connected proper minor N of $M(L(G))$ with $|E(M(L(G))) - E(N)| < 3$.*

Proof. If $G \cong K_4$, then $L(G)$ is isomorphic to the octahedron, and the lemma holds. Thus we may assume that $G \ncong K_4$. Then $|E(G)| \geq 8$ and, as G is internally 4-connected, it has no triangles. Clearly $L(G)$ is simple and 3-connected. If $M(L(G))$ is not internally 4-connected, it is straightforward to show that it has a $(4,3)$-violator (X, Y) such that the subgraphs of $L(G)$ induced by X and Y are connected and have exactly three common vertices. The edges of G corresponding to these vertices must form a triad in $M(G)$ but these edges do not meet at a common vertex of G. This implies that G is not internally 4-connected; a contradiction. We conclude that $M(L(G))$ is internally 4-connected.

Corresponding to every edge e of G, there is a bowtie in $L(G)$ induced by the vertices of $L(G)$ that correspond to e and its incident edges in G. It follows that every edge f of $L(G)$ is in two bowties that overlap in a triangle containing f. Now let N be an internally 4-connected proper minor of $M(L(G))$. We may assume that N is a minor of $M(L(G)))\backslash f$ for some edge f, otherwise N is a minor of some $M(L(G))/g$ and the latter has parallel edges, at least one of which must be deleted to produce N. Now $M(L(G)))\backslash f$ has two edge-disjoint 4-element fans, and no single-element deletion or contraction of it will destroy both of these fans. Hence $|E(M(L(G))) - E(N)| \geq 2$. □

The last lemma suggests that determining all the matroids from which exactly three elements need to be removed to recover internal 4-connectivity is likely to be complicated.

4. Small Matroids

In this section, we prove the main theorem for matroids with at most twelve elements.

Theorem 4.1. *Let M be an internally 4-connected binary matroid with at least seven and at most twelve elements. Then M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 3$ unless M or its dual is the cycle matroid of K_5, the cube or the octahedron. If M is isomorphic* *to* $M(K_5)$ *or* $M^*(K_5)$*, then* M *has an internally* 4*-connected minor* N *with* $|E(M)|-|E(N)| = 4$*. If* M *is the cycle matroid of the cube or the octahedron, then* M *has an* $M(K_4)$*-minor but has no internally* 4*-connected proper minor* N *with* $|E(M)| - |E(N)| < 6$*.*

The proof of this theorem will use the following result of Qin and Zhou [14, Theorem 1.3]. We could use results of Zhou [17] to give more details about the minimum number of elements we need to remove from M to recover internal 4-connectivity. But, since it is not needed for the proof of the theorem, we omit these details.

Theorem 4.2. *Let* M *be an internally* 4*-connected binary matroid with no minor isomorphic to any of* $M(K_{3,3})$*,* $M^*(K_{3,3})$*,* $M(K_5)$*, or* $M^*(K_5)$*. Then either* M *is isomorphic to the cycle matroid of a planar graph, or* M *is isomorphic to* F_7 *or* F_7^**.*

Proof of Theorem 4.1. Suppose first that M has a minor M' isomorphic to one of $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, or $M^*(K_5)$. If M' is a proper minor of M, then the theorem holds since $|E(M)| \leq 12$. If $M' = M$, then M has an $M(K_4)$-minor and again the theorem holds. We may now assume that M has no minor isomorphic to any of $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, or $M^*(K_5)$. Then, it follows by Theorem 4.2 that we may assume that M is isomorphic to the cycle matroid of a planar graph G.

Suppose that M has no minor isomorphic to the cycle matroid of the 5-spoked wheel, $\mathcal{W}_5$. Then, by a result of Oxley [9, p.297], G is isomorphic to the octahedron, $K_{2,2,2}$, or its dual, the cube. The result is easily checked in these cases. Hence we may assume that $M(G)$ has an $M(\mathcal{W}_5)$-minor. Since G is planar and $\mathcal{W}_5$ is isomorphic to its dual, we may, by switching to the dual if necessary, assume that $M(G)$ has the unique planar 3-connected single-element extension, M_1, of $M(\mathcal{W}_5)$ as a minor. This extension is not internally 4-connected, so we may assume that $M(G)$ is a single-element extension or coextension of M_1. But every such extension or coextension has a 4-element fan and hence is not internally 4-connected. □

5. A Triangle Theorem

In this section, we prove the following result.

Theorem 5.1. *Let* T *be a triangle of an internally* 4*-connected binary matroid* M *with* $|E(M)| \geq 13$*. Then either*

(i) T *is the central triangle of a rotor;*
(ii) T *contains an element* e *such that* $M\backslash e$ *is* $(4, 4, S)$*-connected.*

Our proof of this theorem will use the following theorem of Geelen and Zhou [4, Corollary 5.4].

Theorem 5.2. *Let* T *be a triangle of an internally* 4*-connected binary matroid* M *with* $|E(M)| \geq 13$*. Then either*

(i) *T is the central triangle of a rotor;*
(ii) *T contains an element e such that $M\backslash e$ is weakly 4-connected.*

The proof of Theorem 5.1 has much in common with the proof of [13, Theorem 6.3]. A 3-separation (X, Y) of a 3-connected matroid M is a $(4, k, S)$-*violator* if either

(i) $|X|, |Y| \geq k + 1$; or
(ii) (X, Y) is non-sequential.

The next lemma [13, Lemma 2.11] simplifies the task of identifying a $(4, 4, S)$-violator.

Lemma 5.3. *Let N be a 3-connected matroid. Then (X, Y) is a $(4, 4, S)$-violator if and only if*

(i) $|X|, |Y| \geq 5$*; or*
(ii) *X and Y are non-sequential and at least one is a quad.*

Proof of Theorem 5.1. Let $T = \{x, y, z\}$ and assume the theorem fails. If $f \in T$ and (F_1, F_2) is a $(4, 4, S)$-violator for $M\backslash f$, then, as neither $(F_1 \cup f, F_2)$ nor $(F_1, F_2 \cup f)$ is a 3-separation of M, we must have that $T \cap F_1 \neq \emptyset \neq T \cap F_2$. Let (X_y, X_z), (Y_x, Y_z), and (Z_x, Z_y) be $(4, 4, S)$-violators for $M\backslash x$, $M\backslash y$, and $M\backslash z$, respectively. Then we may assume that $y \in X_y$ and $z \in X_z$, that $x \in Y_x$ and $z \in Y_z$, and that $x \in Z_x$ and $y \in Z_y$.

Lemma 5.4. *If $(X_y, X_z) \cong (X_y \cup f, X_z - f)$ for some element f of X_z, then $(X_y \cup f, X_z - f)$ is a $(4, 3, S)$-violator of $M\backslash x$.*

Proof. If (X_y, X_z) is non-sequential, then so is $(X_y \cup f, X_z - f)$. If (X_y, X_z) is sequential, then $|X_y|, |X_z| \geq 5$, so $|X_y \cup f|, |X_z - f| \geq 4$. □

Lemma 5.5. *The element y is in $\mathrm{cl}(X_y - y)$.*

Proof. Suppose $y \notin \mathrm{cl}(X_y - y)$. Then y is a coloop of $(M\backslash x)|X_y$, so $(X_y - y, X_z \cup y) \cong (X_y, X_z)$. But $X_z \cup y$ contains $\{y, z\}$, so $(X_y - y, X_z \cup y \cup x)$ is a 3-separation of M. By Lemma 5.4, $(X_y - y, X_z \cup y)$ is a $(4, 3, S)$-violator of $M\backslash x$ so $|X_y - y|, |X_z \cup y| \geq 4$. Thus $(X_y - y, X_z \cup y \cup x)$ is a $(4, 3, S)$-violator of M; a contradiction. □

Lemma 5.6. $X_z \cap Y_x \neq \emptyset$

Proof. Suppose that $X_z \cap Y_x = \emptyset$. By Lemma 5.5 and symmetry, $x \in \mathrm{cl}(Y_x - x)$. But $Y_x - x \subseteq X_y$, so $x \in \mathrm{cl}(X_y)$; a contradiction. □

Lemma 5.7. *The element x is not in $\mathrm{cl}(Z_y)$.*

Proof. As $y \in Z_y$, if $x \in \mathrm{cl}(Z_y)$, then $z \in \mathrm{cl}(Z_y)$; a contradiction. □

Lemma 5.8. *Either*

(i) $|X_z \cap Y_x| \geq 2$*; or*
(ii) *Y_x is a 5-element fan of $M\backslash y$ and $X_y \cap Y_x$ is a triangle.*

Proof. Assume that $|X_z \cap Y_x| < 2$. Then, by Lemma 5.6, $X_z \cap Y_x = \{e\}$ for some element e. Suppose first that $e \in \mathrm{cl}(X_z \cap Y_z)$. Then $e \in \mathrm{cl}(Y_z)$, so $(Y_x, Y_z) \cong (Y_x - e, Y_z \cup e)$. By Lemma 5.4, $(Y_x - e, Y_z \cup e)$ is a $(4, 3, S)$-violator for $M\backslash y$. Thus $|Y_x - e| \geq 4$. If x is not a coloop of $M|(Y_x - e)$, then $x \in \mathrm{cl}(Y_x - e - x)$. But $Y_x - e - x \subseteq X_y$, so $x \in \mathrm{cl}(X_y)$; a contradiction. Hence x is a coloop of $M|(Y_x - e)$. Thus (Y_x, Y_z) and $(Y_x - e - x, Y_z \cup e \cup x)$ are equivalent 3-separations of $M\backslash y$. Since $x \in \mathrm{cl}^*{}_{M\backslash y}(Y_z \cup e)$, we deduce that $x \in \mathrm{cl}^*{}_{M\backslash y}(Y_x - e - x)$. Also, as $\{x, z\} \subseteq Y_z \cup e \cup x$, we have that $y \in \mathrm{cl}(Y_z \cup e \cup x)$. Hence $(Y_x - e - x, Y_z \cup e \cup x \cup y)$ is a 3-separation of M. Thus $|Y_x - e - x| \leq 3$. As $|Y_x - e| \geq 4$, we deduce that $|Y_x| = 5$ and $Y_x - e - x$ is a triangle or a triad of M. Since $x \in \mathrm{cl}^*{}_{M\backslash y}(Y_x - e - x)$ and M is binary, $Y_x - e - x$ is a triangle. Hence Y_x is a 5-element sequential set in $M\backslash y$, and $Y_x - e - x$, which equals $X_y \cap Y_x$, is a triangle. Since $M\backslash y$ is binary, it follows that Y_x is a 5-element fan of $M\backslash y$. Thus (ii) holds when $e \in \mathrm{cl}(X_z \cap Y_z)$.

We may now assume that $e \notin \mathrm{cl}(X_z \cap Y_z)$. Then $(X_y, X_z) \cong (X_y \cup e, X_z - e)$ in $M\backslash x$ and, by Lemma 5.4, $|X_z - e| \geq 4$. By Lemma 5.5, $x \in \mathrm{cl}(Y_x - x)$, so $x \in \mathrm{cl}(X_y \cup e)$. Thus $(X_y \cup e \cup x, X_z - e)$ is a 3-separation of M. Hence $|X_z - e| \leq 3$; a contradiction. □

Lemma 5.9. *The set X_y is not a quad of $M\backslash x$.*

Proof. Suppose that X_y is a quad of $M\backslash x$. We shall argue that this implies that $M\backslash y$ or $M\backslash z$ is $(4, 4, S)$-connected; a contradiction. Using Lemma 5.8 and symmetry, we have, since X_y is not a 5-element fan of $M\backslash x$, that $|X_y \cap Y_z| \geq 2$. Now $X_y \cap Y_x \neq \emptyset$ otherwise, since $y \in \mathrm{cl}(X_y - y)$, we have that $y \in \mathrm{cl}(Y_z)$; a contradiction. Thus $|X_y \cap Y_z| = 2$ and $|X_z \cap Y_x| \geq 2$. Now $\lambda_{M\backslash y}(Y_x) = 2 = \lambda_{M\backslash x}(X_z)$. By Lemma 5.5, $x \in \mathrm{cl}(Y_x - x)$ and $y \in \mathrm{cl}(X_y - y)$, so $\lambda_{M\backslash x,y}(Y_x - x) = 2 = \lambda_{M\backslash x,y}(X_z)$. Thus $\lambda_{M\backslash x,y}((Y_x - x) \cap X_z) + \lambda_{M\backslash x,y}((Y_x - x) \cup X_z) \leq 4$, that is, $\lambda_{M\backslash x,y}(Y_x \cap X_z) + \lambda_{M\backslash x,y}(Y_z \cap X_y) \leq 4$. But $\lambda_{M\backslash x,y}(Y_x \cap X_z) = \lambda_M(Y_x \cap X_z)$ as $z \in E(M) - \{x, y\} - (Y_x \cap X_z)$ and $y \in \mathrm{cl}(X_y - y)$, so $x \in \mathrm{cl}(E(M) - \{x, y\} - (Y_x \cap X_z))$. Similarly, $\lambda_{M\backslash x,y}(Y_z \cap X_y) = \lambda_M(Y_z \cap X_y)$. As $|Y_x \cap X_z|, |Y_z \cap X_y| \geq 2$, we have $\lambda_M(Y_x \cap X_z) = 2 = \lambda_M(Y_z \cap X_y)$. Thus $|Y_x \cap X_z| \leq 3$.

Let $X_y \cap Y_x = \{1\}$ and $X_y \cap Y_z = \{2, 3\}$. We shall show next that we may choose Y_x to be a quad of $M\backslash y$. This is certainly true if $|Y_x \cap X_z| = 2$, so we assume that $|Y_x \cap X_z| = 3$. Then $Y_x \cap X_z$ is a triangle or a triad of M, and $|Y_x| = 5$. Suppose that Y_x is a sequential 3-separating set in $M\backslash y$. Then Y_x is a 5-element fan in $M\backslash y$. If $Y_x \cap X_z$ is a triad of M, then there is a triangle contained in Y_x meeting this triad, so M has a 4-element fan; a contradiction. Hence $Y_x \cap X_z$ is a triangle $\{4, 5, 6\}$ of M. Now $x \notin \mathrm{cl}(Y_z)$ otherwise $y \in \mathrm{cl}(Y_z)$. Moreover, by Lemma 5.5, $x \in \mathrm{cl}(Y_x - x)$. Thus we may assume that $\{4, 5, x\}$ is a triad of $M\backslash y$ and $\{4, x, 1\}$ is a triangle. Then M has $\{4, 5, x, y\}$ as a cocircuit. But $\{1, 2, 3, y\}$ is a circuit. Thus we have a contradiction to orthogonality. We may now assume that Y_x is non-sequential. Since $|Y_x| = 5$, it follows that Y_x contains a quad Y'_x of

$M\backslash y$. Since $z \in Y_z$, we must have that $x \in Y_x'$. Thus, by replacing (Y_x, Y_z) by $(Y_x', E(M\backslash y) - Y_x')$, we may indeed assume that Y_x is a quad of $M\backslash y$.

Let $X_z \cap Y_x = \{4,5\}$. Then M has $\{1,2,3,x,y\}, \{1,4,5,x,y\}$, and $\{2,3,4,5\}$ as cocircuits, and $\{1,2,3,y\}$ and $\{1,4,5,x\}$ as circuits. By Lemma 5.6 and symmetry, either $|X_y \cap Z_x| \geq 2$, or $|X_y \cap Z_x| = 1$. In the latter case, by Lemma 5.8, Z_x is a 5-element fan of $M\backslash z$ and $X_z \cap Z_x$ is a triangle. Similarly, either $|Y_x \cap Z_y| \geq 2$, or $|Y_x \cap Z_y| = 1$ and Z_y is a 5-element fan of $M\backslash z$.

Suppose that $|X_y \cap Z_x| = 1$. Then $|Z_x| = 5$. As $|E(M)| \geq 13$, we deduce that $|Z_y| \neq 5$, so $|Y_x \cap Z_y| \geq 2$. Assume that the element, w, of $X_y \cap Z_x$ is also in Y_x, that is, $w = 1$. As X_y is a circuit, $1 \in \mathrm{cl}(X_y - 1)$, so $1 \in \mathrm{cl}(Z_y)$. As $|Y_x \cap Z_y| \geq 2$ and Y_x is a circuit, we deduce that $x \in \mathrm{cl}(Z_y)$; a contradiction to Lemma 5.7. Hence we may assume that $w \in Y_z$, so, without loss of generality, $w = 2$. Then $1 \in Z_y$. Consider Z_x. It is a 5-element fan of $M\backslash z$ having $X_z \cap Z_x$ as a triangle avoiding $\{2, x\}$. As $\{1,4,5,x\}$ is a cocircuit of $M\backslash y$ and $\{1,x\} \cap (X_z \cap Z_x) = \emptyset$, it follows, by orthogonality, that either $\{4,5\} \subseteq X_z \cap Z_x$, or $\{4,5\} \cap (X_z \cap Z_x) = \emptyset$. But $|Y_x \cap Z_y| \geq 2$, so $\{4,5\} \cap Z_y \neq \emptyset$. Hence $\{4,5\} \subseteq Z_y$, so $\{1,4,5\} \subseteq Z_y$ and $x \in \mathrm{cl}(Z_y)$; a contradiction to Lemma 5.7.

We may now assume that $|X_y \cap Z_x| \geq 2$. Since $y \notin \mathrm{cl}(Z_x)$, we must have $|X_y \cap Z_x| = 2$. Hence we need only consider the following cases:

(a) $\{2,3\} \subseteq Z_x$ and $1 \in Z_y$; and
(b) $\{1,3\} \subseteq Z_x$ and $2 \in Z_y$.

Consider case (a). If $\{4,5\} \subseteq Z_x$, then $1 \in \mathrm{cl}(Z_x)$, so $y \in \mathrm{cl}(Z_x)$; a contradiction. If $\{4,5\} \subseteq Z_y$, then $x \in \mathrm{cl}(Z_y)$; a contradiction. Thus we may assume that $4 \in Z_x$ and $5 \in Z_y$. As $\{2,3,4,5\}$ is a cocircuit of M, it follows that $(Z_x, Z_y) \cong (Z_x \cup 5, Z_y - 5)$ in $M\backslash z$. Now $1 \in \mathrm{cl}(Z_x \cup 5)$ and so $y \in \mathrm{cl}(Z_x \cup 5)$. Thus $(Z_x \cup 5 \cup y \cup z, Z_y - 5 - y)$ is a 3-separating partition of M. Hence $|Z_y - 5 - y| \leq 3$. We deduce that Z_y is sequential in $M\backslash z$. Thus, by Lemma 5.3, $|Z_y| = 5$, so $Z_y - 5$ is a 4-element fan in $M\backslash y$. This is a contradiction since $\{1, y\} \subseteq \mathrm{cl}(Z_x \cup 5)$.

Consider case (b). If $\{4,5\} \subseteq Z_y$, then $(Z_x, Z_y) \cong (Z_x - 3, Z_y \cup 3)$ in $M\backslash z$. But $1 \in \mathrm{cl}(Z_y \cup 3)$, so $x \in \mathrm{cl}(Z_y \cup 3)$. Thus $(Z_x, Z_y) \cong (Z_x - 3 - x, Z_y \cup 3 \cup x)$. Hence $|Z_x - 3 - x| \leq 3$ and Z_x is sequential in $M\backslash z$. Thus $|Z_x| = 5$, so $Z_x - 3$ is a 4-element fan in $M\backslash x$. This is a contradiction since $\{1, x\} \subseteq \mathrm{cl}(Z_y \cup 3)$. If $\{4,5\} \subseteq Z_x$, then $Y_x \cap Z_y = \emptyset$; a contradiction. Thus we may assume that $4 \in Z_x$ and $5 \in Z_y$. Then $|Y_x \cap Z_y| = 1$, so Z_y is a 5-element fan in $M\backslash z$ and $Y_z \cap Z_y$ is a triangle. But $\{1,2,3,x,y\}$ is a cocircuit of M meeting the circuit $Y_z \cap Z_y$ in a single element thereby contradicting orthogonality. □

We are now ready to complete the proof of the theorem. As $|E(M)| \geq 13$, by Theorem 5.2, either T is the central triangle of a rotor, or T contains an element e such that $M\backslash e$ is weakly 4-connected. Assume that T is not the central triangle of a rotor. Then, without loss of generality, we may assume that $M\backslash x$ is weakly 4-connected. But none of $M\backslash x, M\backslash y$, or $M\backslash z$

is $(4,4,S)$-connected. By Lemma 5.3, $M\backslash x$ has a quad, which we may take to be X_y. Thus we have a contradiction to Lemma 5.9. □

6. The Quasi Rotor Case

The purpose of this section is to prove the following result.

Theorem 6.1. *Let M be an internally 4-connected binary matroid having $(\{1,2,3\},\{4,5,6\},\{7,8,9\},\{2,3,4,5\},\{5,6,7,8\},\{3,5,7\})$ as a quasi rotor and having at least thirteen elements. Then either*

(i) *$M\backslash 1$, $M\backslash 9$, $M\backslash 1/2$, or $M\backslash 9/8$ is internally 4-connected; or*

(ii) *M has triangles $\{6,8,10\}$ and $\{2,4,11\}$ such that $|\{1,2,\dots,11\}| = 11$, and $M\backslash 3,4/5$ is internally 4-connected.*

Lemma 6.3 below will be useful not only in the proof of the last theorem but also elsewhere in the paper. We shall use the following preliminary result.

Lemma 6.2. *Let $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ and $(\{1,7,8\},\{4,5,6\},\{5,6,7,8\})$ be bowties in an internally 4-connected binary matroid M where $|\{1,2,\dots,8\}| = 8$ and $|E(M)| \geq 13$. Then $M\backslash 1$ is internally 4-connected.*

Proof. Assume that $M\backslash 1$ is not internally 4-connected and let (X_1,Y_1) be a $(4,3)$-violator of it. By symmetry, we may assume that $\{2,7\} \subseteq X_1$ and $\{3,8\} \subseteq Y_1$.

Suppose first that $\{5,6\} \subseteq X_1$. Then, as $(X_1 \cup 8, Y_1 - 8) \cong (X_1,Y_1)$ and $(X_1 \cup 8 \cup 1, Y_1 - 8)$ is a 3-separation of M, we deduce that Y_1 is a 4-element fan of $M\backslash 1$ having a fan ordering $(y_1,y_2,y_3,8)$ where $\{y_2,y_3,8\}$ is a triad. Then $\{y_2,y_3,8,1\}$ is a cocircuit of M. By orthogonality with the triangle $\{1,2,3\}$, we deduce that $3 \in \{y_2,y_3\}$, so we may suppose that $y_3 = 3$. The triangle $\{y_1,y_2,y_3\}$ and the cocircuit $\{2,3,4,5\}$ imply that $4 \in \{y_1,y_2\}$. If $y_2 = 4$, then $\{4,3,8,1\}$ is a cocircuit of M, which contradicts orthogonality with the circuit $\{4,5,6\}$. If $y_1 = 4$, then $\{4,y_2,3\}$ is a triangle and $\{y_2,3,8,1\}$ is a cocircuit. Thus, for $Z = \{1,2,\dots,8,y_2\}$, we have $\lambda(Z) = r(Z) + r^*(Z) - |Z| \leq 5 + 6 - 9 = 2$; a contradiction. We deduce that $\{5,6\} \not\subseteq X_1$. By symmetry, $\{5,6\} \not\subseteq Y_1$.

By symmetry again, we may now assume that $5 \in X_1$ and $6 \in Y_1$. Consider the location of 4. Suppose first that $4 \in X_1$. Then $\{2,7,5,4\} \subseteq X_1$ and $\{3,8,6\} \subseteq Y_1$. Now $(X_1 \cup 6, Y_1 - 6) \cong (X_1,Y_1)$ so we have reduced to the previous case unless Y_1 is a 4-element fan of $M\backslash 1$ having $(y_1,y_2,y_3,6)$ as a fan ordering where $\{y_2,y_3,6\}$ is a triangle. Consider the exceptional case. Since $3 \in \mathrm{cl}^*{}_{M\backslash 1}(X_1)$, we deduce that $y_1 = 3$, so, as $8 \in \{y_2,y_3\}$, we may take $y_3 = 8$. Then $\lambda(\{1,2,\dots,8,y_2\}) \leq 2$; a contradiction.

Finally, with $5 \in X_1$ and $6 \in Y_1$, we may assume that $4 \in Y_1$. Then $(X_1 - 5, Y_1 \cup 5) \cong (X_1,Y_1)$ and we have reduced to an earlier case unless X_1 is a 4-element fan having an ordering $(x_1,x_2,x_3,5)$ where $\{x_2,x_3,5\}$ is a triangle of M. Thus $\{x_1,x_2,x_3,1\}$ is a cocircuit of M. The cocircuit $\{2,3,4,5\}$ implies that $2 \in \{x_2,x_3\}$ so we may take $2 = x_3$. Now $7 \in$

$\{x_1, x_2\}$. If $x_2 = 7$, then $\{7, 2, 5\}$ is a triangle of M and $\lambda(\{1, 2, \ldots, 8\}) \leq 2$; a contradiction. If $x_1 = 7$, then $\lambda(\{1, 2, \ldots, 8, x_2\}) \leq 2$; a contradiction. We conclude that the lemma holds. □

Lemma 6.3. *Let* $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ *be a bowtie in an internally* 4*-connected binary matroid* M *where* $|E(M)| \geq 13$. *Assume that* $M\backslash 1$ *is not internally* 4*-connected. Then one of the following holds:*

(i) $M\backslash 6$ *is internally* 4*-connected;*
(ii) M *has a triangle* $\{7, 8, 9\}$ *and a* 4*-cocircuit* C^* *containing* $\{6, 7, 8\}$ *where* $\{7, 8, 9\} \cap \{1, 2, \ldots, 6\} = \emptyset$ *and* $|C^* \cap \{4, 5\}| = 1$*; or*
(iii) M *has a triangle* $\{a, d, e\}$ *and a cocircuit* $\{a, b, d, 6\}$ *where* d *and* e *are in* $\{4, 5\}$ *and* $\{2, 3\}$*, respectively, and* $|\{1, 2, 3, 4, 5, 6, a, b\}| = 8$.

Moreover, $M\backslash 6$ *is* $(4, 4, S)$*-connected unless* $\{4, 5, 6\}$ *is the central triangle of a quasi rotor whose other triangles are* $\{1, 2, 3\}$*,* $\{x, y, 7\}$*, and* $\{7, 8, 9\}$ *and whose cocircuits are* $\{2, 3, 4, 5\}$ *and* $\{y, 6, 7, 8\}$*, for some* x *in* $\{2, 3\}$ *and some* y *in* $\{4, 5\}$.

Proof. Assume that $M\backslash 6$ is not internally 4-connected. Then $M\backslash 6$ has a $(4, 3)$-violator (X_6, Y_6). If possible, choose (X_6, Y_6) to be a $(4, 4, S)$-violator. As M is internally 4-connected, we may assume that $4 \in X_6$ and $5 \in Y_6$.

6.3.1. *If* X_6 *or* Y_6 *contains* $\{1, 2, 3\}$*, then (ii) holds,* $\min\{|X_6|, |Y_6|\} = 4$*, and* (X_6, Y_6) *is sequential. Moreover,* $M\backslash 6$ *is* $(4, 4, S)$*-connected.*

By symmetry, we may assume that $\{1, 2, 3\} \subseteq X_6$. Then $(X_6 \cup 5, Y_6 - 5) \cong (X_6, Y_6)$ in $M\backslash 6$. Moreover, $(X_6 \cup 5 \cup 6, Y_6 - 5)$ is a 3-separation of M. Thus $|Y_6 - 5| \leq 3$ so $|Y_6| = 4$. Hence, by Lemma 2.8, Y_6 contains a triangle $\{7, 8, 9\}$ of M that is disjoint from $\{1, 2, \ldots, 6\}$, and Y_6 contains a triad T^* of $M\backslash 6$ that contains 5. We may assume that T^* meets $\{7, 8, 9\}$ in $\{7, 8\}$. Then $T^* \cup 6 = \{5, 6, 7, 8\}$ and so (ii) holds in this case. Observe also that $\min\{|X_6|, |Y_6|\} = 4$ and (X_6, Y_6) is sequential. Thus (X_6, Y_6) is not a $(4, 4, S)$-violator. Hence (6.3.1) holds.

Now let $\{x, y, z\} = \{1, 2, 3\}$. Without loss of generality, assume that $\{x, y\} \subseteq X_6$ and $z \in Y_6$. Then, as $|Y_6| \geq 4$, we have that $(X_6 \cup z \cup 5, Y_6 - z - 5) \cong (X_6, Y_6)$. Thus $(X_6 \cup z \cup 5 \cup 6, Y_6 - z - 5)$ is an exactly 3-separating partition of M. Hence $|Y_6 - z - 5| \leq 3$, so $|Y_6| = 4$ or $|Y_6| = 5$.

6.3.2. *If* $|Y_6| = 4$*, then (iii) holds and* $M\backslash 6$ *is* $(4, 4, S)$*-connected.*

The set Y_6 is a fan of $M\backslash 6$ having an ordering of the form (y_1, y_2, y_3, z) where $\{y_2, y_3, z\}$ is a triangle of M. Now $\{2, 3, 4, 5\}$ is a cocircuit of $M\backslash 6$, so $|\{2, 3, 4, 5\} \cap \{y_2, y_3, z\}| \in \{0, 2\}$. Thus, by symmetry, either $z = 2$ and $y_3 = 5$; or $z = 1$ and $y_1 = 5$. In the first case, (iii) holds. In the second case, $\{y_2, y_3, 1\}$ is a triangle of M that is disjoint from $\{2, 3, 4, 5, 6\}$ while $\{5, y_2, y_3, 6\}$ is a cocircuit of M. Hence $(\{1, y_2, y_3\}, \{5, 6, 4\}, \{y_2, y_3, 5, 6\})$ is a bowtie of M. Thus, by Lemma 6.2, $M\backslash 1$ is internally 4-connected; a contradiction. We conclude that (6.3.2) holds.

We may now assume that $|Y_6| = 5$. Then Y_6 has a sequential ordering of the form $(9, 8, 7, 5, z)$. Thus Y_6 is a 5-element fan in $M\backslash 6$ having $\{9, 8, 7\}$ as a triangle that avoids $\{1, 2, 3, 4, 5, 6\}$. We may also assume that $\{7, 5, z\}$ is a triangle and $\{8, 7, 5\}$ is a triad of $M\backslash 6$. Thus $\{5, 6, 7, 8\}$ is a cocircuit of M. By orthogonality and symmetry, we may assume that $z = 3$. Thus M has $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$, and $\{3, 5, 7\}$ as triangles, and $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ as cocircuits. We conclude that $\{4, 5, 6\}$ is the central triangle of a quasi rotor. □

Lemma 6.4. *In an internally* 4*-connected binary matroid* M*, let* $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ *be a quasi rotor. If* $|E(M)| \geq 13$*, then*

(i) *the only triangles containing* 5 *are* $\{3, 5, 7\}$ *and* $\{4, 5, 6\}$*; and*
(ii) *the only cocircuits of* M *contained in* $\{1, 2, \ldots, 9\}$ *are* $\{2, 3, 4, 5\}, \{5, 6, 7, 8\}$*, and* $\{2, 3, 4, 6, 7, 8\}$*.*

Proof. Let $Z = \{1, 2, \ldots, 9\}$. Then the specified triangles and cocircuits imply that $r(Z) \leq 5$ and $r^*(Z) \leq 7$. If Z contains a cocircuit other than the three specified in (ii), then $r^*(Z) \leq 6$, so $\lambda(Z) \leq 2$. On the other hand, if 5 is in a triangle T different from $\{4, 5, 6\}$ or $\{3, 5, 7\}$, then the cocircuits $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ and the other triangles imply that $T = \{2, 5, 8\}$. Thus $r(Z) \leq 4$ and again $\lambda(Z) \leq 2$. We conclude that if (i) or (ii) fails, then $\lambda(Z) \leq 2$. Thus, when $|E(M)| \geq 13$, we contradict the fact that M is internally 4-connected. □

Lemma 6.5. *Let* M *be an internally* 4*-connected binary matroid having* $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ *as a quasi rotor where* $|E(M)| \geq 13$ *and* $M\backslash 9$ *is not internally* 4*-connected. Then* M *has a* 4*-cocircuit* C^* *that meets* $\{1, 2, \ldots, 9\}$ *in* $\{8, 9\}$*. Moreover, either*

(i) M *has a triangle that contains* $\{6, 8\}$ *and an element of* $C^* - \{8, 9\}$*; or*
(ii) M *has a triangle that contains* $C^* - \{8, 9\}$ *and an element* e *that avoids* $\{2, 3, \ldots, 9\}$*.*

Proof. Consider the bowtie $(\{4, 5, 6\}, \{7, 8, 9\}, \{5, 6, 7, 8\})$ in M. By assumption, $M\backslash 9$ is not internally 4-connected. Moreover, $M\backslash 4$ is not internally 4-connected since it has $\{1, 2, 3, 5\}$ as a fan. Thus, by Lemmas 6.3 and 6.4, either

(a) M has a triangle $\{10, 11, 12\}$ disjoint from $\{4, 5, 6, 7, 8, 9\}$ and M has a 4-cocircuit containing $\{9, 10, 11\}$ and exactly one of 7 and 8; or
(b) M has a triangle $\{a, d, 6\}$ and a cocircuit $\{a, b, d, 9\}$ where $d \in \{7, 8\}$ and $\{a, b\} \cap \{4, 5, 6, 7, 8, 9\} = \emptyset$.

In each case, we obtain a 4-cocircuit C^* containing 9 and exactly one of 7 and 8. Moreover, C^* avoids $\{4, 5, 6\}$. Assume $7 \in C^*$. Then, in each case, orthogonality implies that $3 \in C^*$, and 1 or 2 is in C^*. Thus $C^* \subseteq \{9, 7, 3, 1, 2\}$ and we have a contradiction to Lemma 6.4. Hence $8 \in C^*$.

Moreover, C^* meets $\{4,5,6,7,8,9\}$ in $\{8,9\}$. If C^* meets $\{1,2,3\}$, then $|C^* \cap \{1,2,3\}| = 2$ and again we contradict Lemma 6.4. Thus C^* meets $\{1,2,\dots,9\}$ in $\{8,9\}$.

In case (b), the triangle $\{a,d,6\}$ contains $\{6,8\}$, so (i) holds. In case (a), the triangle $\{10,11,12\}$ satisfies the condition in (ii) since the cocircuit $\{2,3,4,5\}$ implies that $12 \notin \{2,3\}$. □

Lemma 6.6. *In an internally* 4*-connected binary matroid* M*, let* $(\{1,2,3\},\{4,5,6\},\{7,8,9\},\{2,3,4,5\},\{5,6,7,8\},\{3,5,7\})$ *be a quasi rotor. Then*

(i) $M\backslash 1$ *is internally* 4*-connected; or*
(ii) $M\backslash 1/2$ *is internally* 4*-connected; or*
(iii) M *has a triangle containing* $\{2,4\}$.

Proof. Assume that neither (i) nor (iii) holds. Then, as $\{2,3,4,5\}$ is a cocircuit and Lemma 6.4 implies that M has no triangle containing $\{2,5\}$, it follows by orthogonality that $\{1,2,3\}$ is the only triangle of M containing 2. Moreover, by Lemma 6.5, M has a 4-cocircuit $\{1,2,a,b\}$ and a triangle $\{a,b,c\}$ where $\{a,b\}$ avoids $\{1,2,\dots,9\}$. Since 3 is in a triangle, $M\backslash 3$ is 3-connected. Now $(2,4,5,6)$ is a fan ordering of a fan in $M\backslash 3$. Since 2 is in no triangles of $M\backslash 3$, it is a fan end in $M\backslash 3$. Thus $M\backslash 3/2$ is 3-connected. But $M/2$ has $\{3,1\}$ as a circuit, so $M/2\backslash 3 \cong M/2\backslash 1$. Hence $M/2\backslash 1$ is 3-connected. Assume it is not internally 4-connected. Then it has a $(4,3)$-violator (X,Y).

If X or Y, say X, contains $\{3,4,5\}$, then $(X\cup 2\cup 1, Y)$ is a 3-separation of M; a contradiction. Hence we may assume that neither X nor Y contains $\{3,4,5\}$. Without loss of generality, $|X \cap \{3,4,5,6,7\}| \geq 3$. Suppose first that $|X \cap \{3,4,5,6,7\}| = 4$. Then $|X \cap \{3,4,5\}| = 2$. Let $\{y\} = Y \cap \{3,4,5,6,7\}$. The triangles $\{3,5,7\}$ and $\{4,5,6\}$ imply that $(X,Y) \cong (X\cup y, X-y)$ in $M\backslash 1/2$. If $|Y-y| \geq 4$, then, as $X\cup y$ contains $\{3,4,5\}$, we obtain a contradiction. Hence $|Y-y| = 3$. Now Y is a 4-element fan in $M\backslash 1/2$ containing a triangle $\{y_2,y_3,y\}$ and a triad $\{y_1,y_2,y_3\}$. Moreover, $\{y_1,y_2,y_3\} \cap \{1,2,\dots,7\} = \emptyset$. Since $\{2,3,4,5\}$ is a cocircuit of M and $y \in \{3,4,5\}$, orthogonality implies that $\{y_2,y_3,y,2\}$ is a circuit of M. The triangle $\{1,2,3\}$ of M avoids $\{y_1,y_2,y_3\}$. Thus $\{y_1,y_2,y_3,1\}$ is not a cocircuit of M, so $\{y_1,y_2,y_3\}$ is a triad of M. Hence $\{y_1,y_2,y_3\}$ avoids $\{1,2,\dots,9\}$. Now M has a cocircuit $\{1,2,a,b\}$ and a triangle $\{a,b,c\}$ where $\{a,b\}$ avoids $\{1,2,\dots,9\}$. Then $\{a,b,c\}$ avoids $\{y_1,y_2,y_3\}$. But $|\{1,2,a,b\} \cap \{y_2,y_3,y,2\}| = 1$, so we contradict orthogonality. We deduce that $|X \cap \{3,4,5,6,7\}| = 3$.

Now either

(a) X spans $\{3,4,5,6,7\}$; or
(b) $X \cap \{3,4,5,6,7\} = \{3,5,7\}$; or
(c) $X \cap \{3,4,5,6,7\} = \{4,5,6\}$.

In case (a), let $Y\cap\{3,4,5,6,7\} = \{y_1,y_2\}$. Then $(X,Y) \cong (X\cup y_1, Y-y_1)$. As $\{y_1,y_2\} \subseteq \mathrm{cl}_{M\backslash 1/2}(X)$, we must have $|Y| \geq 5$. Hence $|Y-y_1| \geq 4$. As

$|(X \cup y_1) \cap \{3,4,5,6,7\}| = 4$, we obtain a contradiction as in the previous paragraph.

Next consider case (b), that is, $\{3,5,7\} \subseteq X$ and $\{4,6\} \subseteq Y$. Suppose that $8 \in X$. The cocircuit $\{5,6,7,8\}$ implies that $(X \cup 6, Y - 6) \cong (X, Y)$. Since $|(X \cup 6) \cap \{3,4,5,6,7\}| = 4$, we again obtain a contradiction as above unless Y is a 4-element fan in $M\backslash 1/2$. In the exceptional case, Y contains a triad T^* of $M\backslash 1/2$ containing 6. As 6 is not in a triad of M, we have that $T^* \cup 1$ is a cocircuit of M meeting the triangle $\{1,2,3\}$ in a single element; a contradiction. We may now assume that $8 \in Y$. In that case, we consider the location of 9 supposing first that it is in X. Then X contains $\{3,5,7,9\}$ and Y contains $\{4,6,8\}$, so $(X,Y) \cong (X \cup 8, Y - 8)$. Thus we have reduced to an earlier case unless Y is a 4-element fan of $M\backslash 1/2$ containing a triangle $\{y_2, y_3, 8\}$ and a triad $\{y_1, y_2, y_3\}$. This triad contains $\{4,6\}$, so $\{y_1, y_2, y_3, 1\}$ is a cocircuit of M meeting the triangle $\{1,2,3\}$ in a single element; a contradiction.

We may now assume that $9 \in Y$, so $\{3,5,7\} \subseteq X$ and $\{4,6,8,9\} \subseteq Y$. Then $(X,Y) \cong (X - 7, Y \cup 7)$. As $Y \cup 7$ spans $\{3,4,5,6,7\}$, we have reduced to case (a), unless $|X - 7| = 3$. In the exceptional case, $X - 7$ is a triad of $M\backslash 1/2$ containing $\{3,5\}$. Hence $(X - 7) \cup 1$ is a 4-cocircuit of M containing $\{1,3,5\}$ but meeting the circuit $\{4,5,6\}$ in a single element; a contradiction. This completes the elimination of case (b).

Now consider case (c), that is, $\{4,5,6\} \subseteq X$ and $\{3,7\} \subseteq Y$. Then $(X,Y) \cong (X - 5, Y \cup 5)$, so we have reduced to a case that is symmetric to case (b) unless $X - 5$ is a triad of $M\backslash 1/2$. In the exceptional case, as $X - 5$ contains $\{4,6\}$, it follows that $(X - 5) \cup 1$ is a cocircuit of M. But this cocircuit meets the triangle $\{1,2,3\}$ in a single element; a contradiction. We conclude that $M\backslash 1/2$ is internally 4-connected. □

We are now ready to prove that, when M contains a quasi rotor, it has a proper internally 4-connected minor N such that $|E(M)| - |E(N)| \leq 3$.

Proof of Theorem 6.1. Assume that none of $M\backslash 1$, $M\backslash 9$, $M\backslash 1/2$, or $M\backslash 9/8$ is internally 4-connected. By Lemma 6.6 and symmetry, M has triangles $\{6,8,10\}$ and $\{2,4,11\}$. Let $Z = \{1,2,\ldots,9\}$. As $|E(M)| \geq 13$, we must have $\lambda_M(Z) \geq 3$. But $r(Z) \leq 5$ and $r^*(Z) \leq 7$, so equality must hold in the last three inequalities. The elements 10 and 11 are distinct, otherwise $\{2,4,6,8\}$ is a circuit of M and $r(Z) \leq 4$; a contradiction. Similarly, neither 10 nor 11 is in $\{1,2,\ldots,9\}$, otherwise $r(Z) \leq 4$.

Next we observe that

6.7.1. *M has no 4-cocircuit containing $\{10,11\}$.*

Assume the contrary. Then the triangles of M imply, by orthogonality, that $\{4,6,10,11\}$ is a cocircuit of M. Let $Z' = \{2,3,4,5,6,7,8,10,11\}$. Then $r^*(Z') \leq 6$ and $r(Z') \leq 5$, so $\lambda_M(Z') \leq 2$; a contradiction since $|E(M)| \geq 13$. Hence (6.7.1) holds.

Now, since 3 is in a triangle of M, the matroid $M\backslash 3$ is 3-connected. This matroid has $\{2,4,5,6\}$ as a fan. Moreover, 6 is a fan end unless $M\backslash 3$ has a triad containing 6 and either 4 or 5. In the exceptional case, M has a 4-cocircuit containing $\{3,6\}$ and either 4 or 5. By orthogonality, this cocircuit also contains 1 or 2, so we contradict Lemma 6.4. We deduce that 6 is a fan end in $M\backslash 3$, so $M\backslash 3,6$ is 3-connected. The last matroid has $\{5,7,8,9\}$ as a fan. Since M has $\{3,5,7\}$ and $\{4,5,6\}$ as its only triangles containing 5, it follows that 5 is a fan end in $M\backslash 3,6$, so $M\backslash 3,6/5$ is 3-connected. As $M/5$ has 4 and 6 in parallel, we deduce that $M\backslash 3,4/5$ is 3-connected.

Suppose that $M\backslash 3,4/5$ is not internally 4-connected. Then this matroid has a $(4,3)$-violator (X,Y). Thus $M/5$ has (X',Y') as a 3-separation where we adjoin 3 to whichever of X and Y contains 7, and 4 to whichever of X and Y contains 6. Then $r(X'\cup 5)+r(Y'\cup 5)=r(M)+3$. If X' or Y', say X', contains $\{6,7,8\}$ or $\{2,3,4\}$, then $(X'\cup 5,Y')$ is a 3-separation of M; a contradiction. Thus, as $M/5$ has $\{3,7\}$ and $\{4,6\}$ as circuits, we may assume that neither X nor Y contains $\{6,7,8\}$ or $\{2,6,7\}$.

Suppose next that $|X\cap\{2,6,7,8\}|=3$ and let $Y\cap\{2,6,7,8\}=\{y\}$. Then $y\in\{6,7\}$ and, as $(X,Y)\cong(X\cup y,Y-y)$, we reduce to the case treated in the last paragraph unless Y is a 4-element fan in $M\backslash 3,4/5$ having $\{y_2,y_3,y\}$ as a triad and $\{y_1,y_2,y_3\}$ as a triangle. If $y=6$, then, since Y avoids $\{2,8\}$, we deduce, by orthogonality with the triangles $\{2,6,11\}$ and $\{6,8,10\}$ of $M\backslash 3,4/5$, that $\{y_2,y_3\}=\{10,11\}$. It follows that M has $\{4,6,10,11\}$ as a cocircuit; a contradiction to (6.7.1). Hence $y\neq 6$, so $y=7$. Then orthogonality with the triangles $\{1,2,7\}$ and $\{7,8,9\}$ of $M\backslash 3,4/5$ implies that $\{y_2,y_3\}=\{1,9\}$. It follows that $\{3,1,7,9\}$ is a cocircuit of M; a contradiction to Lemma 6.4.

We may now assume that $|X\cap\{2,6,7,8\}|=2=|Y\cap\{2,6,7,8\}|$. Then, by symmetry, we may suppose that either

(a) $\{6,7\}\subseteq X$ and $\{2,8\}\subseteq Y$; or
(b) $\{7,8\}\subseteq X$ and $\{2,6\}\subseteq Y$.

Consider case (a) and suppose first that $9\in X$. Then $(X,Y)\cong(X\cup 8,Y-8)$, so we have reduced to an earlier case unless Y is a 4-element fan of $M\backslash 3,4/5$ having $\{y_1,y_2,y_3\}$ as a triad and $\{y_2,y_3,8\}$ as a triangle. Since this triangle avoids $\{5,6,7\}$, orthogonality with the cocircuits $\{5,6,7,8\}$ and $\{2,3,4,5\}$ of M implies that $\{y_2,y_3,8,5\}$ is a circuit of M and $2\in\{y_2,y_3\}$. As $\{y_1,y_2,y_3\}$ is a triad of $M\backslash 3,4/5$, we deduce from the triangles $\{1,2,7\}$ and $\{2,6,11\}$ of $M\backslash 3,4/5$ that $\{y_1,y_2,y_3\}=\{1,2,11\}$. The triangles $\{1,2,3\}$ and $\{2,4,11\}$ of M imply that $\{1,2,11\}$ is a triad of M; a contradiction.

Still in case (a), we may now assume that $9\in Y$ and, by symmetry, that $1\in Y$. Then $(X-7,Y\cup 7)\cong(X,Y)$ and we reduce to an earlier case unless X is a 4-element fan of $M\backslash 3,4/5$. In the exceptional case, X contains a triangle $\{x_2,x_3,7\}$ and a triad $\{x_1,x_2,x_3\}$. Thus $\{x_2,x_3,7\}$ is a triangle of M, otherwise $\{x_2,x_3,7,5\}$ is a circuit of M having a single common element with the cocircuit $\{2,3,4,5\}$. By orthogonality with the cocircuit $\{5,6,7,8\}$ of M, we deduce that $6 \in \{x_2, x_3\}$, so we can take $x_3 = 6$. Then $\{x_1, x_2, 6\}$ is a triad of $M\backslash 3, 4/5$ avoiding $\{2,8\}$. Hence $\{x_1, x_2\} = \{10, 11\}$, so $\{4, 6, 10, 11\}$ is a cocircuit of M, contradicting (6.7.1).

Finally, consider case (b). Suppose first that $1 \in X$. Then X contains $\{7,8,1\}$ and Y contains $\{2,6\}$, so $(X \cup 2, Y - 2) \cong (X,Y)$. Hence we reduce to an earlier case unless Y is a 4-element fan of $M\backslash 3, 4/5$ containing $\{y_2, y_3, 2\}$ as a triangle and $\{y_1, y_2, y_3\}$ as a triad. Moreover, $6 \in \{y_1, y_2, y_3\}$. Then the triangles $\{2,6,11\}$ and $\{6,8,10\}$ of $M\backslash 3, 4/5$ imply that $\{6,10,11\}$ is a cocircuit of $M\backslash 3, 4$. Hence $\{4,6,10,11\}$ is a cocircuit of M; a contradiction to (6.7.1). We may now assume that $1 \in Y$. Then $(X,Y) \cong (X - 7, Y \cup 7)$ and we have reduced to an earlier case unless $M\backslash 3, 4/5$ has $X - 7$ as a triad. In the exceptional case, $X - 7$ contains 8 but avoids 6 and 7. Thus $X - 7$ contains 9 and 10. Hence $\{8,9,10\}$ is a triad of $M\backslash 3, 4/5$. By orthogonality, this set is also a triad of M; a contradiction. We are now able to conclude that $M\backslash 3, 4/5$ is internally 4-connected. □

7. An $M(K_4)$-restriction

In this section, we shall prove the main result when M has an $M(K_4)$-restriction but has no quasi rotor.

Theorem 7.1. *Let M be a binary internally 4-connected matroid having at least 13 elements. Assume that M contains no quasi rotor and that M has a restriction isomorphic to $M(K_4)$. Then M has a proper internally 4-connected minor M' with $|E(M)| - |E(M')| \le 2$.*

Proof. We may assume that no single-element deletion of M is internally 4-connected. Let N be a restriction of M that is isomorphic to $M(K_4)$.

Lemma 7.2. *No cocircuit of M is contained in $E(N)$.*

Proof. We know $r_M(E(N)) = 3$. If $E(N)$ contains a cocircuit of M, then $r^*_M(E(N)) \le 5$, so $\lambda_M(E(N)) \le 2$. Hence $|E(M)| \le 9$; a contradiction. □

We now label N so that its triangles are $\{1,2,3\}$, $\{1,5,6\}$, $\{2,4,6\}$, and $\{3,4,5\}$. By Theorem 5.1, since no triangle of M is the central triangle of a quasi rotor, each triangle contains an element e such that $M\backslash e$ is $(4,4,S)$-connected.

Lemma 7.3. *If $M\backslash 1$ is $(4,4,S)$-connected, then M has a 4-cocircuit that meets $E(N)$ in either $\{1,3,5\}$ or $\{1,2,6\}$. Moreover, $M\backslash 1$ has at most two 4-element fans, including at most one containing $\{3,4,5\}$ and at most one containing $\{2,4,6\}$.*

Proof. Clearly $M\backslash 1$ has a 4-element fan $\{e_1, e_2, e_3, e_4\}$ where $\{e_1, e_2, e_3\}$ is a triangle and $\{e_2, e_3, e_4\}$ is a triad. As M is internally 4-connected, $\{e_2, e_3, e_4, 1\}$ is a cocircuit of M. By orthogonality with the circuits $\{1,5,6\}$ and $\{1,2,3\}$ of M, it follows that $\{e_2, e_3, e_4\}$ contains exactly one of 5 and

6 and contains exactly one of 2 and 3. If $\{e_2, e_3, e_4\}$ contains $\{2, 5\}$, then it must contain 4. Hence M has $\{1, 2, 4, 5\}$ as a cocircuit contradicting Lemma 7.2. By symmetry, we deduce that $\{e_2, e_3, e_4\}$ contains either $\{3, 5\}$ or $\{2, 6\}$. In each case, $4 \in \text{cl}(\{e_1, e_2, e_3, e_4\})$. As $M\backslash 1$ is $(4, 4, S)$-connected, it follows that $4 \in \{e_1, e_2, e_3, e_4\}$. Thus $\{e_1, e_2, e_3\}$ is $\{3, 4, 5\}$ or $\{2, 4, 6\}$, so $e_1 = 4$, and the lemma follows without difficulty. □

Lemma 7.4. *Either*

(i) N has a triangle T such that $M\backslash e$ is $(4, 4, S)$-connected for each e in T and T can be labelled $\{a, b, c\}$ so that M has 4-element cocircuits that contain $\{a, c\}$ and $\{b, c\}$; or

(ii) N has a matching $\{a, b\}$ and another element c such that $M\backslash e$ is $(4, 4, S)$-connected for each e in $\{a, b\}$ and M has 4-element cocircuits that contain $\{a, c\}$ and $\{b, c\}$.

Proof. Let S be the set of elements s of $E(N)$ for which $M\backslash s$ is $(4, 4, S)$-connected. Since every triangle of N contains a member of S, either

(a) S includes the edges of a triangle of N; or
(b) S includes the edges of a matching in (the graph associated with) N.

Then, by repeatedly applying Lemma 7.3, we deduce that (i) or (ii) holds. □

By the last lemma and symmetry, we may assume that:

7.5. *Either*

(A) $M\backslash 1$ and $M\backslash 4$ are $(4, 4, S)$-connected; or

(B) $M\backslash 1$, $M\backslash 2$, and $M\backslash 3$ are $(4, 4, S)$-connected.

Moreover, M has 4-cocircuits $\{1, 3, 5, 7\}$ and $\{2, 3, 4, 8\}$ where $|\{1, 2, \ldots, 8\}| = 8$.

Lemma 7.6. *Let (X_1, Y_1) be a $(4, 3)$-violator of $M\backslash 1$. Then, after a possible permutation of X_1 and Y_1,*

(i) $\{2, 6\} \subseteq X_1$ and $\{3, 5\} \subseteq Y_1$; and

(ii) either X_1 is a 4-element fan of $M\backslash 1$ containing $\{2, 4, 6\}$ and M has a 4-cocircuit containing $\{1, 2, 6\}$; or Y_1 is a 4-element fan of $M\backslash 1$ containing $\{3, 4, 5\}$ and M has a 4-cocircuit containing $\{1, 3, 5\}$.

Proof. As M has $\{1, 3, 5, 7\}$ as a cocircuit, $M\backslash 1$ has $\{4, 3, 5, 7\}$ as a fan which, since $M\backslash 1$ is $(4, 4, S)$-connected, must be maximal. Hence M has no triangle containing $\{3, 7\}$ or $\{5, 7\}$. Thus, by orthogonality with the cocircuit $\{1, 3, 5, 7\}$, the only triangles of M containing 3 or 5 are $\{1, 2, 3\}$, $\{3, 4, 5\}$, and $\{1, 5, 6\}$. As $M\backslash 1$ is $(4, 4, S)$-connected, it has X_1 or Y_1 as a 4-element fan. Without loss of generality, $2 \in X_1$ and $3 \in Y_1$. Moreover, either

(a) $5 \in X_1$ and $6 \in Y_1$; or
(b) $6 \in X_1$ and $5 \in Y_1$.

Suppose that (a) occurs. Then $M\backslash 1$ has a 4-element fan F such that either (I) $\{2,5\} \subseteq F$ and $\{1,3,6\} \cap F = \emptyset$; or (II) $\{3,6\} \subseteq F$ and $\{1,2,5\} \cap F = \emptyset$. These cases are symmetric, so we may assume that (I) holds. Now F contains a triangle of M. But this triangle cannot contain 5. Thus F has $(f_1, f_2, f_3, 5)$ as a fan ordering where $\{f_2, f_3, 5\}$ is a triad of $M\backslash 1$. The triangle $\{3,4,5\}$ of $M\backslash 1$ implies that $4 \in \{f_2, f_3\}$. Since $2 \in \{f_1, f_2, f_3\}$, we deduce that the triangle $\{f_1, f_2, f_3\}$ must be $\{2,4,6\}$ as M has a unique triangle containing $\{2,4\}$. Since $6 \notin F$, this is a contradiction. We conclude that (a) cannot occur, so (b) holds; that is, $\{2,6\} \subseteq X_1$ and $\{3,5\} \subseteq Y_1$.

If Y_1 is a 4-element fan of $M\backslash 1$, then, as $\{4,7\} \subseteq \mathrm{fcl}_{M\backslash 1}(Y_1)$ and $M\backslash 1$ is $(4,4,S)$-connected, it follows that $Y_1 = \{3,4,5,7\}$. Hence, (ii) holds when Y_1 is a 4-element fan.

If X_1 is a 4-element fan of $M\backslash 1$, then, since $4 \in \mathrm{cl}(X_1)$ and $M\backslash 1$ is $(4,4,S)$-connected, $\{2,4,6\} \subseteq X_1$. The triad T^* of $M\backslash 1$ that is contained in X_1 includes 2 or 6. In the latter case, T^* contains $\{2,6\}$ or $\{4,6\}$. But if T^* contains 4, it also contains 3 or 5; a contradiction. Thus T^* contains $\{2,6\}$, so $T^* = \{2,6,9\}$ for some element 9. Hence $\{1,2,6,9\}$ is a cocircuit of M and (ii) holds. □

Recall, by (7.5), that M has $\{1,3,5,7\}$ and $\{2,3,4,8\}$ as cocircuits.

Lemma 7.7. *One of the following holds:*

(i) $M\backslash 5$ *is* $(4,4,S)$*-connected;*
(ii) M *has a triangle* $\{y_1, y_2, 6\}$ *and a cocircuit* $\{y_2, 6, 4, 5\}$ *where* $\{y_1, y_2\} \cap \{1, 2, \dots 8\} = \emptyset$*, and* $M\backslash 4$ *is not* $(4,4,S)$*-connected; or*
(iii) M *has a triangle containing* $\{1,7\}$ *and avoiding* $\{2,3,4,5,6,8\}$ *and* $M\backslash 3$ *is not* $(4,4,S)$*-connected.*

Proof. Suppose that $M\backslash 5$ is not $(4,4,S)$-connected and let (X_5, Y_5) be a $(4,4,S)$-violator for $M\backslash 5$. Then, without loss of generality, $1 \in X_5$ and $6 \in Y_5$. Moreover, either

(a) $3 \in X_5$ and $4 \in Y_5$; or
(b) $4 \in X_5$ and $3 \in Y_5$.

Assume that (b) holds. If $7 \in X_5$, then $(X_5 \cup 3, Y_5 - 3) \cong (X_5, Y_5)$ and, as (X_5, Y_5) is a $(4,4,S)$-violator, $|Y_5 - 3| \geq 4$. Hence $(X_5 \cup 3 \cup 5, Y_5 - 3)$ is a 3-separator of M, a contradiction. Thus we may assume that $7 \in Y_5$. Then $(X_5 - 1, Y_5 \cup 1) \cong (X_5, Y_5)$ and again we get the contradiction that M is not internally 4-connected. Hence (b) does not hold.

We may now assume that $\{1,3\} \subseteq X_5$ and $\{4,6\} \subseteq Y_5$. Consider the location of the elements 2 and 8. Suppose $\{2,8\} \subseteq Y_5$. Then $(X_5 - 3, Y_5 \cup 3) \cong (X_5, Y_5)$. But $5 \in \mathrm{cl}(Y_5 \cup 3)$, so M is not internally 4-connected; a contradiction. Similarly, if $\{2,8\} \subseteq X_5$, then $(X_5 \cup 4, Y_5 - 4) \cong (X_5, Y_5)$ and, as $5 \in \mathrm{cl}(X_5 \cup 4)$, we again get a contradiction.

Now suppose that $8 \in X_5$ and $2 \in Y_5$. Then $(X_5, Y_5) \cong (X_5 \cup 2, Y_5 - 2) \cong (X_5 \cup 2 \cup 4, Y_5 - 2 - 4)$. As M is internally 4-connected, it follows that $Y_5 - 2$ is a 4-element fan of $M\backslash 5$ having $(y_1, y_2, y_3, 4)$ as a fan ordering where $\{y_2, y_3, 4\}$

is a triad. By orthogonality, $y_3 = 6$. Hence $\{y_2, 6, 4, 5\}$ is a cocircuit of M and $\{y_1, y_2, 6\}$ is a triangle of M. Thus $M\backslash 4$ has $\{y_1, y_2, 6, 5, 1\}$ as a fan, so $M\backslash 4$ is not $(4, 4, S)$-connected, and (ii) holds.

Finally, suppose that $2 \in X_5$ and $8 \in Y_5$. Then $(X_5, Y_5) \cong (X_5 - 2, Y_5 \cup 2) \cong (X_5 - 2 - 3, Y_5 \cup 2 \cup 3)$, so, by Lemma 5.3, $X_5 - 2$ is a 4-element fan of $M\backslash 5$ having $(x_1, x_2, x_3, 3)$ as an ordering where $\{x_2, x_3, 3\}$ is a triad. Hence $x_3 = 1$ and $\{x_2, 1, 3, 5\}$ is a cocircuit of M, so $x_2 = 7$. Thus M has $\{x_1, 1, 7\}$ as a triangle avoiding $\{2, 3, 4, 5, 6\}$ and hence avoiding 8. Therefore $M\backslash 3$ has $\{x_1, 1, 7, 5, 6\}$ as a fan so $M\backslash 3$ is not $(4, 4, S)$-connected, and (iii) holds. □

In (7.5), we distinguished two cases. The next two lemmas complete the proof of Theorem 7.1 in case (A).

Lemma 7.8. *Suppose $M\backslash 1$ and $M\backslash 4$ are $(4, 4, S)$-connected. Then either*

(i) $M\backslash 1, 4$ is internally 4-connected; or

(ii) M has a triangle $\{6, y_1, y_2\}$ and a cocircuit $\{1, 3, 4, 6, y_2\}$ where $\{y_1, y_2\}$ avoids $\{1, 2, \dots, 8\}$.

Proof. Consider $M\backslash 1, 4$. As $M\backslash 1$ has $\{3, 4, 5, 7\}$ as a maximal fan and this fan has 4 as an end, $M\backslash 1, 4$ is 3-connected. Assume it is not internally 4-connected. Then it has a $(4, 3)$-violator (X_{14}, Y_{14}).

7.8.1. *Neither X_{14} nor Y_{14} contains $\{3, 5\}$ or $\{2, 6\}$.*

Assume X_{14} contains $\{3, 5\}$ or $\{2, 6\}$. Then $(X_{14} \cup 4, Y_{14})$ is a 3-separation of $M\backslash 1$ with $|X_{14} \cup 4| \geq 5$ and $|Y_{14}| \geq 4$. But neither $X_{14} \cup 4$ nor Y_{14} is a 4-element fan containing $\{2, 4, 6\}$ or $\{3, 4, 5\}$. This contradicts Lemma 7.6. Thus (7.8.1) holds.

By (7.8.1) and symmetry, we have

(a) $\{2, 3\} \subseteq X_{14}$ and $\{5, 6\} \subseteq Y_{14}$; or
(b) $\{6, 3\} \subseteq X_{14}$ and $\{5, 2\} \subseteq Y_{14}$.

In the former case, $(X_{14} \cup 1, Y_{14})$ and $(X_{14}, Y_{14} \cup 1)$ are 3-separations of $M\backslash 4$. Since $|X_{14}|$ or $|Y_{14}|$ is at least 5, we contradict the fact that $M\backslash 4$ is $(4, 4, S)$-connected. We deduce that $\{6, 3\} \subseteq X_{14}$ and $\{5, 2\} \subseteq Y_{14}$, that is, (b) holds. Next, we consider the location of 7 and 8.

Suppose $7 \in X_{14}$ and $8 \in Y_{14}$. Then the cocircuits $\{3, 5, 7\}$ and $\{2, 3, 8\}$ imply that $(X_{14}, Y_{14}) \cong ((X_{14} \cup 5) - 3, (Y_{14} - 5) \cup 3)$. But $\{5, 6\} \subseteq (X_{14} \cup 5) - 3$ and $\{2, 3\} \subseteq (Y_{14} - 5) \cup 3$, so we have returned to case (a) and thereby get a contradiction.

If $7 \in Y_{14}$ and $8 \in X_{14}$, then $(X_{14}, Y_{14}) \cong ((X_{14} \cup 2) - 3, (Y_{14} - 2) \cup 3)$. But $\{2, 6\} \subseteq (X_{14} \cup 2) - 3$, so we get a contradiction by (7.8.1).

Suppose $\{7, 8\} \subseteq X_{14}$. Then both 2 and 5 are in $\mathrm{cl}^*_{M\backslash 1,4}(X_{14})$. Hence $|Y_{14}| \geq 5$. Thus $(X_{14} \cup 5 \cup 1 \cup 4, Y_{14} - 5)$ is a 3-separation of M with $|Y_{14} - 5| \geq 4$; a contradiction.

Finally, suppose $\{7, 8\} \subseteq Y_{14}$. Then $(X_{14} - 3, Y_{14} \cup 3) \cong (X_{14}, Y_{14})$. As $\{1, 4\} \subseteq \mathrm{cl}(Y_{14} \cup 3)$, we deduce that $|X_{14}| = 4$ otherwise M is not internally 4-connected. Thus X_{14} is a 4-element fan of $M\backslash 1, 4$ containing a triangle

$\{y_1, y_2, y_3\}$ avoiding $\{1,2,3,4,5,7,8\}$ and a triad $\{y_2, y_3, 3\}$. The circuit $\{2,3,5,6\}$ of $M\backslash 1,4$ implies that $6 \in \{y_2, y_3\}$, say $6 = y_3$. Since $\{1,2,\ldots,6\}$ does not contain a cocircuit of M, the element y_2 is not in $\{1,2,\ldots,6\}$. The circuits $\{3,4,5\}$ and $\{1,6,5\}$ imply that $\{y_2, 6, 3, 4, 1\}$ is a cocircuit of M. Thus (ii) holds. □

Lemma 7.9. *Suppose $M\backslash 1$ and $M\backslash 4$ are $(4,4,S)$-connected. Then $M\backslash 1,4$ or $M\backslash 2,5$ is internally 4-connected.*

Proof. Assume that $M\backslash 1,4$ is not internally 4-connected. Then, by Lemma 7.8, M has a triangle $\{6, y_1, y_2\}$ and a cocircuit $\{1,3,4,6,y_2\}$ where $\{y_1, y_2\}$ avoids $\{1,2,\ldots,8\}$. Suppose (iii) of Lemma 7.7 holds. Then M has a triangle $\{1,7,a\}$ that avoids $\{2,3,4,5,6,8\}$. By orthogonality with the cocircuit $\{1,3,4,6,y_2\}$, we deduce that $a = y_2$. Consider $Z = \{1,2,3,4,5,6,7,y_2\}$. Then $r(Z) \le 4$ and $r^*(Z) \le 6$, so $\lambda(Z) \le 2$; a contradiction to the fact that $|E(M)| \ge 13$. Thus (iii) of Lemma 7.7 does not hold. Moreover, (ii) of Lemma 7.7 does not hold as $M\backslash 4$ is $(4,4,S)$-connected. Hence (i) of Lemma 7.7 holds, that is, $M\backslash 5$ is $(4,4,S)$-connected.

Now, by applying the permutation $(7,8)(2,5)(1,4)$ of $\{1,2,\ldots,8\}$ to Lemma 7.7, we deduce that either $M\backslash 2$ is $(4,4,S)$-connected, or M has a triangle containing $\{4,8\}$ and avoiding $\{1,2,3,5,6,7\}$. In the latter case, the triangle $\{4,8,b\}$ and the cocircuit $\{1,3,4,6,y_2\}$ imply that $y_2 = b$. Then, for $Z = \{1,2,3,4,5,6,8,y_2\}$, we have $\lambda(Z) \le 2$; a contradiction. We conclude that $M\backslash 2$ is $(4,4,S)$-connected.

Since both $M\backslash 2$ and $M\backslash 5$ are $(4,4,S)$-connected, by applying the permutation $(1,2)(4,5)(7,8)$ to $\{1,2,\ldots,8\}$, we deduce from Lemma 7.8 that $M\backslash 2,5$ is internally 4-connected, or M has a cocircuit $\{2,3,5,6,z_2\}$ and a triangle $\{6,z_1,z_2\}$ where $\{z_1,z_2\} \cap \{1,2,\ldots,8\} = \emptyset$. We may assume that $M\backslash 2,5$ is not internally 4-connected. Then the symmetric difference of the cocircuits $\{1,3,5,7\}$, $\{2,3,4,8\}$, $\{1,3,4,6,y_2\}$, and $\{2,3,5,6,z_2\}$ is the disjoint union of a set of cocircuits of M. This symmetric difference is $\{7,8\} \cup (\{y_2\} \triangle \{z_2\})$, so it equals $\{7,8,y_2,z_2\}$, a cocircuit of M. By orthogonality with the triangle $\{6,z_1,z_2\}$, we deduce that $z_1 = y_2$. Similarly, $z_2 = y_1$.

Now $M\backslash y_1$ and $M\backslash y_2$ have $\{2,3,5,6\}$ and $\{1,3,4,6\}$, respectively, as quads. Thus neither $M\backslash y_1$ nor $M\backslash y_2$ is $(4,4,S)$-connected. Hence, as $\{y_1,y_2,6\}$ is not the central triangle of a quasi rotor, $M\backslash 6$ is $(4,4,S)$-connected. Thus M has a 4-cocircuit containing $\{4,5,6\}$ or $\{1,2,6\}$. Under the permutation $(7,8)(2,5)(1,4)$ of $\{1,2,\ldots,8\}$, these two possibilities are symmetric. Thus we let this cocircuit be $\{4,5,6,x\}$. Then the symmetric difference of $\{1,3,4,6,y_2\}$, $\{4,5,6,x\}$, and $\{1,3,5,7\}$ is $\{y_2\} \triangle \{x\} \triangle \{7\}$ and so equals $\{y_2,x,7\}$, a triad of M. But $\{6,y_1,y_2\}$ is a triangle of M, so y_2 is in a 4-element fan of M; a contradiction. □

The next lemma completes the proof of Theorem 7.1 in case (B).

Lemma 7.10. *Suppose that each of $M\backslash 1$, $M\backslash 2$, and $M\backslash 3$ is $(4,4,S)$-connected. Then $M\backslash 1,4$ or $M\backslash 2,5$ or $M\backslash 6,3$ is internally 4-connected.*

Proof. By Lemma 7.9 and symmetry, if $M\backslash 5$ is $(4,4,S)$-connected, the lemma holds. Hence we may assume that $M\backslash 5$ is not $(4,4,S)$-connected. Then, by Lemma 7.7, M has a triangle $\{y_1, y_2, 6\}$ and a cocircuit $\{y_2, 6, 4, 5\}$ where $\{y_1, y_2\} \cap \{1, 2, \dots, 8\} = \emptyset$.

Now $M\backslash 3$ has $\{7, 1, 5, 6\}$ as a maximal fan with 6 as an end, so $M\backslash 3, 6$ is 3-connected. Moreover, by Lemma 7.3, $M\backslash 3$ has exactly two 4-element fans, $\{1, 5, 6, 7\}$ and $\{2, 4, 6, 8\}$. Suppose $M\backslash 3, 6$ is not internally 4-connected and let (X_{36}, Y_{36}) be a $(4,3)$-violator of it. If $\{1, 5\}$ or $\{2, 4\}$ or $\{y_1, y_2\}$ is contained in X_{36}, then $(X_{36} \cup 6, Y_{36})$ is a 3-separation of $M\backslash 3$ but neither $X_{36} \cup 6$ nor Y_{36} is $\{1, 5, 6, 7\}$ or $\{2, 4, 6, 8\}$. Hence we may assume that exactly one of y_1 and y_2 is in each of X_{36} and Y_{36}, and either

(i) $\{1, 4\} \subseteq X_{36}$ and $\{2, 5\} \subseteq Y_{36}$; or
(ii) $\{1, 2\} \subseteq X_{36}$ and $\{4, 5\} \subseteq Y_{36}$.

Assume (i) holds and suppose $y_2 \in Y_{36}$. Then $(X_{36} - 4, Y_{36} \cup 4) \cong (X_{36}, Y_{36})$ so $(X_{36} - 4, Y_{36} \cup 4 \cup 6)$ is a 3-separation of $M\backslash 3$ with $\{1, y_1\} \subseteq X_{36} - 4$. Then neither $X_{36} - 4$ nor $Y_{36} \cup 4 \cup 6$ is $\{1, 5, 6, 7\}$ or $\{2, 4, 6, 8\}$ so $|X_{36} - 4| = 3$. Then X_{36} is a 4-element fan of $M\backslash 3, 6$ having a fan ordering of the form $(g_1, g_2, g_3, 4)$ where $\{g_1, g_2, g_3\}$ is a triangle of M containing 1. This triangle is not $\{1, 5, 6\}$ or $\{1, 2, 3\}$ so it contains $\{1, 7\}$. Then $M\backslash 3$ has a 5-element fan; a contradiction.

Now suppose (i) holds but $y_2 \in X_{36}$. Then $(X_{36} \cup 5, Y_{36} - 5) \cong (X_{36}, Y_{36})$ and $\{2, y_1\} \subseteq Y_{36} - 5$. Then, as in the last paragraph, we get that M has a triangle containing 2 that is different from $\{2, 3, 1\}$ and $\{2, 4, 6\}$. Hence this triangle contains $\{2, 8\}$ and so $M\backslash 3$ is not $(4,4,S)$-connected.

We may now assume that (ii) holds. Suppose $7 \in Y_{36}$. Then $(X_{36} - 1, Y_{36} \cup 1) \cong (X_{36}, Y_{36})$. Again M has a triangle containing 2 and y_1 or y_2, so this triangle contains 8 and $M\backslash 3$ is not $(4,4,S)$-connected. If $7 \in X_{36}$, then $(X_{36} \cup 5, Y_{36} - 5) \cong (X_{36}, Y_{36})$. This time, M has a triangle containing 4 and y_1 or y_2. This triangle must also contain 8 and again $M\backslash 3$ is not $(4,4,S)$-connected. □

In (7.5), we noted that (A) or (B) may be assumed to occur. The last two lemmas establish that Theorem 7.1 holds in these two cases, so the theorem is proved. □

8. Building Structure

In this section, we establish a number of lemmas that are basic tools for building structure in a binary internally 4-connected matroid. The first of these begins with the structure shown in Figure 3.

Lemma 8.1. *In a binary internally 4-connected matroid M, assume that $\{1, 2, 3\}$ and $\{3, 4, 5\}$ are triangles, and $\{2, 3, 4, 6\}$ and $\{1, 3, 5, 7\}$ are cocircuits. Suppose $|E(M)| \geq 11$. Then*

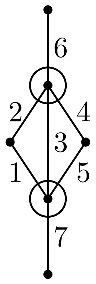

FIGURE 3. The initial configuration in Lemma 8.1.

(i) *M has a triangle containing* 6 *and exactly one of* 2 *and* 4*, but avoiding* 3*; or*
(ii) $M/6$ *is internally* 4*-connected; or*
(iii) M *has a circuit* $\{y_2, y_3, 4, 6\}$ *and a triad* $\{y_1, y_2, y_3\}$ *where* $y_1, y_2, y_3, 1, 2, \dots, 7$ *are distinct except that, possibly,* $y_1 = 7$*; or*
(iv) M *has a circuit* $\{3, 6, 7, y_2\}$ *and a triad* $\{7, y_2, y_1\}$ *where* $|\{1, 2, \dots, 7, y_1, y_2\}| = 9$*; or*
(v) M *has a circuit* $\{x_2, x_3, 2, 6\}$ *and a triad* $\{x_1, x_2, x_3\}$ *where* $x_1, x_2, x_3, 1, 2, \dots, 7$ *are distinct except that, possibly,* $x_1 = 7$.

Proof. First observe that, since M has no 4-element fans, none of $1, 2, 3, 4, 5$ is in a triad. Suppose that M has a triangle T containing 6. Then $|\{2, 3, 4\} \cap T| = 1$. If $3 \in T$, then $7 \in T$, and $\lambda(\{1, 2, \dots, 7\}) \le 2$; a contradiction since $|E(M)| \ge 11$. Thus T contains 2 or 4, and, since M is binary, $|T \cap \{2, 4\}| = 1$.

Assume next that M has no triangles containing 6. Now $M\backslash 2$ has $\{3, 4, 5, 6\}$ as a fan, and 6 is an end of this fan because 6 is in no triangles. Thus $M\backslash 2/6$ is 3-connected. Hence so is $M/6$. Assume that $M/6$ is not internally 4-connected letting (X, Y) be a $(4, 3)$-violator of it. Then neither X nor Y contains $\{2, 3, 4\}$. Thus we have the following three cases to consider:

(a) $\{2, 3\} \subseteq X$ and $4 \in Y$;
(b) $\{2, 4\} \subseteq X$ and $3 \in Y$;
(c) $\{3, 4\} \subseteq X$ and $2 \in Y$.

Consider case (a). Suppose first that $5 \in X$. Then $(X, Y) \cong (X \cup 4, Y - 4)$. Thus we may assume that Y is a 4-element fan of $M/6$ containing a triangle $\{y_2, y_3, 4\}$ and a triad $\{y_1, y_2, y_3\}$. Then $\{y_2, y_3, 4, 6\}$ is a circuit of M and $\{y_1, y_2, y_3\}$ is a triad. We know that $\{y_1, y_2, y_3\} \cap \{1, 2, 3, 4, 5\} = \emptyset$, and $6 \notin \{y_1, y_2, y_3\}$. By orthogonality, $7 \notin \{y_2, y_3\}$, but possibly $y_1 = 7$. Thus (iii) holds.

We may now assume, in case (a), that $5 \in Y$. If $1 \in Y$, then $(X \cup 1, Y - 1) \cong (X, Y)$. If $|Y - 1| = 3$, then $M/6$, and hence M, has a triad meeting $\{4, 5\}$; a contradiction. Thus $|Y - 1| \ge 4$, so we may assume that $1 \in X$. Suppose $7 \in X$. Then $(X, Y) \cong (X \cup 5, Y - 5)$ and we have reduced to

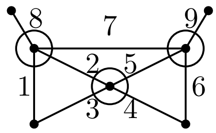

FIGURE 4. The initial configuration in Lemma 8.2.

a previous case unless $|Y - 5| = 3$. In the exceptional case, by the dual of Lemma 2.8, the element 5 is in a triad of M; a contradiction. We may now assume that $7 \in Y$. Then $\{2,3,1\} \subseteq X$ and $\{4,5,7\} \subseteq Y$. Hence $(X,Y) \cong (X-3, Y \cup 3) \cong (X-3-1, Y \cup 3 \cup 1)$. As $1 \in \text{cl}^*_{M/6}(Y \cup 3)$, it follows by Lemmas 2.9 and 2.8 that $|X-3| \geq 6$ otherwise 1 is in a triad of M. Hence $(X,Y) \cong (X-3-1-2, Y \cup 3 \cup 1 \cup 2)$; a contradiction. This completes the proof of case (a).

Consider case (b), that is, $\{2,4\} \subseteq X$ and $3 \in Y$. If 1 or 5 is in X, then $(X,Y) \cong (X \cup 3, Y-3)$, so Y is a 4-element fan of $M/6$ containing a triangle $\{y_2, y_3, 3\}$ and a triad $\{y_1, y_2, y_3\}$. Now $\{y_1, y_2, y_3\}$ avoids $\{1,2,3,4,5\}$ and $\{y_2, y_3, 3, 6\}$ is a circuit of M. Hence, by orthogonality, $7 \in \{y_2, y_3\}$, so we may take $y_3 = 7$. Then (iv) holds. We may now assume that Y contains $\{1,5\}$. Hence $\{2,4\} \subseteq X$ and $\{1,3,5\} \subseteq Y$. Then $\{2,4\} \subseteq \text{cl}_{M/6}(Y)$. Hence $|X| \geq 5$ and, as $(X,Y) \cong (X-2, Y \cup 2)$, we have reduced to a case symmetric to case (a).

Finally, assume that case (c) occurs, that is, $\{3,4\} \subseteq X$ and $2 \in Y$. Then, by symmetry with case (a), we deduce that (v) holds. □

In the next lemma, we begin with the structure in Figure 4.

Lemma 8.2. *In a binary internally 4-connected matroid M, assume that $(\{1,2,3\}, \{4,5,6\}, \{2,3,4,5\})$ is a bowtie, and that $\{2,5,7\}$ is a triangle and $\{1,2,7,8\}$ and $\{5,6,7,9\}$ are cocircuits. Assume that M has no $M(K_4)$-restriction and $|E(M)| \geq 13$. Then $|\{1,2,\ldots,9\}| = 9$. Moreover,*

(i) *M has a triangle containing $\{1,8\}$; or*
(ii) *$M/8$ is internally 4-connected; or*
(iii) *M has a circuit $\{y_2, 9, 7, 8\}$ and a triad $\{y_1, y_2, 9\}$ where $|\{1,2,\ldots,9,y_1,y_2\}| = 11$; or*
(iv) *M has a circuit $\{x_2, x_3, 1, 8\}$ and a triad $\{x_1, x_2, x_3\}$ where $x_1, x_2, x_3, 1, 2, \ldots, 9$ are distinct except that, possibly, $x_1 = 9$.*

Proof. Clearly $|\{1,2,\ldots,6\}| = 6$. Moreover, $\{7,8,9\} \cap \{1,2,\ldots,6\} = \emptyset$ by orthogonality. Thus $|\{1,2,\ldots,9\}| = 9$ otherwise $8 = 9$ and $\lambda(\{1,2,\ldots,8\}) \leq 2$; a contradiction. We now apply the preceding lemma to the triangles $\{1,2,3\}$ and $\{2,5,7\}$ and the cocircuits $\{1,2,7,8\}$ and $\{2,3,4,5\}$. If (i) of that lemma holds, then M has a triangle T containing 8. If T contains 7, then it contains 5, 6, or 9, so $\lambda(\{1,2,\ldots,9\}) \leq 2$; a contradiction. Hence T must contain 1, so (i) holds. If (ii) of the last lemma holds, then (ii) of this lemma holds. If (iii) of the last lemma holds, then M has a circuit $\{y_2, y_3, 7, 8\}$ and a triad $\{y_1, y_2, y_3\}$, where $y_1, y_2, y_3, 1, 2, 3, 4, 5, 7, 8$ are distinct since $y_1 \neq 4$ as M has no 4-element fans. The triangle $\{4, 5, 6\}$ implies that $6 \notin \{y_1, y_2, y_3\}$. The cocircuit $\{5, 6, 7, 9\}$ and circuit $\{y_2, y_3, 7, 8\}$ imply that $9 \in \{y_2, y_3\}$. Thus we may take $9 = y_3$. Then $|\{y_1, y_2, 1, 2, \dots, 9\}| = 11$ and (iii) holds.

If part (iv) of the last lemma holds, then 4 is in a triad of M, which is not so. If part (v) of the last lemma holds, then M has a circuit $\{x_2, x_3, 1, 8\}$ and a triad $\{x_1, x_2, x_3\}$ where $x_1, x_2, x_3, 1, 2, 3, 4, 5, 7, 8$ are distinct as $x_1 \neq 4$. Also $6 \notin \{x_1, x_2, x_3\}$. The cocircuit $\{5, 6, 7, 9\}$ and circuit $\{x_2, x_3, 1, 8\}$ imply that $9 \notin \{x_2, x_3\}$ but possibly $9 = x_1$. Hence (iv) holds. □

Lemma 8.3. *Assume that $|E(M)| \geq 13$. Let M have, as triangles, $\{a, b, c\}$, $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$ and, as cocircuits, $\{1, 2, a, b\}$, $\{4, 5, b, c\}$, and $\{7, 8, a, c\}$. Assume that $a, b, c, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are distinct except that, possibly, $3 = 9$ or $3 = 6$ or $6 = 9$. Then either*

(i) *$\{a, b, c\}$ is the central triangle of a quasi rotor; or*
(ii) *$M/a, b, c$ is internally 4-connected.*

Proof. Assume that $\{a, b, c\}$ is not the central triangle of a quasi rotor. The triangle $\{a, b, c\}$ has an element e such that $M\backslash e$ is $(4, 4, S)$-connected. By symmetry, we may assume that e is a.

8.3.1. *$M/a, b, c$ is 3-connected.*

Assume that $M/a, b, c$ is not 3-connected. Now $M\backslash a$ is $(4, 4, S)$-connected having $\{b, 1, 2, 3\}$ as a maximal fan. Since b is a fan end, $M\backslash a/b$ is 3-connected. Now $M\backslash a/b$ has $\{c, 7, 8, 9\}$ as a fan and $M\backslash a/b/c = M/a, b, c$. As the last matroid is not 3-connected, it follows, by Tutte's Triangle Lemma, that c is in a triangle T of $M\backslash a/b$. It follows by orthogonality with the cocircuit $\{c, 7, 8\}$ that T contains 7 or 8. By symmetry, we may assume that $T = \{c, 7, x\}$ for some element x.

Suppose that $\{c, 7, x\}$ is a circuit of M. Then $x \in \{4, 5\}$ by orthogonality with the cocircuit $\{4, 5, b, c\}$. This implies that $6 = 9$ otherwise $\{a, b, c\}$ is the central triangle of a quasi rotor. Let $X = \{a, b, c, 4, 5, 6, 7, 8\}$. Then X contains at least two cocircuits of M, so $r^*(X) \leq 6$. Moreover, X is spanned by $\{a, 5, 6, 7\}$, so $r(X) \leq 4$. Hence $\lambda(X) \leq 2$, so $|E(M)| \leq 11$; a contradiction.

We may now assume that $\{c, 7, x, b\}$ is a circuit of M. Then $\{c, 7, x, b\} \triangle \{a, b, c\}$, which equals $\{a, 7, x\}$, is a circuit of M. Using symmetry with the argument in the last paragraph, we conclude that (8.3.1) holds.

Suppose $M/a, b, c$ has a $(4, 3)$-violator (X, Y). Then $r_{M/a,b,c}(X) + r_{M/a,b,c}(Y) = r(M/a, b, c) + 2$. Thus

8.3.2. $r(X \cup \{a, b, c\}) + r(Y \cup \{a, b, c\}) - 2 = r(M) + 2.$

Next we show that:

8.3.3. *Neither X nor Y contains $\{1, 2, 4, 5\}$, $\{1, 2, 7, 8\}$, or $\{4, 5, 7, 8\}$.*

Assume X contains $\{1,2,4,5\}$. Then each of $\{a,b\}$ and $\{b,c\}$ is a cocircuit of $M\backslash\{1,2,4,5\}$, so $\{a,b,c\}$ is a component of $M\backslash\{1,2,4,5\}$. Hence $r(Y \cup \{a,b,c\}) = r(Y) + 2$ and (8.3.2) implies the contradiction that $(X \cup \{a,b,c\}, Y)$ is a 3-separation of M. Thus (8.3.3) holds.

Now, by symmetry, we may assume that X contains at least two elements of each of $\{1,2,3\}$ and $\{4,5,6\}$. Suppose first that $\{1,2\} \subseteq X$. Then, since X does not contain $\{1,2,4,5\}$, we may assume that $\{5,6\} \subseteq X$ and $4 \in Y$. We get the contradiction that $(X \cup 4, Y - 4)$ is a $(4,3)$-violator of $M/a,b,c$ unless $|Y| = 4$. In the exceptional case, Y is a fan of $M/a,b,c$ with ordering $(y_1, y_2, y_3, 4)$ where $\{y_2, y_3, 4\}$ is a triangle. Then $\{y_1, y_2, y_3\}$ is a triad of M so it avoids $\{a,b,c,1,2,\dots,9\}$ since M is internally 4-connected. Now $M/a,b,c = M\backslash b/a,c$. Thus M has a circuit C that contains $\{y_2,y_3,4\}$ and is contained in $\{y_2,y_3,4,a,c\}$. By orthogonality with the cocircuits $\{b,c,4,5\}$ and $\{a,c,7,8\}$, we deduce that $C = \{y_2,y_3,4,a,c\}$. This contradicts orthogonality with the cocircuit $\{1,2,a,b\}$.

We conclude that $\{1,2\} \not\subseteq X$. By symmetry, this eliminates the case in which $\{1,3\} \subseteq X$ and $\{4,5\} \subseteq X$. It remains to consider the case when $\{1,3\} \subseteq X$ and $\{4,6\} \subseteq X$. Then $\{2,5\} \subseteq Y$. Now $(X \cup 2, Y - 2) \cong (X,Y)$. If $|Y - 2| \geq 4$, then we have reduced to the case in which X contains $\{1,2\}$. If $|Y - 2| = 3$, then $Y - 2$ is a triad of $M/a,b,c$ and hence of M. Since this triad contains 5, we have a contradiction. □

Lemma 8.4. *Let M be an internally 4-connected binary matroid having $\{1,2,3\}$, $\{a,b,c\}$, and $\{4,5,6\}$ as circuits, and $\{2,3,a,b\}$ and $\{4,5,b,c\}$ as cocircuits where $|\{a,b,c,1,2,3,4,5,6\}| = 9$. Then*

(i) *$M/a,b,c$ is internally 4-connected; or*
(ii) *$\{a,b,c\}$ is the central triangle of a quasi rotor; or*
(iii) *$M/b\backslash a$ is internally 4-connected; or*
(iv) *M has a cocircuit $\{a,c,z_1,z_2\}$ and a triangle containing exactly one of z_1 and z_2 and either*
  (a) *a and exactly one of 2 and 3; or*
  (b) *c and exactly one of 4 and 5.*
  Moreover, $\{z_1,z_2\}$ avoids $\{1,2,3,4,5,6\}$.

Proof. Assume that neither (i) nor (ii) holds. By Lemma 2.5, as a is not in a triad of M, the matroid $M\backslash a$ is 3-connected. This matroid has $\{b,1,2,3\}$ as a fan with b as an end. Thus $M\backslash a/b$ is 3-connected unless b is in a triangle other than $\{a,b,c\}$. In the exceptional case, if T is such a triangle, then $T = \{b,x,y\}$ where $x \in \{2,3\}$ and $y \in \{4,5\}$. Thus $\{a,b,c\}$ is the central triangle of a quasi rotor, which is not so. We conclude that $M\backslash a/b$ is 3-connected.

Now assume that $M\backslash a/b$ is not internally 4-connected. Then $M\backslash a/b$ has a $(4,3)$-violator (X,Y). Without loss of generality, $c \in X$. As $\{a,c\}$ is a circuit of M/b, it follows that $M/b\backslash c$ has $((X-c) \cup a, Y)$ as a $(4,3)$-violator. Thus we have symmetry in M between $(1,\{2,3\},c)$ and $(6,\{4,5\},a)$.

Now X or Y contains at least two elements of $\{1,2,3\}$, and X or Y contains at least two elements of $\{4,5,6\}$. Hence, by symmetry, we need only consider the following cases:

(a) $\{2,3\} \subseteq X$;
(b) $\{1,2\} \subseteq X$ and $3 \in Y$;
(c) $\{2,3,4,5\} \subseteq Y$;
(d) $\{2,3,5,6\} \subseteq Y$ and $4 \in X$; and
(e) $\{1,2,5,6\} \subseteq Y$ and $\{3,4\} \subseteq X$.

In case (a), as $c \in X$ and $\{a,c\}$ is a circuit of M/b, we have $(X \cup a, Y)$ as a 3-separation of M/b. But $\{2,3,a,b\}$ is a cocircuit of M, so $(X \cup a \cup b, Y)$ is a 3-separation of M; a contradiction.

In case (b), $(X \cup 3, Y-3) \cong (X,Y)$ so we reduce to case (a) unless $Y-3$ is a triad of $M\backslash a/b$. Consider the exceptional case. Then 4 or 5, say 4, is in Y otherwise we have reduced to a case symmetric to (a). Then $Y-3$ is not a cocircuit of M. Thus M has a cocircuit $\{4,a,y_1,y_2\}$. But $\{y_1,y_2\}$ avoids $\{b,c\}$ so we contradict orthogonality.

Next consider case (c), that is, $\{2,3,4,5\} \subseteq Y$. As $\{2,3,4,5,c\}$ is a cocircuit of $M/b\backslash a$, we have $(X-c, Y \cup c) \cong (X,Y)$. Thus $|X-c| = 3$, otherwise we have reduced to a case symmetric to case (a). Therefore X is a fan of $M/b\backslash a$ having an ordering of the form (x_1,x_2,x_3,c) where $\{x_2,x_3,c\}$ is a triad. Thus, by orthogonality, $\{x_2,x_3,c,a\}$ is a cocircuit of M. As $\{2,3,a,b\}$ is a cocircuit of M that avoids $\{x_1,x_2,x_3\}$, we deduce that $\{x_1,x_2,x_3\}$ is a circuit of M. Then $x_1,x_2,x_3,1,2,3,a,b,c,4,5$, and 6 are distinct, or $x_1 \in \{1,6\}$. In each case, it follows, by Lemma 8.3, that $M/a,b,c$ is internally 4-connected.

Now consider case (d), that is, $\{2,3,5,6\} \subseteq Y$ and $4 \in X$. Then $(X-4, Y \cup 4) \cong (X,Y)$ and we have reduced to case (c) unless X is a 4-element fan in $M/b\backslash a$ having a fan ordering $(x_1,x_2,x_3,4)$ where $\{x_1,x_2,x_3\}$ is a triad. Consider the exceptional case. As $c \in \{x_1,x_2,x_3\}$, it follows that $\{x_1,x_2,x_3,a\}$ is a cocircuit of M containing $\{a,c\}$ but avoiding $\{4,5\}$. Moreover, orthogonality between the cocircuit $\{2,3,4,5,c\}$ and the triangle $\{x_2,x_3,4\}$ of $M/b\backslash a$ implies that $c \in \{x_2,x_3\}$ so we may take $c = x_3$. The cocircuit $\{b,c,4,5\}$ of M implies that $\{x_2,c,4\}$ is a circuit of M. We conclude that (iv) of the lemma holds.

Finally, consider case (e), that is, $\{1,2,5,6\} \subseteq Y$ and $\{3,4\} \subseteq X$. As $\{3,4\} \subseteq \mathrm{cl}_{M/b\backslash a}(Y)$ and $|X| \geq 4$, it follows that $|X| \geq 5$. Thus $|X-3| \geq 4$. As $(X-3, Y \cup 3) \cong (X,Y)$, we have reduced to case (d). □

9. The Non-bowtie Case

The purpose of this section is to prove the following result.

Theorem 9.1. *Let M be an internally 4-connected binary matroid having at least 13 elements. Suppose that M has no bowties and has no $M(K_4)$-restriction. Assume that M has triangles $\{1,2,3\}$ and $\{3,4,5\}$ and a cocircuit $\{2,3,4,6\}$, and that $M\backslash 4$ is $(4,4,S)$-connected. Then $M\backslash 1$, $M\backslash 3$, or $M\backslash 1,4$ is internally 4-connected.*

To prove this theorem, we shall establish a sequence of lemmas. All are subject to the same hypotheses as the theorem.

Lemma 9.2. *Suppose $M\backslash 1$ is not internally 4-connected. Then*

(i) *every $(4,3)$-violator (X_1,Y_1) of $M\backslash 1$ has $\{2,6\} \subseteq U_1$ and $\{3,4\} \subseteq V_1$ where $\{U_1,V_1\} = \{X_1,Y_1\}$; and*

(ii) *if (X_1,Y_1) is a $(4,3)$-violator of $M\backslash 1$ and $\{2,6\} \subseteq X_1$ and $\{3,4\} \subseteq Y_1$, then either $5 \in Y_1$; or $5 \in X_1$ and $|X_1| \geq 5$ and $(X_1-5, Y_1 \cup 5)$ is a 3-separation of $M\backslash 1$.*

Proof. Let (X_1,Y_1) be a $(4,3)$-violator of $M\backslash 1$. Then we may assume that $2 \in X_1$ and $3 \in Y_1$. We shall consider the location of 4 and 6.

Suppose $\{4,6\} \subseteq X_1$. Then $(X_1 \cup 3, Y_1 - 3) \cong (X_1,Y_1)$. Thus Y_1 is a 4-element fan otherwise we obtain a contradiction. Hence Y_1 contains a triad $\{y_2,y_3,3\}$ of $M\backslash 1$. Thus $\{y_2,y_3,3,1\}$ is a cocircuit of M. By orthogonality with the triangle $\{3,4,5\}$, we deduce that $5 \in \{y_2,y_3\}$ so we may take $y_3 = 5$. Then $(\{y_1,y_2,5\},\{1,2,3\},\{y_2,5,3,1\})$ is a bowtie of M; a contradiction.

Next, suppose that $\{4,6\} \subseteq Y_1$. Since $(X_1 - 2, Y_1 \cup 2) \cong (X_1,Y_1)$, we obtain a contradiction unless X_1 is a 4-element fan of $M\backslash 1$ containing a triangle $\{x_1,x_2,x_3\}$ and a triad $\{x_2,x_3,2\}$. In the exceptional case, $(\{x_1,x_2,x_3\},\{1,2,3\},\{x_2,x_3,2,1\})$ is a bowtie of M; a contradiction.

Now suppose that $4 \in X_1$ and $6 \in Y_1$. Assume first that $5 \in X_1$. Then $3 \in \mathrm{cl}(X_1)$ so $(X_1,Y_1) \cong (X_1 \cup 3, Y_1 - 3)$ and we must have that Y_1 is a 4-element fan of $M\backslash 1$ containing a triangle $\{y_2,y_3,3\}$ and a triad $\{y_1,y_2,y_3\}$. Hence $\{y_1,y_2,y_3,1\}$ is a cocircuit of M that meets the triangle $\{1,2,3\}$ in a single element; a contradiction. We conclude that $5 \in Y_1$. Then $(X_1,Y_1) \cong (X_1 - 4, Y_1 \cup 4)$. Thus we have reduced to an earlier case unless X_1 is a 4-element fan having an ordering of the form $(x_1,x_2,x_3,4)$ where $\{x_2,x_3,4\}$ is a triangle of M. Consider the exceptional case. As $\{2,3,4,6\}$ is a cocircuit of M, we deduce that $2 \in \{x_2,x_3\}$. Thus M has a triangle containing $\{2,4\}$, so M has an $M(K_4)$-restriction using $\{1,2,3,4,5\}$ and one other element; a contradiction. Since we have eliminated every other possibility, we conclude that $6 \in X_1$ and $4 \in Y_1$; that is, (i) holds.

For (ii), suppose (X_1,Y_1) is a $(4,3)$-violator of $M\backslash 1$ with $\{2,6,5\} \subseteq X_1$ and $\{3,4\} \subseteq Y_1$. Then $(X_1 - 5, Y_1 \cup 5) \cong (X_1,Y_1)$. Suppose $|X_1| = 4$. Then X_1 has a fan ordering of the form $(x_1,x_2,x_3,5)$ where $\{x_2,x_3,5\}$ is a circuit of M avoiding $\{3,4\}$ and meeting $\{2,6\}$. By orthogonality with the cocircuit $\{2,3,4,6\}$, we deduce that $\{x_2,x_3\} = \{2,6\}$. Then $\{2,6,5\} \triangle \{3,4,5\}$, which equals $\{2,3,4,6\}$, is a circuit of M, and so is a quad of M; a contradiction. We conclude that (ii) holds. □

Lemma 9.3. *Suppose that $M\backslash 3$ is not internally 4-connected. Then*

(i) *every $(4,3)$-violator (X_3, Y_3) of $M\backslash 3$ has $\{1,5\} \subseteq U_3$ and $\{2,4\} \subseteq V_3$ where $\{U_3, V_3\} = \{X_3, Y_3\}$; and*

(ii) *if (X_3, Y_3) is a $(4,3)$-violator of $M\backslash 3$ and $\{1,5\} \subseteq X_3$ and $\{2,4\} \subseteq Y_3$, then either $6 \in Y_3$; or $6 \in X_3$ and $|X_3| \geq 5$ and $(X_3 - 6, Y_3 \cup 6)$ is a 3-separation of $M\backslash 3$.*

Proof. Let (X_3, Y_3) be a $(4,3)$-violator of $M\backslash 3$. Then, without loss of generality, we may assume that $1 \in X_3$ and $2 \in Y_3$. Moreover, each of X_3 and Y_3 contains one of 4 and 5. Suppose first that $4 \in X_3$ and $5 \in Y_3$. Consider the location of 6. Suppose $6 \in X_3$. Since $(X_3, Y_3) \cong (X_3 \cup 2, Y_3 - 2)$, it follows that Y_3 is a 4-element fan in $M\backslash 3$ having $(y_1, y_2, y_3, 2)$ as a fan ordering where $\{y_2, y_3, 2\}$ is a triad. Then $\{y_2, y_3, 2, 3\}$ is a cocircuit of M which, by orthogonality, must contain 5. Hence we may assume $y_3 = 5$. Also, $\{y_1, y_2, 5\}$ is a triangle of M, so $(\{y_1, y_2, 5\}, \{1,2,3\}, \{y_2, 5, 2, 3\})$ is a bowtie; a contradiction. We conclude that $6 \notin X_3$. A symmetric argument establishes that $6 \notin Y_3$. Therefore we must have that $4 \in Y_3$ and $5 \in X_3$; that is, (i) holds.

Now let (X_3, Y_3) be a $(4,3)$-violator of $M\backslash 3$ and suppose that $\{1,5,6\} \subseteq X_3$ and $\{2,4\} \subseteq Y_3$. Then $(X_3 - 6, Y_3 \cup 6) \cong (X_3, Y_3)$. If $|X_3| = 4$, then X_3 is a fan having an ordering of the form $(x_1, x_2, x_3, 6)$ where $\{x_1, x_2, x_3\}$ is a triangle containing $\{1,5\}$. We conclude that M has an $M(K_4)$-restriction; a contradiction. Hence (ii) holds. □

Lemma 9.4. *If neither $M\backslash 1$ nor $M\backslash 3$ is internally 4-connected, then either*

(i) *$M\backslash 1, 3$ is 3-connected; or*

(ii) *$M\backslash 1, 3$ has a unique 2-separation $(\{5, a\}, E(M\backslash 1, 3) - \{5, a\})$ and $\{1, 3, 5, a\}$ is a cocircuit of M.*

Proof. Let (J, K) be a 2-separation of $M\backslash 1, 3$. Assume that $|J| \geq |K| \geq 3$. As $|E(M)| \geq 13$, we have $|J| \geq 4$. Now $(J, K \cup 1)$ and $(J, K \cup 3)$ are 3-separations of $M\backslash 3$ and $M\backslash 1$, respectively. Then, by Lemmas 9.2 and 9.3, $\{1,5\} \subseteq K \cup 1$ and $\{2,4\} \subseteq J$, while $\{2,6\} \subseteq J$ and $\{3,4\} \subseteq K \cup 3$. Hence $4 \in J \cap K$; a contradiction. We deduce that (J, K) is a minimal 2-separation of $M\backslash 1, 3$. Thus $M\backslash 1, 3$ has a 2-cocircuit, $\{a, b\}$. Hence $\{1, 3, a, b\}$ is a cocircuit of M. By orthogonality, we may assume that $b \in \{4, 5\}$. If $b = 4$, then $M\backslash 4$ has $\{1, 3, a\}$ as a triad, so $M\backslash 4$ has $\{a, 1, 3, 2, 6\}$ as a fan, contradicting the fact that $M\backslash 4$ is $(4, 4, S)$-connected. Thus $b = 5$ and M has $\{1, 3, 5, a\}$ as a cocircuit. □

Lemma 9.5. *Assume that neither $M\backslash 1$ nor $M\backslash 3$ is internally 4-connected. Then M has a 4-element cocircuit that contains $\{1,3,5\}$ and some element 7 not in $\{1, 2, \ldots, 6\}$.*

Proof. Suppose M has a 4-cocircuit containing $\{1,3,5\}$. Then the fourth element of this cocircuit is not in $\{1, 2, \ldots, 6\}$ otherwise $r(\{1, 2, \ldots, 6\}) + r^*(\{1, 2, \ldots, 6\}) - |\{1, 2, \ldots, 6\}| \leq 4 + 4 - 6 = 2$, that is, $\lambda_M(\{1, 2, \ldots, 6\}) \leq 2$.

This is a contradiction since $|E(M)| \geq 13$. We may now assume that M has no 4-cocircuit containing $\{1,3,5\}$. Then, by the last lemma, $M\backslash 1,3$ is 3-connected. Let (X_1, Y_1) and (X_3, Y_3) be $(4,3)$-violators of $M\backslash 1$ and $M\backslash 3$, respectively. Then, by Lemmas 9.2 and 9.3, we may assume that $\{2,6\} \subseteq X_1$ and $\{3,4,5\} \subseteq Y_1$, while $\{1,5\} \subseteq X_3$ and $\{2,4,6\} \subseteq Y_3$. Observe that, as $M\backslash 1,3$ is 3-connected, $\lambda_{M\backslash 1}(X_1) = 2 = \lambda_{M\backslash 1,3}(X_1)$. Moreover, $\lambda_{M\backslash 3}(X_3) = 2 \geq \lambda_{M\backslash 3,1}(X_3 - 1) \geq 2$. Thus $r(X_3 - 1) = r(X_3)$, that is,

9.5.1. $1 \in \text{cl}(X_3 - 1)$.

We show next that

9.5.2. $|X_3 \cap Y_1| \geq 2$.

Assume that $|X_3 \cap Y_1| < 2$. Then $X_3 \cap Y_1 = \{5\}$. Thus $|X_1 \cap X_3| \geq 2$ and $|Y_1 \cap Y_3| \geq 2$. As $M\backslash 1,3$ is 3-connected, we deduce that $\lambda_{M\backslash 1,3}(X_1 \cap X_3) \geq 2$ and $\lambda_{M\backslash 1,3}(Y_1 \cap Y_3) \geq 2$. But $\lambda_{M\backslash 1,3}(X_1) = 2 = \lambda_{M\backslash 1,3}(X_3 - 1)$. Thus, by the submodularity of λ and its invariance under taking complements, we have $\lambda_{M\backslash 1,3}(X_1 \cap X_3) = 2 = \lambda_{M\backslash 1,3}(Y_1 \cap Y_3)$. Now $1 \in \text{cl}(X_3 - 1)$, so $3 \in \text{cl}((X_3 - 1) \cup X_1)$. Hence $\lambda_M(Y_1 \cap Y_3) = 2$. Thus $|Y_1 \cap Y_3| \leq 3$. If $|Y_1 \cap Y_3| = 3$, then $Y_1 \cap Y_3$ is a triangle or a triad of M. As $4 \in Y_1 \cap Y_3$ and 4 is in a triangle of M, the set $Y_1 \cap Y_3$ is not a triad. Hence it is a triangle having a single element in common with the cocircuit $\{2,3,4,6\}$; a contradiction. We deduce that $|Y_1 \cap Y_3| \leq 2$. Hence $|Y_1| = 4$. As Y_1 contains a triangle of $M\backslash 1$, there is also a triad of $M\backslash 1$ contained in Y_1. Thus M has a 4-cocircuit C^* contained in $\{1,3,4,5,y\}$ and containing 1 where $Y_1 \cap Y_3 = \{4, y\}$. By orthogonality, C^* contains 3 and exactly one of 4 and 5. If C^* contains 4, then $M\backslash 4$ is not $(4,4,S)$-connected. Hence C^* is $\{1,3,5,y\}$; a contradiction. We conclude that (9.5.2) holds.

As $M\backslash 1,3$ is 3-connected, (9.5.2) implies that $\lambda_{M\backslash 1,3}(X_3 \cap Y_1) \geq 2$. As $|X_1 \cap Y_3| \geq 2$, we also have $\lambda_{M\backslash 1,3}(X_1 \cap Y_3) \geq 2$. Since $\lambda_{M\backslash 1,3}(X_1) = 2 = \lambda_{M\backslash 1,3}(Y_3)$, it follows by submodularity that $\lambda_{M\backslash 1,3}(X_3 \cap Y_1) = 2 = \lambda_{M\backslash 1,3}(X_1 \cap Y_3)$. Thus, by (9.5.1), $\lambda_M(X_1 \cap Y_3) = 2$, so $|X_1 \cap Y_3| \leq 3$. If $|X_1 \cap Y_3| = 3$, then $X_1 \cap Y_3$ is a triangle or a triad of M containing $\{2,6\}$. But 2 is not in a triad of M, and $\{2,6\}$ is not in a triangle otherwise $M\backslash 4$ is not $(4,4,S)$-connected. Thus $|X_1 \cap Y_3| < 3$ so $|X_1 \cap Y_3| = 2$. Hence $|X_1 \cap X_3| \geq 2$ and $|Y_1 \cap Y_3| \geq 2$. It follows, by the submodularity of λ, that $\lambda_{M\backslash 1,3}(Y_1 \cap Y_3) = 2 = \lambda_{M\backslash 1,3}(X_1 \cap X_3)$. Thus $\lambda_M(Y_1 \cap Y_3) = 2$ since $1 \in \text{cl}(X_3 - 1)$ by (9.5.1), and $3 \in \text{cl}((X_3 - 1) \cup X_1)$. Hence $|Y_1 \cap Y_3| \leq 3$. If $|Y_1 \cap Y_3| = 3$, then $Y_1 \cap Y_3$ is a triad of M containing 4, or $Y_1 \cap Y_3$ is a triangle of M meeting the cocircuit $\{2,3,4,6\}$ in a single element. Both possibilities lead to contradictions. Thus $|Y_1 \cap Y_3| \leq 2$. Hence $|Y_1 \cap Y_3| = 2$, so $|Y_3| = 4$. Thus $Y_3 = \{2,4,6,y\}$ where $Y_1 \cap Y_3 = \{4, y\}$. Now $M\backslash 3$ has $\{2,4,6\}$ as a cocircuit, so Y_3 contains a triangle T containing y and two elements of $\{2,4,6\}$. But $M\backslash 4$ is $(4,4,S)$-connected and M has no $M(K_4)$-restriction, so T contains neither $\{2,6\}$ nor $\{2,4\}$. Thus T contains $\{4,6\}$. Hence $(X_1, Y_1) \cong (X_1 - 6, Y_1 \cup 6)$ in $M\backslash 1$. This gives a contradiction unless

X_1 is a 4-element fan containing a triangle T' of M where $6 \in T'$. It follows by orthogonality with the cocircuit $\{2,3,4,6\}$ that T' contains 2. Thus M has a triangle containing $\{2,6\}$, a contradiction. □

Lemma 9.6. *The only triangles of M meeting $\{1,2,\dots,7\}$ are $\{1,2,3\}$ and $\{3,4,5\}$.*

Proof. The cocircuits $\{1,3,5,7\}$ and $\{2,3,4,6\}$ mean that a triangle T meeting $\{1,2,\dots,7\}$ must contain at least two elements of this set. In this proof, we will not use the fact that $M\backslash 4$ is $(4,4,S)$-connected, so we have symmetry between the cocircuits $\{1,3,5,7\}$ and $\{2,3,4,6\}$. Hence we may assume that T meets $\{1,3,5,7\}$. If T contains 3 and is not $\{1,3,5\}$ or $\{3,4,5\}$, then, by orthogonality, it meets $\{2,4,6\}$ and $\{1,5,7\}$, so it is contained in $\{1,2,\dots,7\}$. Thus $r(\{1,2,\dots,7\}) \le 4$, so $\lambda(\{1,2,\dots,7\}) \le 2$; a contradiction since $|E(M)| \ge 13$. Hence we may assume that $3 \notin T$. If T contains $\{1,5\}$, then M has an $M(K_4)$-restriction. If T contains $\{1,7\}$, then M has $(T,\{3,4,5\},\{1,3,5,7\})$ as a bowtie. Similarly, if T contains $\{5,7\}$, then M has a bowtie. □

Lemma 9.7. *Let (X_4,Y_4) be a $(4,3)$-violator of $M\backslash 4$. Then X_4 or Y_4 is $\{1,2,3,6\}$.*

Proof. Since $M\backslash 4$ is $(4,4,S)$-connected, either X_4 or Y_4, say X_4, is a 4-element fan of $M\backslash 4$. Thus X_4 contains a triad T^* and a triangle T of $M\backslash 4$. Hence $T^* \cup 4$ is a cocircuit of M. By orthogonality with the triangle $\{3,4,5\}$, we deduce that exactly one of 3 and 5 is in T^*.

Suppose that $3 \in T^*$. Then, by orthogonality, 1 or 2 is in T^*. If $1 \in T^*$, then $T^* \cup 4 = \{1,3,4,y\}$ for some element y. Thus $M\backslash 4$ has a 5-element fan, $\{1,2,3,6,y\}$; a contradiction to the fact that $M\backslash 4$ is $(4,4,S)$-connected. Hence we may assume that $2 \in T^*$. Then $T^* \cup 4$ contains $\{2,3,4\}$, so $T^* \cup 4 = \{2,3,4,6\}$ and therefore $X_4 = \{1,2,3,6\}$.

We may now assume that $3 \notin T^*$, so $5 \in T^*$. By Lemma 9.6, if the triangle T in X_4 is not $\{3,4,5\}$ or $\{1,2,3\}$, then it avoids $\{1,2,\dots,7\}$. Thus $(T,\{3,4,5\},T^* \cup 4)$ is a bowtie in M; a contradiction. Clearly X_4 does not contain $\{3,4,5\}$. Hence we may assume that X_4 contains $\{1,2,3\}$. Thus $X_4 = \{1,2,3,5\}$ so $X_4 \cup 4$ contains a cocircuit of M containing 4. Hence $r^*(\{1,2,\dots,6\}) \le 4$, so $\lambda(\{1,2,\dots,6\}) \le 2$; a contradiction. □

Lemma 9.8. *If $M\backslash 1$ is not internally 4-connected, then $M\backslash 1,4$ is internally 4-connected unless M has a triangle $\{a,b,c\}$ and a cocircuit $\{1,3,4,a,b\}$ where $\{a,b,c\}$ is disjoint from $\{1,2,\dots,7\}$.*

Proof. As $M\backslash 4$ is $(4,4,S)$-connected having $\{1,2,3,6\}$ as a maximal fan with 1 as an end, $M\backslash 4\backslash 1$ is 3-connected. Suppose that $M\backslash 1,4$ is not internally 4-connected and let (X_{14},Y_{14}) be a $(4,3)$-violator of it. If X_{14} contains $\{2,3\}$, then $(X_{14} \cup 1, Y_{14})$ is a 3-separation of $M\backslash 4$ contradicting Lemma 9.7. Hence we may assume that $2 \in X_{14}$ and $3 \in Y_{14}$. If Y_{14} contains $\{3,5\}$, then $(X_{14}, Y_{14} \cup 4)$ is a 3-separation of $M\backslash 1$ so, by Lemma 9.2, $\{2,6\} \subseteq X_{14}$.

Now $(X_{14} \cup 3, Y_{14} - 3) \cong (X_{14}, Y_{14})$ in $M\backslash 1,4$, so we get a contradiction as above unless Y_{14} is a 4-element fan of $M\backslash 1,4$ having $(y_1, y_2, y_3, 3)$ as a fan ordering where $\{y_1, y_2, y_3\}$ is a triangle. In the exceptional case, $\{y_1, y_2, y_3\}$ contains 5 but is not $\{3, 4, 5\}$; a contradiction to Lemma 9.6. We deduce that $\{2, 5\} \subseteq X_{14}$ and $3 \in Y_{14}$.

Suppose $6 \in Y_{14}$. Then $(X_{14} - 2, Y_{14} \cup 2) \cong (X_{14}, Y_{14})$ in $M\backslash 1,4$. Hence we have a contradiction unless X_{14} is a 4-element fan in $M\backslash 1,4$ having $(x_1, x_2, x_3, 2)$ as a fan ordering and $\{x_1, x_2, x_3\}$ as a triangle. Since this triangle contains 6, we have a contradiction to Lemma 9.6.

We may now assume that $6 \in X_{14}$. Then $(X_{14} \cup 3, Y_{14} - 3) \cong (X_{14}, Y_{14})$ in $M\backslash 1,4$. Thus we have a contradiction unless Y_{14} is a fan of $M\backslash 1,4$ having $\{y_1, y_2, y_3\}$ as a triangle, and $\{y_2, y_3, 3\}$ as a triad. Then, by Lemma 9.6, $\{y_1, y_2, y_3\}$ avoids $\{1, 2, \dots, 7\}$. By orthogonality, $\{y_2, y_3, 3, 1, 4\}$ is a cocircuit of M. Thus the lemma is proved. □

Lemma 9.9. *If neither $M\backslash 1$ nor $M\backslash 1,4$ is internally 4-connected, then $M\backslash 3$ is internally 4-connected.*

Proof. By the last lemma, M has a cocircuit $\{1, 3, 4, a, b\}$ and a triangle $\{a, b, c\}$ where $\{a, b, c\}$ avoids $\{1, 2, \dots, 7\}$. Suppose that $M\backslash 3$ has a $(4,3)$-violator (X_3, Y_3). Then, by Lemma 9.3, we may assume that $\{1, 5\} \subseteq X_3$ and $\{2, 4, 6\} \subseteq Y_3$. If $7 \in Y_3$, then $(X_3 \cup 7, Y_3 - 7) \cong (X_3, Y_3)$ in $M\backslash 3$. Suppose $|Y_3| = 4$. Then $Y_3 = \{2, 4, 6, 7\}$ and Y_3 has $\{2, 4, 6\}$ as a triangle; a contradiction. Thus we may assume that $\{1, 5, 7\} \subseteq X_3$ and $\{2, 4, 6\} \subseteq Y_3$.

Now M has $\{1, 3, 4, a, b\}$ as a cocircuit, so $M\backslash 3$ has $\{1, 4, a, b\}$ as a cocircuit. If $\{a, b\} \subseteq X_3$, then $(X_3 \cup 4, Y_3 - 4) \cong (X_3, Y_3)$ in $M\backslash 3$, so we have a contradiction otherwise Y_3 is a 4-element fan containing a triangle containing $\{2, 6\}$, which does not happen. If $\{a, b\} \subseteq Y_3$, then $(X_3 - 1, Y_3 \cup 1) \cong (X_3, Y_3)$ in $M\backslash 3$, so we have a contradiction otherwise X_3 contains a triangle containing $\{5, 7\}$. We deduce that we may assume that $a \in X_3$ and $b \in Y_3$.

Consider the location of c. Either X_3 or Y_3 contains c, so we can move b or a, respectively, to obtain an equivalent 3-separation in which $\{a, b\}$ is contained in the same set. This move reduces us to an earlier case unless it leaves a set with fewer than four elements. In the exceptional case, there is a triangle of $M\backslash 3$ containing c and some element of $\{1, 5, 7\}$ or $\{2, 4, 6\}$, a contradiction to Lemma 9.6. □

Proof of Theorem 9.1. This theorem follows immediately by combining the preceding lemmas. □

10. Bowties

The purpose of this section is to prove the following result.

Theorem 10.1. *Let M be a binary internally 4-connected matroid with $|E(M)| \geq 13$. Assume that*

(i) M does not contain a quasi rotor; and
(ii) no restriction of M is isomorphic to $M(K_4)$; and

(iii) M has a bowtie (T_1, T_2, C^*) but has no bowtie of the form (T_1, T_3, D^*) where $T_1 \cap C^* \neq T_1 \cap D^*$.

Then M has a proper internally 4-connected minor N with $|E(M)-E(N)| \leq 4$. Moreover, equality is attained in this bound if and only if M is isomorphic to the cycle matroid of a terrahawk.

The proof of this theorem is a long case analysis and we now give a brief outline of it. Assume that the hypotheses of the theorem hold and that M has no proper internally 4-connected minor N with $|E(M) - E(N)| \leq 3$. We begin with a bowtie $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$. By Theorem 5.1 and Lemma 6.3, as M contains no quasi rotor, $M\backslash 1$ is $(4,4,S)$-connected. We distinguish the cases when $M\backslash 1$ has a unique 4-element fan, and when $M\backslash 1$ has more than one such fan. In Figures 5 and 6, we indicate, for these two cases, the lemmas that take us to the structure in Figure 7(a). In Lemma 10.13, we show that the structure in Figure 7(a) forces the structure in Figure 7(b). Finally, Lemma 10.15 shows that if the last structure arises, then M is the cycle matroid of a terrahawk. A more formal description of the proof is given at the end of the section.

Lemma 10.2. *Let M be an internally 4-connected binary matroid in which no single-element deletion is internally 4-connected. Suppose $|E(M)| \geq 13$ and let $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ and $(\{1,2,3\},\{7,8,9\},\{1,2,7,8\})$ be bowties. Then $|\{1,2,\ldots,9\}| = 9$.*

Proof. Certainly $|\{1,2,3,4,5,6\}| = 6 = |\{1,2,3,7,8,9\}|$. If $\{4,5,6\} = \{7,8,9\}$, then $\lambda(\{1,2,\ldots,6\}) \leq 2$; a contradiction. If $\{4,5,6\} \triangle \{7,8,9\}$ is a circuit of M, then it is a 4-circuit and $\lambda(\{1,2,\ldots,9\}) = r(\{1,2,\ldots,9\}) + r^*(\{1,2,\ldots,9\}) - |\{1,2,\ldots,9\}| \leq 4+6-8 = 2$; a contradiction. We conclude that $\{4,5,6\} \cap \{7,8,9\} = \emptyset$, as required. □

For the remainder of this section, we shall assume that M contains no quasi rotor and that M has no $M(K_4)$-restriction.

Lemma 10.3. *Suppose that M has $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ as a bowtie and has $\{2,5,7\}$ as a triangle and $\{1,2,7,8\}$ as a cocircuit where $|\{1,2,\ldots,8\}| = 8$. Assume that $M\backslash 1$ has a unique 4-element fan and that $M\backslash 6$ is not internally 4-connected. Then either*

(i) M has a cocircuit $\{5,6,7,9\}$ where $9 \notin \{1,2,\ldots,8\}$; or
(ii) $M/4\backslash 6$ is internally 4-connected; or
(iii) M has a triangle $\{3,4,9\}$ and a cocircuit $\{4,6,9,10\}$ where $|\{1,2,\ldots,10\}| = 10$.

Proof. By Lemma 6.3, M has a 4-cocircuit C^* containing $\{5,6\}$ or $\{4,6\}$. If $C^* \subseteq \{1,2,\ldots,8\}$, then $\lambda(\{1,2,...,8\}) \leq 2$; a contradiction. Thus, by orthogonality, $C^* = \{5,6,7,9\}$ or $C^* = \{4,6,9,10\}$, where $9 \notin \{1,2,\ldots,8\}$ and $10 \notin \{1,2,3,4,5,6,7,9\}$. In particular, if M has a 4-cocircuit containing $\{5,6\}$, then (i) holds.

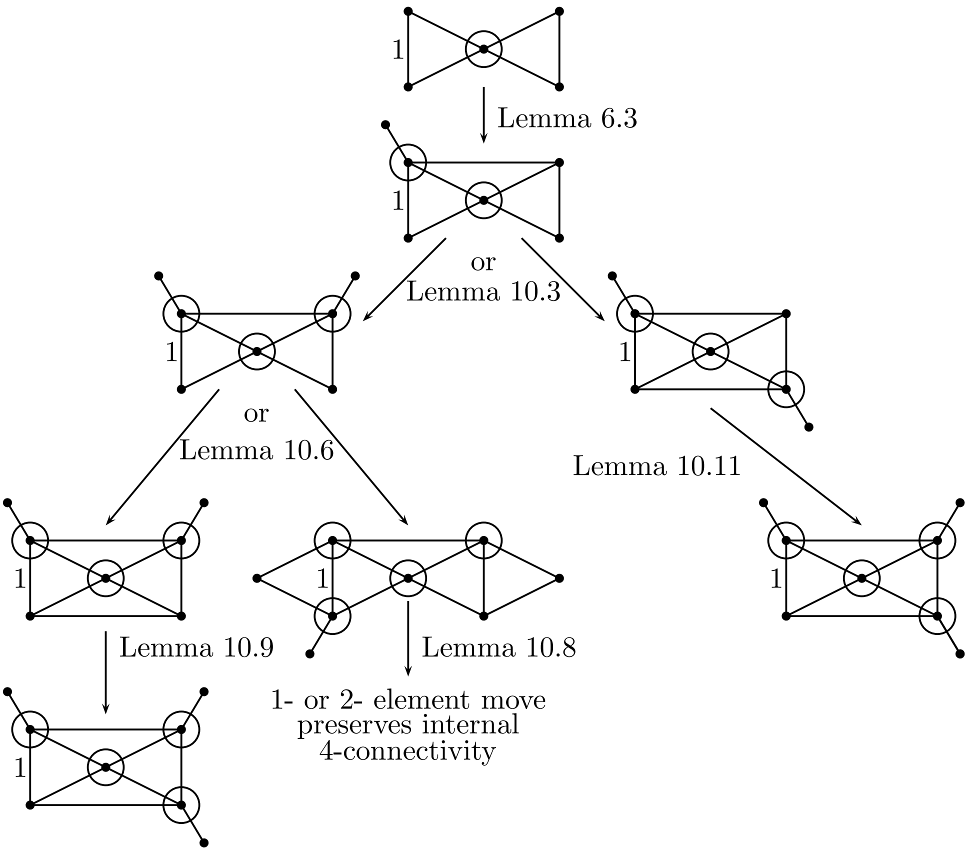

FIGURE 5. The steps when $M\backslash 1$ has a unique 4-element fan.

We may now assume that $C^* = \{4, 6, 9, 10\}$. Either (ii) or (iii) of Lemma 6.3 holds. Assume (iii) holds. Then M has a triangle $\{9, 4, e\}$ where $e \in \{2, 3\}$. As M has no $M(K_4)$-restriction, $e \neq 2$, so $e = 3$. Hence (iii) of the lemma holds because $10 \neq 8$ otherwise $\lambda_M(\{1, 2, \dots, 9\}) \leq 2$. Next we assume that (ii) of Lemma 6.3 holds. Then M has a triangle $\{9, 10, 11\}$ that is disjoint from $\{1, 2, 3, 4, 5, 6\}$. Thus we have $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ and $(\{4, 5, 6\}, \{9, 10, 11\}, \{4, 6, 9, 10\})$ as bowties. Thus, by Lemma 8.4, since $M/4, 5, 6$ is not 3-connected and $\{4, 5, 6\}$ is not the central triangle of a quasi rotor, either $M/4\backslash 6$ is internally 4-connected and so (ii) holds, or M has a 4-cocircuit containing $\{5, 6\}$, and (i) holds. □

Lemma 10.4. *Suppose that M has $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ as a bowtie and that M has $\{2, 5, 7\}$ as a triangle and $\{1, 2, 7, 8\}$ and $\{5, 6, 7, 9\}$ as cocircuits. Then $\{2, 5, 7\}$ is the only triangle of M containing 7.*

Proof. Suppose that M has a triangle T containing 7 but different from $\{2, 5, 7\}$. By orthogonality, $T \subseteq \{1, 2, 5, 6, 7, 8, 9\}$, so $\lambda_M(\{1, 2, \dots, 9\}) \leq 2$; a contradiction. □

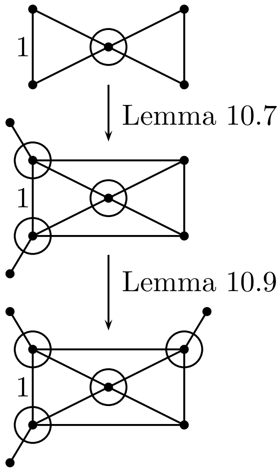

FIGURE 6. When $M\backslash 1$ has more than one 4-element fan.

Lemma 10.5. *Let* $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ *be a bowtie in a binary internally* 4*-connected matroid* M*. Let* $\{2,5,7\}$ *be a circuit and* $\{1,2,7,8\}$ *be a cocircuit. Suppose* $M\backslash 1$ *has a unique* 4*-element fan and that* $M\backslash 6$ *is not internally* 4*-connected. Then* $M\backslash 1/8$ *is* 3*-connected. Let* (X_{18}, Y_{18}) *be a* $(4,3)$*-violator of* $M\backslash 1/8$*. Then each of* X_{18} *and* Y_{18} *meets* $\{2,7\}$*.*

- *(i) If* $2 \in X_{18}$ *and* $\{5,7\} \subseteq Y_{18}$*, then* M *has a triangle* $\{1,8,9\}$ *and a cocircuit* $\{1,3,9,10\}$ *where* $|\{1,2,\ldots,10\}| = 10$*.*
- *(ii) If* $\{2,5\} \subseteq X_{18}$ *and* $7 \in Y_{18}$*, then* Y_{18} *is a* 4*-element fan of* $M\backslash 1/8$ *having* $(y_1, y_2, y_3, 7)$ *as a fan ordering where* $\{y_2, y_3\} \cap \{1,2,\ldots,8\} = \emptyset$*; and* M *has a circuit* $\{y_2, y_3, 7, 8\}$*; and either* $\{3, y_2, y_3, 1\}$ *is a cocircuit of* M*, or* $\{y_1, y_2, y_3\}$ *is a cocircuit of* M *and* $y_1 \notin \{1,2,\ldots,8\}$*.*

Proof. By Lemma 6.3, $M\backslash 1$ is $(4,4,S)$-connected. Since $M\backslash 1$ has $\{8,2,7,5\}$ as its unique 4-element fan and this fan has 8 as an end; $M\backslash 1/8$ is 3-connected. Let (X_{18}, Y_{18}) be a $(4,3)$-violator of $M\backslash 1/8$. Since $M\backslash 1$ has

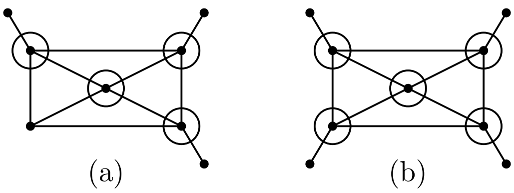

FIGURE 7

$\{2,7,8\}$ as a cocircuit, if $\{2,7\} \subseteq X_{18}$, then $(X_{18} \cup 8, Y_{18})$ is a 3-separation of $M\backslash 1$. But neither $X_{18} \cup 8$ nor Y_{18} equals $\{8,2,7,5\}$, so we have a contradiction. Thus we may assume that $2 \in X_{18}$ and $7 \in Y_{18}$. Consider the location of 5.

Suppose first that $5 \in Y_{18}$. Then $2 \in X_{18}$ and $\{5,7\} \subseteq Y_{18}$, and $(X_{18}, Y_{18}) \cong (X_{18} - 2, Y_{18} \cup 2)$ in $M\backslash 1/8$. Thus we get a contradiction unless X_{18} is a 4-element fan in $M\backslash 1/8$ containing a triangle $\{x_2, x_3, 2\}$ and a triad $\{x_1, x_2, x_3\}$. Consider the exceptional case. Orthogonality with the cocircuit $\{1,2,7,8\}$ implies that $\{x_2, x_3, 2, 8\}$ is a circuit of M. The cocircuit $\{2,3,4,5\}$ of M implies that $|\{3,4\} \cap \{x_2, x_3\}| = 1$. Thus $\{x_1, x_2, x_3, 1\}$ is a cocircuit of M, so $3 \in \{x_1, x_2, x_3\}$ by orthogonality with the circuit $\{1,2,3\}$. If $\{4,6\} \cap \{x_1, x_2, x_3\} \neq \emptyset$, then orthogonality with the circuit $\{4,5,6\}$ implies that $\{x_1, x_2, x_3, 1\} = \{1,3,4,6\}$. Then $\lambda_M(\{1,2,\dots,6\}) \leq 2$; a contradiction. We deduce that $\{4,6\} \cap \{x_1, x_2, x_3\} = \emptyset$. Thus we may assume that $3 = x_3$. Then M has $\{x_1, x_2, 3, 1\}$ as a cocircuit and $\{x_2, 3, 2, 8\}$ as a circuit. But $\{1,2,3\}$ is also a circuit, so $\{1, 8, x_2\}$ is a circuit. Now let $x_2 = 9$ and $x_1 = 10$. Then $9 \notin \{1,2,\dots 8\}$ otherwise $\lambda_M(\{1,2,\dots,8\}) \leq 2$. If $10 \in \{1,2,\dots,9\}$, then $10 \notin \{1,3,8,9\}$. Orthogonality with known triangles implies that $10 \notin \{2,4,5,6,7\}$. We conclude that (i) holds.

Suppose next that $5 \in X_{18}$. Then $(X_{18} \cup 7, Y_{18} - 7) \cong (X_{18}, Y_{18})$ in $M\backslash 1/8$. Thus Y_{18} is a 4-element fan of $M\backslash 1/8$ having $(y_1, y_2, y_3, 7)$ as a fan ordering where $\{y_2, y_3, 7\}$ is a triangle of $M\backslash 1/8$ and hence of $M/8$, while $\{y_1, y_2, y_3\}$ is a triad of $M\backslash 1$. Orthogonality with the cocircuit $\{1,2,7,8\}$ of M implies that $\{y_2, y_3, 7, 8\}$ is a circuit of M. Now either $\{y_1, y_2, y_3\}$ or $\{y_1, y_2, y_3, 1\}$ is a cocircuit of M. Since M is internally 4-connected, the first possibility implies that $\{y_1, y_2, y_3\} \cap \{1,2,\dots,8\} = \emptyset$, so (ii) holds. Consider the second possibility. From it, we deduce that $3 \in \{y_1, y_2, y_3\}$. If $3 \in \{y_2, y_3\}$, then the cocircuit $\{2,3,4,5\}$ implies that $4 \in \{y_2, y_3\}$. Hence $\{3,4,7,8\}$ is a circuit of M, and $\lambda_M(\{1,2,\dots,8\}) \leq 2$; a contradiction. Thus $3 = y_1$. Evidently, $\{y_2, y_3\} \cap \{1,2,3,5,7,8\} = \emptyset$. Moreover, $\{y_2, y_3\}$ avoids $\{4,6\}$ otherwise, by orthogonality, $\{y_2, y_3\} = \{4,6\}$ and $\lambda_M(\{1,2,\dots,6\}) \leq 2$. Hence $\{y_2, y_3\} \cap \{1,2,\dots,8\} = \emptyset$ and (ii) holds. □

Lemma 10.6. *Suppose that M has $(\{1,2,3\}, \{4,5,6\}, \{2,3,4,5\})$ as a bowtie and that M has $\{2,5,7\}$ as a triangle and $\{1,2,7,8\}$ and $\{5,6,7,9\}$ as cocircuits. If $M\backslash 1$ has a unique 4-element fan, then either*

- *(i) $M\backslash 1, 5$ or $M\backslash 1/8$ is internally 4-connected; or*
- *(ii) M has a triangle containing $\{3,4\}$; or*
- *(iii)*
  - *(a) M has a triangle containing $\{6,9\}$ and an element 10 not in $\{1,2,\dots,9\}$; and*
  - *(b) M has a triangle $\{1,8,11\}$ and a cocircuit $\{1,3,11,12\}$ where $|\{1,2,\dots,12\}| = 12$.*

Proof. By assumption, $\{1,2,3\}$ is not the central triangle of a quasi rotor. Assume also that M has no triangle containing $\{3,4\}$. By Lemma 6.3, $M\backslash 1$ is $(4,4,S)$-connected. Now $\{8,2,7,5\}$ is the unique 4-element fan of $M\backslash 1$

and it has 5 as an end, so $M\backslash 1,5$ is 3-connected. Assume that $M\backslash 1,5$ is not internally 4-connected letting (X_{15}, Y_{15}) be a $(4,3)$-violator of it.

If $\{4,6\}$ or $\{2,7\}$ is contained in X_{15}, then $(X_{15} \cup 5, Y_{15})$ is a 3-separation of $M\backslash 1$ with $|X_{15} \cup 5| \geq 5$ and $|Y_{15}| \geq 4$, so neither $X_{15} \cup 5$ nor Y_{15} is $\{8,2,7,5\}$; a contradiction. Hence we may assume that $4 \in X_{15}$ and $6 \in Y_{15}$. Also, exactly one of 2 and 7 is in X_{15}. Thus we have two cases to consider:

(I) $\{4,7\} \subseteq X_{15}$ and $\{6,2\} \subseteq Y_{15}$; or
(II) $\{4,2\} \subseteq X_{15}$ and $\{6,7\} \subseteq Y_{15}$.

Consider case (I). Suppose first that $3 \in X_{15}$. Then $(X_{15} \cup 2, Y_{15} - 2) \cong (X_{15}, Y_{15})$ in $M\backslash 1,5$, so Y_{15} is a 4-element fan in $M\backslash 1,5$ otherwise we have a contradiction. Now Y_{15} has $(y_1, y_2, y_3, 2)$ as a fan ordering where $\{y_1, y_2, y_3\}$ is a triangle containing 6, and $\{y_2, y_3, 2\}$ is a triad of $M\backslash 1,5$. The circuit $\{2,4,6,7\}$ of $M\backslash 1,5$ implies that $6 \in \{y_2, y_3\}$, so $6 = y_3$, say. The cocircuit $\{6,7,9\}$ of $M\backslash 1,5$ implies that $9 \in \{y_1, y_2\}$. But $\{2,6,9\}$ is not a cocircuit of $M\backslash 1,5$, so $9 = y_1$. Now $y_2 \notin \{1,2,\dots,9\}$, otherwise $\lambda_M(\{1,2,\dots,9\}) \leq 2$. Taking $y_2 = 10$, we have that (iii)(a) holds.

Next in case (I), assume that $3 \in Y_{15}$. Then $(X_{15}, Y_{15}) \cong (X_{15} - 4, Y_{15} \cup 4)$ in $M\backslash 1,5$ and we have a contradiction unless X_{15} is a 4-element fan in $M\backslash 1,5$ containing a triangle that contains 7 but is different from $\{2,5,7\}$. The exceptional case contradicts Lemma 10.4.

Now consider case (II) and look at the location of 8. Suppose $8 \in X_{15}$. Then $(X_{15} \cup 7, Y_{15} - 7) \cong (X_{15}, Y_{15})$ in $M\backslash 1,5$. Thus Y_{15} is a 4-element fan in $M\backslash 1,5$ otherwise we obtain a contradiction. We have $(y_1, y_2, y_3, 7)$ as a fan ordering of Y_{15} where $\{y_1, y_2, y_3\}$ is a triangle and $\{y_2, y_3, 7\}$ is a triad. The circuit $\{2,4,6,7\}$ of $M\backslash 1,5$ implies that $6 \in \{y_2, y_3\}$. Since $\{6,7,9\}$ is a cocircuit of $M\backslash 1,5$, we deduce that $\{y_2, y_3\} = \{6,9\}$. By orthogonality, y_1 is not in $\{1,2,\dots,9\}$, so (iii)(a) holds with $10 = y_1$.

Finally, in case (II), suppose that $8 \in Y_{15}$. Then $(X_{15}, Y_{15}) \cong (X_{15} - 2, Y_{15} \cup 2)$ in $M\backslash 1,5$. Therefore X_{15} is a 4-element fan of $M\backslash 1,5$ having $(x_1, x_2, x_3, 2)$ as a fan ordering with $\{x_1, x_2, x_3\}$ as a triangle containing 4. It follows, by orthogonality with the cocircuit $\{2,3,4\}$ of $M\backslash 1,5$, that the triangle $\{x_1, x_2, x_3\}$ contains 3 and so is $\{1,2,3\}$; a contradiction.

To complete the proof, we need to show that when (iii)(a) holds, so does (iii)(b). To establish this, assume that (iii)(a) holds and consider $M\backslash 1/8$. By Lemma 10.5, it is 3-connected. Moreover, if (X_{18}, Y_{18}) is a $(4,3)$-violator of it, then we may assume $2 \in X_{18}$ and $7 \in Y_{18}$.

Suppose that $5 \in X_{18}$. Then, by Lemma 10.5, Y_{18} is a 4-element fan of $M\backslash 1/8$ having $(y_1, y_2, y_3, 7)$ as a fan ordering where $\{y_2, y_3\} \cap \{1,2,\dots,8\} = \emptyset$; and $\{y_2, y_3, 7, 8\}$ is a circuit of M, while $\{3, y_2, y_3, 1\}$ or $\{y_1, y_2, y_3\}$ is a cocircuit of M. The cocircuit $\{5,6,7,9\}$ of M implies that $9 \in \{y_2, y_3\}$. But 9 is in a triangle of M. Hence $\{y_1, y_2, y_3\}$ is not a cocircuit of M, so $\{3, y_2, y_3, 1\}$ is. Moreover, since $9 \in \{y_2, y_3\}$, the triangle $\{6,9,10\}$ implies that $\{y_2, y_3\} = \{9,10\}$. Therefore $\{3,9,10,1\}$ is a cocircuit of M, so $\lambda_M(\{1,2,\dots,10\}) \leq 2$; a contradiction.

We may now assume that $5 \in Y_{18}$. Then, by Lemma 10.5, M has a triangle $\{1,8,11\}$ and a cocircuit $\{1,3,11,12\}$ where $|\{1,2,\dots,8,11,12\}| = 10$. Thus $|\{1,2,\dots,12\}| = 12$ provided $\{9,10\} \cap \{11,12\} = \emptyset$. The triangle $\{6,9,10\}$ and the cocircuit $\{1,3,11,12\}$ imply that $|\{9,10\} \cap \{11,12\}|$ is 0 or 2. In the latter case, $\lambda_M(\{1,2,\dots,7,9,10\}) \leq 2$; a contradiction. □

Lemma 10.7. *Let $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ be a bowtie in M. Suppose that no single-element deletion of M is internally 4-connected and that $M\backslash 1$ has at least two 4-element fans. Then*

(i) for $\{a,b\} = \{4,5\}$, there are distinct elements x_{12}, x_{2a}, x_{13} and x_{3b} not in $\{1,2,\dots,6\}$ such that $\{2,a,x_{2a}\}$ and $\{3,b,x_{3b}\}$ are circuits of M, and $\{1,2,x_{12},x_{2a}\}$ and $\{1,3,x_{13},x_{3b}\}$ are cocircuits of M; or

(ii) M has a triangle $\{7,8,9\}$ disjoint from $\{1,2,\dots,6\}$ such that $(\{1,2,3\},\{7,8,9\},C^*)$ is a bowtie and C^* contains 1.

Proof. By Lemma 6.3, $M\backslash 1$ is $(4,4,S)$-connected. Let (y_1,y_2,y_3,y_4) be a fan ordering of a fan in $M\backslash 1$ where $\{y_2,y_3,y_4\}$ is a triad. Then $\{y_2,y_3,y_4,1\}$ is a cocircuit of M, so $|\{y_2,y_3,y_4\} \cap \{2,3\}| = 1$. By symmetry, we may suppose that $2 \in \{y_2,y_3,y_4\}$ and indeed that $2 \in \{y_3,y_4\}$.

First we show:

10.7.1. *If $2 = y_3$, then M has $\{y_1,y_2,2\}$ as a triangle and $\{y_2,2,y_4,1\}$ as a cocircuit where $y_1 \in \{4,5\}$, and $\{y_2,y_4\}$ avoids $\{1,2,\dots,6\}$.*

Since M is binary, $|\{y_1,y_2\} \cap \{4,5\}| = 1$. If 4 or 5 is y_2, then $\{4,5\} = \{y_2,y_4\}$, so $\{y_2,2,y_4,1\} \subseteq \{1,2,\dots,6\}$. Thus $r^*(\{1,2,\dots,6\}) \leq 4$, so $\lambda(\{1,2,\dots,6\}) \leq 2$; a contradiction. Hence y_1 is 4 or 5. Certainly $\{y_2,y_4\}$ avoids $\{1,2,3\}$. If $\{y_2,y_4\}$ meets $\{4,5,6\}$ then $\{y_2,y_4\} \subseteq \{4,5,6\}$ and again we get the contradiction that $\lambda(\{1,2,\dots,6\}) \leq 2$. Thus (10.7.1) holds.

10.7.2. *If $2 = y_4$, then $(\{1,2,3\},\{y_1,y_2,y_3\},\{1,2,y_2,y_3\})$ is a bowtie and $|\{1,2,\dots,6,y_1,y_2,y_3\}| = 9$.*

Certainly $(\{1,2,3\},\{y_1,y_2,y_3\},\{1,2,y_2,y_3\})$ is a bowtie. The fact that $|\{1,2,\dots,6,y_1,y_2,y_3\}| = 9$ follows immediately from Lemma 10.2.

We now assume that (ii) of the lemma does not hold. Then, by (10.7.2) and symmetry, we get that if (y_1,y_2,y_3,y_4) is a fan ordering of a fan in $M\backslash 1$ having $\{y_1,y_2,y_3\}$ as a triangle, then $y_4 \notin \{2,3\}$. Suppose $M\backslash 1$ has two distinct 4-element fans having fan orderings $(y_1,y_2,2,y_4)$ and $(z_1,z_2,2,z_4)$ where $\{y_1,y_2,2\}$ and $\{z_1,z_2,2\}$ are triangles. Then, by (10.7.1), each of y_1 and z_1 is in $\{4,5\}$. If $y_1 = z_1$, then $y_2 = z_2$, so $y_4 = z_4$, and the fans are equal. Thus we may assume that $y_1 \neq z_1$. Then, without loss of generality, $y_1 = 4$ and $z_1 = 5$. It follows that $M|\{2,4,5,6,y_2,z_2\} \cong M(K_4)$; a contradiction.

It remains to consider when $M\backslash 1$ has fans with fan orderings $(y_1,y_2,2,y_4)$ and $(z_1,z_2,3,z_4)$ where $\{y_1,y_2,2\}$ and $\{z_1,z_2,3\}$ are triangles. If $y_i = z_i$ for some i in $\{1,2\}$, then we get an $M(K_4)$-restriction of M containing $\{1,2,3,y_i\}$. Hence we may assume that $y_1 \neq z_1$ and $y_2 \neq z_2$. Thus, by (10.7.1), we may suppose that $y_1 = 5$ and $z_1 = 4$.

We show next that $|\{1, 2, \ldots, 6, y_2, y_4, z_2, z_4\}| = 10$. If $y_2 = z_4$, then the triangle $\{5, y_2, 2\}$ and triad $\{z_2, 3, y_2\}$ of $M\backslash 1$ imply that $z_2 \in \{2, 5\}$, which does not occur. Thus $y_2 \neq z_4$ and, by symmetry, $y_4 \neq z_2$. Finally, suppose $y_4 = z_4$. Then let $Z = \{1, 2, 3, 4, 5, y_2, z_2, y_4\}$. We have $r(Z) \leq 5$ and $r^*(Z) \leq 5$, so $\lambda(Z) \leq 2$; a contradiction. We conclude that $|\{1, 2, \ldots, 6, y_2, y_4, z_2, z_4\}| = 10$. Hence (i) holds. □

Lemma 10.8. *Suppose that M has $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ as a bowtie and that $\{2, 5, 7\}$, $\{6, 9, 10\}$, and $\{1, 8, 11\}$ are triangles while $\{1, 2, 7, 8\}$, $\{5, 6, 7, 9\}$, and $\{1, 3, 11, 12\}$ are cocircuits, where $|\{1, 2, \ldots, 12\}| = 12$. Assume that $M\backslash 1$ has a unique 4-element fan. Then $M\backslash 11/12$ or $M/12$ is internally 4-connected.*

Proof. The matroid M has $(\{1, 8, 11\}, \{2, 5, 7\}, \{1, 2, 7, 8\})$ as a bowtie and has $\{1, 2, 3\}$ as a triangle and $\{1, 3, 11, 12\}$ and $\{2, 3, 4, 5\}$ as cocircuits. Suppose $M\backslash 11$ has a unique 4-element fan. Then, by Lemma 10.6, $M\backslash 11, 2$ or $M\backslash 11/12$ is internally 4-connected; or M has a triangle containing $\{7, 8\}$; or M has a triangle containing $\{11, 12\}$. Since $M\backslash 11, 2$ has a 4-element fan while the lemma holds if $M\backslash 11/12$ is internally 4-connected, we may assume that one of the last two possibilities occurs. The second last possibility contradicts Lemma 10.4, while the last contradicts the fact that $M\backslash 1$ has a unique 4-element fan.

We may now assume that $M\backslash 11$ has more than one 4-element fan. Then, by Lemma 10.7 applied to the bowtie $(\{1, 8, 11\}, \{2, 5, 7\}, \{1, 2, 7, 8\})$, either

(a) M has triangles containing $\{1, 7\}$ and $\{2, 8\}$; or
(b) M has a triangle containing $\{7, 8\}$; or
(c) M has a triangle $\{13, 14, 15\}$ disjoint from $\{1, 8, 11, 2, 5, 7\}$, and M has a bowtie $(\{1, 8, 11\}, \{13, 14, 15\}, C^*)$ where C^* contains $\{11, 13, 14\}$.

In the first case, M has an $M(K_4)$-restriction, which is not so. The second case is excluded by Lemma 10.4. Hence the third case occurs. Then C^* contains $\{1, 11\}$ or $\{8, 11\}$. Suppose $\{1, 11\} \subseteq C^*$. The circuit $\{1, 2, 3\}$ implies that $3 \in C^*$. Thus C^* contains $\{1, 3, 11\}$, so $C^* = \{1, 3, 11, 12\}$ and $\{13, 14\} = \{3, 12\}$. Hence 12 is in a triangle of $M\backslash 1$, which contradicts the fact that $M\backslash 1$ has a unique 4-element fan. Thus C^* contains $\{8, 11\}$.

The matroid $M\backslash 3$ is 3-connected and has $\{1, 8, 11, 12\}$ as a fan. Now 12 is not in a triangle of M otherwise M has an $M(K_4)$-restriction or $M\backslash 1$ does not have a unique 4-element fan. Thus 12 is a fan end in $M\backslash 3$. Hence $M\backslash 3/12$ is 3-connected. It follows that $M/12$ is 3-connected since 12 is not in a triangle of M. Assume that $M/12$ is not internally 4-connected. Let (X_{12}, Y_{12}) be a $(4, 3)$-violator of $M/12$. Then neither X_{12} nor Y_{12} contains $\{1, 3, 11\}$. Thus we may assume that one of the following possibilities occurs:

(I) $\{1, 11\} \subseteq X_{12}$ and $3 \in Y_{12}$; or
(II) $\{3, 11\} \subseteq X_{12}$ and $1 \in Y_{12}$; or
(III) $\{1, 3\} \subseteq X_{12}$ and $11 \in Y_{12}$.

Consider case (I). Suppose first that $2 \in X_{12}$. Then $(X_{12} \cup 3, Y_{12} - 3) \cong (X_{12}, Y_{12})$. Thus we may assume that Y_{12} is a 4-element fan in $M/12$ having $(y_1, y_2, y_3, 3)$ as a fan ordering where $\{y_2, y_3, 3\}$ is a triangle. As $M/12$ has $\{2, 3, 4, 5\}$ as a cocircuit, the triad $\{y_1, y_2, y_3\}$ meets $\{2, 4, 5\}$; a contradiction. Hence we may assume that $2 \in Y_{12}$. Next consider the location of 8. Suppose $8 \in Y_{12}$. Then $\{1, 11\} \subseteq \mathrm{cl}_{M/12}(Y_{12})$. Thus $(X_{12}, Y_{12}) \cong (X_{12} - 11, Y_{12} \cup 11) \cong (X_{12} - 11 - 1, Y_{12} \cup 11 \cup 1)$. Hence $|X_{12} - 11 - 1| = 3$ otherwise we get a contradiction. Thus $X_{12} - 11 - 1$ is a triad $\{x_1, x_2, x_3\}$ of $M/12$, and $X_{12} - 11$ is a 4-element fan having $(x_1, x_2, x_3, 1)$ as a fan ordering. The triangle $\{x_2, x_3, 1\}$ of $M/12$ implies that $\{2, 7, 8\}$ meets the triad $\{x_1, x_2, x_3\}$; a contradiction.

We may now assume that $8 \in X_{12}$, so $\{1, 11, 8\} \subseteq X_{12}$ and $\{2, 3\} \subseteq Y_{12}$. If $7 \in X_{12}$, then $(X_{12} \cup 2, Y_{12} - 2) \cong (X_{12}, Y_{12})$ so we get a contradiction unless Y_{12} is a 4-element fan in $M/12$. In the exceptional case, $(y_1, y_2, y_3, 2)$ is a fan ordering of Y_{12} with $\{y_2, y_3, 2\}$ as a triad; a contradiction. We deduce that $7 \in Y_{12}$. Then $(X_{12}, Y_{12}) \cong (X_{12} - 1, Y_{12} \cup 1) \cong (X_{12} - 1 - 8, Y_{12} \cup 1 \cup 8) \cong (X_{12} - 1 - 8 - 11, Y_{12} \cup 1 \cup 8 \cup 11)$. Hence we get a contradiction if $|X_{12}| \geq 7$. If $|X_{12}| \leq 6$, then $|X_{12} - 1| \leq 5$ so $X_{12} - 1$ is a fan of $M/12$. Since $8 \in \mathrm{cl^*}_{M/12}(Y_{12} \cup 1)$, it follows that 8 is in a triad of M; a contradiction. We conclude that case (I) does not occur.

Next consider case (II). Suppose first that $2 \in X_{12}$, so $\{2, 3, 11\} \subseteq X_{12}$ and $1 \in Y_{12}$. Then, as $(X_{12} \cup 1, Y_{12} - 1) \cong (X_{12}, Y_{12})$, we must have that Y_{12} is a 4-element fan of $M/12$ having $(y_1, y_2, y_3, 1)$ as a fan ordering where $\{y_2, y_3, 1\}$ is a triangle of $M/12$. The cocircuit $\{1, 2, 7, 8\}$ implies that $\{y_2, y_3\}$ meets $\{2, 7, 8\}$. Thus the triad $\{y_1, y_2, y_3\}$ meets $\{2, 7, 8\}$; a contradiction. Hence we may assume that $2 \in Y_{12}$. Then $(X_{12} - 3, Y_{12} \cup 3) \cong (X_{12}, Y_{12})$ and we have reduced to case (I) unless X_{12} is a 4-element fan of $M/12$ having a fan ordering $(x_1, x_2, x_3, 3)$ where $\{x_1, x_2, x_3\}$ is a triad. In the exceptional case, the triangle $\{x_2, x_3, 3\}$ and cocircuit $\{2, 3, 4, 5\}$ of $M/12$ imply that the triad $\{x_1, x_2, x_3\}$ meets $\{2, 4, 5\}$; a contradiction. We conclude that case (II) does not occur.

Finally, consider case (III). Suppose first that $8 \in X_{12}$. Then X_{12} contains $\{1, 3, 8\}$ and Y_{12} contains 11. As $(X_{12} \cup 11, Y_{12} - 11) \cong (X_{12}, Y_{12})$, we deduce that $M/12$ has $(y_1, y_2, y_3, 11)$ as a fan ordering of Y_{12} having $\{y_2, y_3, 11\}$ as a triangle. Then the triad $\{y_1, y_2, y_3\}$ meets $\{8, 13, 14\}$; a contradiction. We may now suppose that $8 \in Y_{12}$. Then $(X_{12}, Y_{12}) \cong (X_{12} - 1, Y_{12} \cup 1)$ and we get a contradiction unless $X_{12} - 1$ is a triad of $M/12$. But $X_{12} - 1$ contains 3 so it cannot be a triad. □

Lemma 10.9. *Suppose M has $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ as a bowtie and that M has $\{2, 5, 7\}$ and $\{3, 4, 10\}$ as circuits and $\{1, 2, 7, 8\}$ and $\{1, 3, 9, 10\}$ as cocircuits where $|\{1, 2, \ldots, 10\}| = 10$. If neither $M\backslash 1$ nor $M\backslash 6$ is internally 4-connected, then M has a 4-element cocircuit that contains $\{4, 6, 10\}$ or $\{5, 6, 7\}$ and has its fourth element outside of $\{1, 2, \ldots, 10\}$.*

Proof. Either (ii) or (iii) of Lemma 6.3 holds. Assume the former. If M has a triangle $\{11,12,13\}$ disjoint from $\{1,2,\dots,6\}$ and a cocircuit C^* containing $\{6,11,12\}$ and one of 4 and 5, then, by symmetry, we may assume that $5 \in C^*$. Then 2 or 7 is in C^*, so $7 \in \{11,12\}$. Thus $7 = 11$, without loss of generality, and $|\{11,12,13\} \cap \{1,2,7,8\}| = 2$, so $\{7,8\} \subseteq \{11,12,13\}$. If $\{7,8\} = \{11,12\}$, then $\{5,6,11,12\} \triangle \{1,2,7,8\}$, which equals $\{1,2,5,6\}$, is a cocircuit of M, so $\lambda_M(\{1,2,\dots,6\}) \leq 2$; a contradiction. Thus $8 = 13$. Hence $\lambda(\{1,2,3,4,5,6,7,8,12\}) \leq 2$; a contradiction. We conclude that (ii) of Lemma 6.3 does not occur.

We may now assume that (iii) of Lemma 6.3 occurs. Then M has a 4-cocircuit $\{a,b,d,6\}$ and a triangle $\{c,d,e\}$ where $|\{1,2,\dots,6,a,b\}| = 8$ and $c \in \{a,b\}$, $d \in \{4,5\}$, and $e \in \{2,3\}$. Without loss of generality, assume that $d = 5$. Then, since $\{2,5\}$ is contained in a triangle but M has no $M(K_4)$-restriction, M has no triangle containing $\{3,5\}$, so $e = 2$, and $c = 7$. Thus, without loss of generality, M has $\{a,5,6,7\}$ as a cocircuit. Then $a \notin \{1,2,3,4,5,6,7,10\}$ by orthogonality. If $a = 8$, then $\lambda(\{1,2,3,4,5,6,7,8\}) \leq 2$; a contradiction. If $a = 9$, then $\lambda(\{1,2,3,4,5,6,7,9,10\}) \leq 2$; a contradiction. □

Lemma 10.10. *Suppose that M has bowties $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ and $(\{2,5,7\},\{3,4,11\},\{2,3,4,5\})$, and M has cocircuits $\{1,2,7,8\}$ and $\{4,6,10,11\}$. Then each of $M\backslash 1$, $M\backslash 6$, $M\backslash 7$, and $M\backslash 11$ is $(4,4,S)$-connected, and each of 1, 6, 7, and 11 is in a unique triangle of M. Moreover,*

(i) M has a 4-cocircuit containing $\{5,6,7\}$; or
(ii) $M\backslash 6$ has a unique 4-element fan, and
  (a) $M\backslash 3,6,7$ is internally 4-connected; or
  (b) M has a triangle $\{g_1,g_2,g_3\}$ and a cocircuit $\{5,6,7,g_2,g_3\}$ where $\{g_1,g_2,g_3\} \cap \{1,2,3,4,5,6,7,8,10,11\} = \emptyset$.

Proof. By Lemma 6.3, for each e in $\{1,6,7,11\}$, the matroid $M\backslash e$ is $(4,4,S)$-connected. Thus each of 1, 6, 7, and 11 is in a unique triangle of M. Now assume that M has no 4-cocircuit containing $\{5,6,7\}$. First we show:

10.10.1. *If (X_6,Y_6) is a $(4,3)$-violator for $M\backslash 6$, then X_6 or Y_6 is $\{3,4,10,11\}$.*

This assertion is equivalent to the assertion that $M\backslash 6$ has $\{3,4,10,11\}$ as its unique 4-element fan. Suppose the latter fails. Then M has a triangle $\{a_1,a_2,a_3\}$ and a 4-cocircuit C^* that is $\{4,6,a_2,a_3\}$ or $\{5,6,a_2,a_3\}$. Suppose $C^* = \{4,6,a_2,a_3\}$. Then, the triangle $\{3,4,11\}$ implies that C^* contains 11 or 3. In the first case, $\{a_2,a_3\} = \{10,11\}$. Then $M\backslash 6$ is not $(4,4,S)$-connected; a contradiction. On the other hand, if $3 \in C^*$, then 1 or 2 is in C^* and $\lambda(\{1,2,\dots,6\}) \leq 2$; a contradiction. Hence we may suppose that C^* is $\{5,6,a_2,a_3\}$. Then $\{a_2,a_3\}$ meets $\{2,7\}$. But $2 \notin C^*$ otherwise $C^* \subseteq \{1,2,\dots,6\}$ and $\lambda(\{1,2,\dots,6\}) \leq 2$. Moreover, $7 \notin \{a_2,a_3\}$ otherwise C^* contains $\{5,6,7\}$. Thus (10.10.1) holds.

Next we show that

10.10.2. $M\backslash 7\backslash 3\backslash 6$ *is* 3*-connected.*

Certainly $M\backslash 7$ is 3-connected having $\{8,1,2,3\}$ as a maximal fan. As 3 is an end of this fan, $M\backslash 7\backslash 3$ is 3-connected and has $\{2,4,5,6\}$ as a fan. The element 6 is an end of a maximal fan unless $M\backslash 7\backslash 3$ has a triad T^* avoiding 2 but containing $\{4,6\}$ or $\{5,6\}$. Suppose T^* contains $\{4,6\}$. Then either $T^* = \{4,6,11\}$; or T^* avoids 11 and $T^* \cup 3$ is a cocircuit of $M\backslash 7$. In the first case, $\{4,6,11\}$ is a triad of $M\backslash 7$. But $\{4,6,11,10\}$ is a cocircuit of M, so we have a contradiction. Thus $T^* \cup 3$ is a cocircuit of $M\backslash 7$. Then 1 or 2 is in T^*. Hence T^* is $\{4,6,1\}$ or $\{4,6,2\}$, so M has $\{4,6,1,3\}$ or $\{4,6,2,3,7\}$ as a cocircuit. The first possibility implies that $\lambda_M(\{1,2,\dots,6\}) \leq 2$. The second implies that $\{4,6,2,3,7\} \triangle \{2,3,4,5\}$, that is, $\{5,6,7\}$ is a cocircuit of M; a contradiction.

Now suppose that $\{5,6\} \subseteq T^*$. As $2 \notin T^*$, it follows that $T^* \cup 7$ is a cocircuit of $M\backslash 3$ containing $\{5,6,7\}$. Since M has no 4-cocircuit containing $\{5,6,7\}$, we deduce that $T^* \cup 7 \cup 3$ is a cocircuit of M, which must equal $\{1,5,6,7,3\}$. Then $\lambda_M(\{1,2,\dots,7\}) \leq 2$; a contradiction. We conclude that 6 is indeed an end of a maximal fan in $M\backslash 7\backslash 3$, so $M\backslash 7\backslash 3\backslash 6$ is 3-connected.

Clearly $M\backslash 3$ is not internally 4-connected. A $(4,3)$-violator (X_3, Y_3) of $M\backslash 3$ must have exactly one of 1 and 2 in X_3, and exactly one of 4 and 11 in X_3. Assume that $M\backslash 3,6,7$ is not internally 4-connected and let (X_{367}, Y_{367}) be a $(4,3)$-violator of it. Suppose $\{2,4,5\} \subseteq X_{367}$. Then $(X_{367} \cup 6 \cup 7, Y_{367})$ is a 3-separation of $M\backslash 3$. Then Y_{367} contains $\{1,11\}$.

Consider the location of 10. Suppose first that $10 \in X_{367}$. Then $(X_{367} \cup 11, Y_{367} - 11) \cong (X_{367}, Y_{367})$ in $M\backslash 3,6,7$, and $(X_{367} \cup 11 \cup \{3,6,7\}, Y_{367} - 11)$ is a 3-separation of M. Hence we may assume that Y_{367} is a 4-element fan in $M\backslash 3,6,7$ containing a triad $\{g_2, g_3, 11\}$ and a triangle $\{g_1, g_2, g_3\}$ where the latter contains 1. By orthogonality, $8 \in \{g_1, g_2, g_3\}$. But $M\backslash 7$ has no triangle containing $\{1,8\}$ since it is $(4,4,S)$-connected. This contradiction implies that $10 \in Y_{367}$.

We now have $\{2,4,5\} \subseteq X_{367}$ and $\{1,11,10\} \subseteq Y_{367}$. Then $(X_{367} - 4, Y_{367} \cup 4) \cong (X_{367}, Y_{367})$ in $M\backslash 3,6,7$. This means that $((X_{367} - 4) \cup 7, (Y_{367} \cup 4) \cup 3)$ is a 3-separation of $M\backslash 6$ in which each side has at least four elements and neither side is $\{3,4,10,11\}$ contradicting (10.10.1).

We now know that neither X_{367} nor Y_{367} contains $\{2,4,5\}$. Let $\{2,4,5\} = \{\alpha, \beta, \gamma\}$ and assume that $\{\alpha, \beta\} \subseteq X_{367}$ and $\gamma \in Y_{367}$. If $|Y_{367}| > 4$, then $(X_{367} \cup \gamma, Y_{367} - \gamma)$ has $\{2,4,5\}$ in $X_{367} \cup \gamma$ and has $|Y_{367} - \gamma| \geq 4$, so we have reduced to the previous case. Thus we may assume that Y_{367} is a 4-element fan in $M\backslash 3,6,7$ containing a triangle $\{g_1, g_2, g_3\}$ and a triad $\{g_2, g_3, \gamma\}$. Hence $\{g_1, g_2, g_3\}$ avoids $\{2,3,4,5,6,7\}$. As M has $\{1,2,4,11\}$ as a circuit, if $\gamma \in \{2,4\}$, then $\{g_2, g_3\}$ meets $\{1,11\}$. But $\{g_1, g_2, g_3\}$ is not $\{1,2,3\}$ or $\{3,4,11\}$, the only triangles of M containing 1 and 11, respectively. Hence $\gamma \notin \{2,4\}$, so $\gamma = 5$. Thus $\{g_2, g_3, 5\}$ is a triad of $M\backslash 3,6,7$. As $\{g_2, g_3\} \cap \{2,4\} = \emptyset$, we deduce that $\{g_2, g_3, 5, 6, 7\}$ is a cocircuit of $M\backslash 3$, by orthogonality. Hence $\{g_2, g_3, 5, 6, 7\}$ is a cocircuit of M. We know $\{g_1, g_2, g_3\}$ avoids $\{2, 3, 4, 5, 6, 7\}$. It also avoids $\{1, 11\}$ since each of 1 and 11 is in a unique triangle. Finally, $\{g_1, g_2, g_3\}$ avoids $\{8, 10\}$ by orthogonality. We conclude that (ii)(b) holds. □

Lemma 10.11. *Suppose that M has bowties $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ and $(\{2, 5, 7\}, \{3, 4, 11\}, \{2, 3, 4, 5\})$ and cocircuits $\{1, 2, 7, 8\}$ and $\{4, 6, 10, 11\}$. Then $|\{1, 2, \dots, 8, 10, 11\}| = 10$. Moreover,*

(i) M has a 4-cocircuit containing $\{1, 3, 11\}$ or $\{5, 6, 7\}$; or
(ii) $M\backslash 3, 6, 7$ is internally 4-connected; or
(iii) $M\backslash 1/8$ is internally 4-connected.

Proof. One easily checks that $|\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11\}| = 10$. Suppose that (i) does not hold. Then, by Lemma 10.10 and symmetry, each of $M\backslash 1$, $M\backslash 6$, $M\backslash 7$, and $M\backslash 11$ has a unique 4-element fan. Suppose that $M\backslash 1/8$ is not internally 4-connected, letting (X_{18}, Y_{18}) be a $(4, 3)$-violator of it. Then, by Lemma 10.5, we may assume that $2 \in X_{18}$ and $7 \in Y_{18}$. Moreover, if $5 \in Y_{18}$, then M has a triangle $\{1, 8, 12\}$. This is a contradiction since, by Lemma 10.10, there is a unique triangle containing 1.

We may now assume that $5 \in X_{18}$. In that case, Lemma 10.5 implies that $\{7, 8, y_2, y_3\}$ is a circuit of M; and $\{3, y_2, y_3, 1\}$ or $\{y_1, y_2, y_3\}$ is a cocircuit of M where $\{y_1, y_2, y_3\} \cap \{1, 2, \dots, 8\} = \emptyset$. If $\{3, y_2, y_3, 1\}$ is a cocircuit of M, then the circuit $\{3, 4, 11\}$ implies, since $\{1, 3, 4, 6\}$ is not a cocircuit, that $11 \in \{y_2, y_3\}$, so (i) holds. We deduce that $\{y_1, y_2, y_3\}$ is a cocircuit of M. This cocircuit avoids $\{1, 2, 3, 4, 5, 6, 7, 8, 11\}$. If $10 \in \{y_1, y_2, y_3\}$, then orthogonality implies that $10 = y_1$. Hence $1, 2, \dots, 8, 10, 11, y_1, y_2, y_3$ are distinct except that, possibly, $10 = y_1$.

Now assume that $M\backslash 3, 6, 7$ is not internally 4-connected. Then, by Lemma 10.10, M has a triangle $\{g_1, g_2, g_3\}$ and a cocircuit $\{5, 6, 7, g_2, g_3\}$ where $\{g_1, g_2, g_3\} \cap \{1, 2, \dots, 8, 10, 11\} = \emptyset$. Now $\{2, 5, 8, y_2, y_3\}$ is a circuit of $M\backslash 3, 6, 7$. Thus $\{g_2, g_3\}$ meets $\{y_2, y_3\}$. But $\{y_1, y_2, y_3\}$ is a triad of M so it avoids $\{g_1, g_2, g_3\}$; a contradiction. □

Lemma 10.12. *Suppose that M has $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ and $(\{2, 5, 7\}, \{3, 4, 11\}, \{2, 3, 4, 5\})$ as bowties, and has $\{1, 2, 7, 8\}$, $\{5, 6, 7, 9\}$, and $\{4, 6, 10, 11\}$ as cocircuits. Then*

(i) $M/9$ is internally 4-connected; or
(ii) M has a 4-circuit $\{7, 8, 9, e_2\}$ and a triad $\{8, e_1, e_2\}$; or
(iii) M has a 4-circuit $\{6, 9, 10, f_2\}$ and a triad $\{10, f_1, f_2\}$.

Proof. We apply Lemma 10.10. Since $M\backslash 6$ is $(4, 4, S)$-connected having 9 in a triad at the end of a 4-element fan, $M\backslash 6/9$ is 3-connected. But M has no triangle containing 9, so $M/9$ is 3-connected. Assume it is not internally 4-connected. Then it has a $(4, 3)$-violator (X_9, Y_9). Clearly neither X_9 nor Y_9 contains $\{5, 6, 7\}$. Thus we may assume that one of the following holds:

(a) $\{5, 6\} \subseteq X_9$ and $7 \in Y_9$; or
(b) $\{5, 7\} \subseteq X_9$ and $6 \in Y_9$; or

(c) $5 \in X_9$ and $\{6, 7\} \subseteq Y_9$.

Consider case (a). If $4 \in Y_9$, then $(X_9 \cup 4, Y_9 - 4) \cong (X_9, Y_9)$ and $|Y_9 - 4| \geq 4$ unless $Y_9 - 4$ is a triad of $M/9$, and hence of M, containing 7. Since this does not arise, we may assume that $4 \in X_9$. Now assume that $2 \in X_9$. Then $(X_9 \cup 7, Y_9 - 7) \cong (X_9, Y_9)$. Hence Y_9 is a 4-element fan in $M/9$ having $(y_1, y_2, y_3, 7)$ as a fan ordering where $\{y_2, y_3, 7\}$ is a triangle. By Lemma 10.4, $\{2, 5, 7\}$ is the unique triangle of M containing 7. Thus $\{y_2, y_3, 7, 9\}$ is a circuit of M. By orthogonality, $8 \in \{y_2, y_3\}$, so we may take $8 = y_3$. Thus $\{y_2, 7, 8, 9\}$ is a circuit of M and $\{8, y_2, y_1\}$ is a cocircuit of M. Hence, when $2 \in X_9$, part (ii) of the lemma holds.

We may now suppose that $2 \in Y_9$. If $3 \in X_9$, then $(X_9 \cup 2, Y_9 - 2) \cong (X_9, Y_9)$, so we may assume that Y_9 is a 4-element fan of $M/9$ having a triad containing 2; a contradiction. Thus $3 \in Y_9$. Hence $\{5, 6, 4\} \subseteq X_9$ and $\{7, 2, 3\} \subseteq Y_9$. Now $(X_9, Y_9) \cong (X_9 - 5, Y_9 \cup 5) \cong (X_9 - 5 - 4, Y_9 \cup 5 \cup 4)$. Since we move 5 by closure and 4 by coclosure, but 4 is not in a triad of M, the set $X_9 - 5$ has at least six elements. But $(X_9 - 5 - 4 - 6, Y_9 \cup 5 \cup 4 \cup 6) \cong (X_9, Y_9)$ and $|X_9 - 5 - 4 - 6| \geq 4$, so we have a contradiction. We deduce that if case (a) arises, then (ii) of the lemma holds. By symmetry, if case (b) arises, then (iii) of the lemma holds.

Finally, consider case (c). If $4 \in X_9$, then $(X_9 \cup 6, Y_9 - 6) \cong (X_9, Y_9)$ so we reduce to case (a) unless $Y_9 - 6$ is a triad containing 7, which is not so. Thus we may assume that $4 \in Y_9$. Then $(X_9 - 5, Y_9 \cup 5) \cong (X_9, Y_9)$ so we may assume that X_9 is a 4-element fan of $M/9$ having 5 in a triangle. This triangle must contain 2, 3, or 4, so one of these elements is in a triad of M, a contradiction. □

Lemma 10.13. *Suppose that M has $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ and $(\{2, 5, 7\}, \{3, 4, 11\}, \{2, 3, 4, 5\})$ as bowties, and has $\{1, 2, 7, 8\}$, $\{5, 6, 7, 9\}$, and $\{4, 6, 10, 11\}$ as cocircuits. Then $|\{1, 2, \ldots, 11\}| = 11$. Moreover, either*

(i) $M/9$ is internally 4-connected; or
(ii) $M\backslash 1/8$ or $M\backslash 11/10$ is internally 4-connected; or
(iii) $M\backslash 1, 11, 5$ is internally 4-connected; or
(iv) M has a 4-cocircuit containing $\{1, 3, 11\}$.

Proof. First observe that one easily checks that

10.13.1. $|\{1, 2, \ldots, 11\}| = 11$.

Assume that none of (i)-(iv) holds. Then M has no 4-cocircuit containing $\{1, 3, 11\}$. By orthogonality and the fact that $\lambda(\{1, 2, \ldots, 6\}) \geq 3$, it follows that M has no 4-cocircuit containing $\{1, 3\}$. By Lemma 10.10, $M\backslash 1$, $M\backslash 7$, $M\backslash 6$ and $M\backslash 11$ are all $(4, 4, S)$-connected and M has no triangles containing 1, 7, 6, or 11 beyond those in the bowties listed. As $M\backslash 1$ has a unique 4-element fan and $M\backslash 1/8$ is not internally 4-connected, Lemma 10.5 implies that M has a circuit $\{y_2, y_3, 7, 8\}$ and a cocircuit $\{y_1, y_2, y_3\}$ where $\{y_1, y_2, y_3\}$ avoids $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The cocircuit $\{5, 6, 7, 9\}$ implies that

$9 \in \{y_2, y_3\}$ so we may take $9 = y_3$, without loss of generality. Hence we have $\{7, 8, 9, y_2\}$ as a circuit and $\{y_1, y_2, 9\}$ as a triad. Moreover,

10.13.2. $|\{1, 2, \ldots, 9, y_1, y_2\}| = 11$.

By symmetry, M has a circuit $\{6, 9, 10, z_2\}$ and a triad $\{z_1, z_2, 9\}$, and $|\{1, 2, \ldots, 8, 10, z_1, z_2\}| = 11$. Now $|\{z_1, z_2, 9\} \cap \{7, 8, 9, y_2\}| = 2$. But $\{7, 8\} \cap \{z_1, z_2\} = \emptyset$, so

$$\{z_1, z_2, 9\} = \{y_1, y_2, 9\}.$$

By Lemma 10.10 and symmetry, since $M\backslash 1, 11, 5$ is not internally 4-connected, M has a triangle $\{g_1, g_2, g_3\}$ and a cocircuit $\{1, 3, 11, g_2, g_3\}$ where $\{g_1, g_2, g_3\} \cap \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11\} = \emptyset$. Thus, by (10.13.2) and orthogonality,

10.13.3. $|\{1, 2, \ldots, 9, y_1, y_2, g_1, g_2, g_3, 11\}| = 15$.

Also $10 \notin \{1, 2, \ldots, 9, 11, g_1, g_2, g_3\}$. By symmetry,

$$|\{1, 2, \ldots, 7, 9, 10, z_1, z_2, g_1, g_2, g_3, 11\}| = 15.$$

Suppose $y_2 = z_2$. The circuits $\{7, 8, 9, y_2\}$ and $\{6, 9, 10, z_2\}$ imply that $\{6, 7, 8, 10\}$ is a circuit. Then $\lambda(\{1, 2, \ldots, 11\}) \leq 2$, so we have a contradiction since $|E(M)| \geq 15$. Thus $(y_2, z_2) = (z_1, y_1)$. Then, by Lemma 10.12 and symmetry, we may assume that M has 4-circuit $\{7, 8, 9, e_2\}$ and a triad $\{8, e_1, e_2\}$. As $\{7, 8, 9, y_2\}$ is a circuit, we must have that $e_2 = y_2$. Thus the cocircuits of M include $\{8, e_1, y_2\}$ and $\{9, y_1, y_2\}$, while the circuits include $\{7, 8, 9, y_2\}$ and $\{6, 9, 10, y_1\}$. Next we observe that

10.13.4. $|\{1, 2, \ldots, 11, y_1, y_2, g_1, g_2, g_3\}| = 16$.

By (10.13.3), if this fails, then $10 \in \{y_1, y_2\}$. But $10 \neq y_1$ as $\{6, 9, 10, y_1\}$ is a 4-circuit; and $10 \neq y_2$ otherwise the triad $\{9, y_1, y_2\}$ is contained in the circuit $\{6, 9, 10, y_1\}$.

Now $e_1 \notin \{1, 2, \ldots, 7, 11, g_1, g_2, g_3\}$ since M has no 4-element fans. By construction, $e_1 \notin \{8, y_2\}$ and, by orthogonality between $\{8, e_1, y_2\}$ and $\{6, 9, 10, y_1\}$, we have $e_1 \notin \{9, 10, y_1\}$. Thus $|\{1, 2, \ldots, 11, y_1, y_2, g_1, g_2, g_3, e_1\}| \geq 17$. As $\lambda_M(\{1, 2, \ldots, 11, y_1, y_2\}) \leq 7 + 8 - 13 = 2$, this gives a contradiction. □

The next lemma will be used in the proof of the subsequent lemma.

Lemma 10.14. *Let (X, Y) be a 3-separation of $M(\mathcal{W}_k)$ for some $k \geq 3$. Then X and Y are fans.*

Proof. This is certainly true if $k = 3$, so assume that $k \geq 4$. Then the set of spokes of $\mathcal{W}_k$ is uniquely defined and neither X nor Y contains this set of spokes. Let $\mathcal{W}_k$ be labelled as in Figure 8. We show first that:

10.14.1. *If s_1 and s_2 are in X, then $w_1 \in X$.*

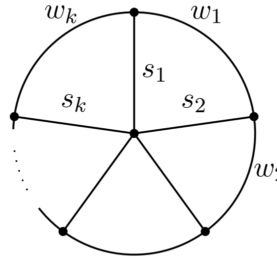

FIGURE 8. A labelled wheel.

Suppose $w_1 \notin X$. Then, since $(X \cup w_1, Y - w_1) \cong (X, Y)$, we must have that $w_1 \in \mathrm{cl}(Y - w_1)$. Thus w_k and w_2 are in Y. If Y contains $\{w_1, w_2, \ldots, w_k\}$, then, as X does not contain the set of spokes, $r(Y) = k$ and $r(X) = |X|$, so $r(X) + r(Y) \neq r(M) + 2$; a contradiction. Thus Y does not contain $\{w_1, w_2, \ldots, w_k\}$. Hence Y contains $s_{i+1}, w_{i+1}, \ldots, w_k, w_1, \ldots, w_j, s_{j+1}$ for some $i > j$. Then s_1 is a coloop of $M|X$. But $s_1 \in \mathrm{cl}(Y)$. Thus $(X - s_1, Y \cup s_1)$ is a 2-separation of M; a contradiction.

By (10.14.1) and duality, if two consecutive rim elements are in X, then so is the spoke element that forms a triad with these two rim elements. Hence if X is a fan, so is Y. Assume that X is not a fan. Then we can partition the spokes of $\mathcal{W}_k$ into maximal consecutive sets that are contained entirely in X or entirely in Y. As there must be at least four such sets, it follows easily that $r(X) + r(Y) \geq r(M) + 4$; a contradiction. □

Lemma 10.15. *Suppose M has bowties $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ and $(\{2,5,7\},\{3,4,11\},\{2,3,4,5\})$, and M has cocircuits $\{1,2,7,8\}$, $\{5,6,7,9\}$, $\{4,6,11,10\}$, and $\{1,3,11,12\}$. If $|E(M)| \geq 13$, then $M.\{1,2,\ldots,12\}$ is isomorphic to the cycle matroid of the octahedron. Moreover,*

- (i) *M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 2$; or*
- (ii) *after a possible symmetric relabelling, M has a 4-circuit $\{7,8,9,y_2\}$ and has triads $\{z_1,y_2,8\}$ and $\{y_1,y_2,9\}$ where $|\{1,2,\ldots,12,y_2,z_1,y_1\}| = 15$, and $M/8,9\backslash y_2$ is internally 4-connected; or*
- (iii) *M is isomorphic to the cycle matroid of a terrahawk.*

Proof. Let $Z = \{1,2,\ldots,12\}$, $R = E(M) - Z$, and $S = \{1,2,\ldots,7,11\}$. By Lemma 10.10, $|Z| = 12$. The symmetric difference of the cocircuits $\{2,3,4,5\}$, $\{1,2,7,8\}$, $\{5,6,7,9\}$, $\{4,6,11,10\}$, and $\{1,3,11,12\}$ is $\{8,9,10,12\}$, which must be a cocircuit of M. Consider $M\backslash\{8,9,10,12\}$. We have $\lambda_{M\backslash\{8,9,10,12\}}(S) = 0$. Hence $M\backslash\{8,9,10,12\} = (M|S) \oplus (M|R)$. Thus $M\backslash\{8,9,10,12\}/R = M|S \cong M(\mathcal{W}_4)$. Since M/R has $\{8,9,10,12\}$ as a cocircuit, we deduce that M/R has rank 5.

Observe that $r_{M/R}(\{8,9,10,12\}) = 4$ otherwise M has a circuit C containing some non-empty subset X of $\{8,9,10,12\}$ such that $C \subseteq R \cup X$. But the cocircuits of M contained in Z prevent the existence of such a circuit. As $\{2,3,4,5\}$ is a cocircuit of M/R, it follows that $M/R\backslash\{2,3,4,5\}$ has $\{8,9,10,12\}$ as a basis. By orthogonality, the fundamental circuit of 7 with respect to this basis avoids $\{10,12\}$ and contains $\{8,9\}$. Hence it is $\{7,8,9\}$. Likewise, M/R has $\{6,9,10\}$, $\{11,10,12\}$, and $\{1,8,12\}$ as circuits. We deduce, since M is binary, that M/R is isomorphic to the cycle matroid of the octahedron.

We now know that if $13 \leq |E(M)| \leq 14$, then M has a proper internally 4-connected minor N such that $|E(M)|-|E(N)| \leq 2$. Hence we may assume that $|E(M)| \geq 15$ and that M has no proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 2$.

By Lemma 10.10, each of $M\backslash 1$, $M\backslash 7$, $M\backslash 6$, and $M\backslash 11$ is $(4,4,S)$-connected, and M has no triangles containing any of 1, 7, 6, or 11 except for those in the two specified bowties. Since M contains no quasi rotor and $\{1,3,11,12\}$ is a cocircuit, it follows by Lemma 8.2 and symmetry that we may assume that M has a 4-circuit $\{y_2,9,7,8\}$ and a triad $\{y_1,y_2,9\}$ where $|\{1,2,\dots,9,y_1,y_2,11\}| = 12$.

Now apply Lemma 8.2 again, this time focussed on the element 9. We get that M has either

(a) a 4-circuit $\{7,8,9,z_2\}$ and a triad $\{z_1,z_2,8\}$; or
(b) a 4-circuit $\{6,9,10,z_2\}$ and a triad $\{z_1,z_2,10\}$ where $z_2 \notin \{1,2,\dots,10\}$.

In case (a), we see immediately that $z_2 = y_2$.

Next we show the following:

10.15.1. *If $\{6,9,10,z_2\}$ is a circuit and z_2 is in a triad of M where $z_2 \notin \{1,2,\dots,10\}$, then $z_2 = y_1$.*

Observe that $\{9,y_1,y_2\}$ meets $\{6,9,10,z_2\}$ in exactly two elements and $6 \notin \{y_1,y_2\}$. Thus $|\{y_1,y_2\} \cap \{10,z_2\}| = 1$. Suppose first that $z_2 = y_2$. Then $\{7,8,9,y_2\} \triangle \{6,9,10,y_2\} = \{6,7,8,10\}$, a circuit of M. Thus $\lambda(\{1,2,3,4,5,6,7,8,10,11\}) \leq 5+7-10 = 2$, so we have a contradiction since $|E(M)| \geq 15$.

Next observe that $y_2 \neq 10$ otherwise the circuit $\{y_2,9,7,8\}$ and the cocircuit $\{8,9,10,12\}$ have three common elements; a contradiction. Now suppose that $10 = y_1$. Then M has $\{y_2,9,7,8\}$ as a circuit and $\{9,10,y_2\}$ as a cocircuit. Thus $\lambda_M(\{1,2,\dots,11,y_2\}) \leq 7+7-12 = 2$, so $|E(M)| \leq 15$. Hence $|E(M)| = 15$, so $E(M) - \{1,2,\dots,11,y_2\}$ is a triangle or a triad of M. But $E(M) - \{1,2,\dots,11,y_2\}$ contains z_2, which is in a triad of M, so $E(M) - \{1,2,\dots,11,y_2\}$ is a triad of M. This triad meets the circuit $\{6,9,10,z_2\}$ in a single element; a contradiction. We conclude that we must have $z_2 = y_1$, that is, (10.15.1) holds.

Note that if $\{6,9,10,y_1\}$ is a circuit and $\{z_1,y_1,10\}$ is a cocircuit, then, as $\{y_1,y_2,9\}$ is also a cocircuit, we have, by symmetry, a special case of case (a). Hence we may suppose that case (a) occurs. Then $z_2 = y_2$, so

10.15.2. *M has a 4-circuit $\{7, 8, 9, y_2\}$ and has triads $\{z_1, y_2, 8\}$ and $\{y_1, y_2, 9\}$.*

By orthogonality, $y_2 \notin \{10, 12\}$. Thus $|\{1, 2, \ldots, 12, y_2\}| = 13$. Moreover, $\{z_1, y_1\} \cap \{1, 2, \ldots, 7, 8, 9, 11\} = \emptyset$. The symmetric difference of the cocircuits $\{8, 9, 10, 12\}$, $\{8, z_1, y_2\}$, and $\{9, y_1, y_2\}$ is $\{10, 12\} \triangle \{z_1\} \triangle \{y_1\}$. Thus either $|\{1, 2, \ldots, 12, y_2, z_1, y_1\}| = 15$, or $\{z_1, y_1\} = \{10, 12\}$.

10.15.3. $\{z_1, y_1\} \neq \{10, 12\}$ *and* $|\{1, 2, \ldots, 12, y_2, z_1, y_1\}| = 15$.

Assume the contrary. Then the symmetric difference of $\{8, y_2, z_1\}$ with whichever of $\{1, 3, 11, 12\}$ and $\{4, 6, 10, 11\}$ contains z_1 implies that either $\{8, y_2, 1, 3, 11\}$ or $\{8, y_2, 4, 6, 11\}$ is a cocircuit of M. Hence $r^*(\{1, 2, \ldots, 9, 11, y_2\}) \leq 8$ and $r(\{1, 2, \ldots, 9, 11, y_2\}) \leq 6$, so $\lambda(\{1, 2, \ldots, 9, 11, y_2\}) \leq 2$; a contradiction since $|E(M)| \geq 15$. Thus (10.15.3) holds.

Now apply Lemma 8.2 focussing on the element 10. Then M has either

(I) a 4-circuit $\{6, 9, 10, u_2\}$ and a cocircuit $\{u_1, u_2, 9\}$; or
(II) a 4-circuit $\{10, 11, 12, u_2\}$ and a cocircuit $\{u_1, u_2, 12\}$.

Next we show the following:

10.15.4. *If M has a circuit $\{6, 9, 10, u_2\}$, then M is isomorphic to the cycle matroid of a terrahawk.*

First, observe, by (10.15.1), that $u_2 = y_1$. Then $\lambda(\{1, 2, \ldots, 11, y_1, y_2\}) \leq 2$, so we get a contradiction if $|E(M)| \geq 17$. But $|E(M)| \geq 15$. Suppose $|E(M)| = 15$. Then, as M is spanned by $\{2, 3, 4, 5, 8, 9, 10\}$ and cospanned by $\{1, 2, 3, 4, 6, 8, 10, y_2\}$, we deduce that $r(M) = 7$. But $r(M.\{1, 2, \ldots, 12\}) = 5$. Hence $r(\{y_2, z_1, y_1\}) = 2$, so y_2 is in both a triangle and a triad; a contradiction. We deduce that $|E(M)| = 16$.

Now M is 3-connected. Let $Z' = \{1, 2, \ldots, 11, y_1, y_2\}$. Then

$$\lambda(Z') = r(Z') + r^*(Z') - |Z'| \leq 7 + 8 - 13 = 2.$$

Since $|E(M)| = 16$, we must have that equality holds throughout the last chain of inequalities. Let w_1 be the element of M not in $\{1, 2, \ldots, 12, y_1, y_2, z_1\}$. Then $\{12, w_1, z_1\}$ is a triangle or a triad. The fact that z_1 is in a triad implies that $\{12, w_1, z_1\}$ is a triad of M. As $r(\{1, 2, \ldots, 11, y_1, y_2\}) = 7$, it follows that $r(M) = 8$. As $\{8, 9, 10, 12\}$ is a cocircuit of M, we have $r(M \backslash \{8, 9, 10, 12\}) = 7$. But, in the notation from the start of the proof of the lemma, $M \backslash \{8, 9, 10, 12\} = (M|S) \oplus M|\{y_1, y_2, z_1, w_1\}$. As $r(M|S) = 4$, it follows that $r(\{y_1, y_2, z_1, w_1\}) = 3$. Since none of y_1, y_2, or z_1 is in a triangle, we deduce that $\{y_1, y_2, z_1, w_1\}$ is a circuit of M.

Now M has $\{2, 3, 4, 5, 8, 9, 10, 12\}$ as a basis B since this set spans $\{1, 7, 6, 11, y_2, y_1\}$, and M is 3-connected having 16 elements. Clearly the fundamental circuit of y_2 with respect to B is $\{y_2, 8, 2, 5, 9\}$. By orthogonality, $C(z_1, B)$, the fundamental circuit of z_1 with respect to B, is

$\{z_1, 8, 2, 3, 12\}$. The fundamental circuits of y_1 and w_1 contain $\{9, 5\}$ and $\{12, 3\}$. Moreover, $C(y_1, B)$ contains exactly one of $\{2, 8\}$, $\{3, 12\}$, and $\{4, 10\}$, and avoids the other two sets. As $C(y_2, B) = \{y_2, 8, 2, 5, 9\}$, it follows that $\{2, 8\} \not\subseteq C(y_1, B)$. Thus $C(y_1, B)$ contains exactly one of $\{3, 12\}$ and $\{4, 10\}$. If $\{3, 12\} \subseteq C(y_1, B)$, then $C(y_1, B) \triangle C(z_1, B) \triangle C(y_2, B) = \{y_1, z_1, y_2\}$. But the last set is not a circuit of M, so $C(y_1, B) = \{y_1, 9, 5, 4, 10\}$. Then the symmetric difference of $C(y_1, B)$, $C(z_1, B)$, $C(y_2, B)$, and $\{y_1, y_2, z_1, w_1\}$ is $\{w_1, 12, 3, 10, 4\}$, which must be the fundamental circuit of w_1. We conclude that M is, indeed, the cycle matroid of a terrahawk. This completes the proof of (10.15.4).

By the last result and symmetry, we may now assume that

10.15.5. *M has no 4-circuit containing $\{6, 9, 10\}$ or $\{12, 1, 8\}$.*

We deduce from the above that (II) holds, that is, M has a circuit $\{10, 11, 12, u_2\}$ and a cocircuit $\{u_1, u_2, 12\}$. By applying Lemma 8.2 relative to the element 12, we get that M has a cocircuit $\{v_1, u_2, 10\}$.

Since 8 is in a triad, $M/8$ is 3-connected. This matroid has $\{7, 9, y_2, y_1\}$ contained in a maximal fan having 7 as an end that is in a triangle. Thus $M/8\backslash 7$ is 3-connected. It has $\{9, 5, 6, 4\}$ as a fan. Suppose this fan is not contained in a maximal fan having 9 as an end. Then $M/8\backslash 7$ has a triangle T containing 9 and one of 5 and 6. Then $T \cup 8$ is a 4-circuit of M containing $\{8, 9, 5\}$ or $\{8, 9, 6\}$. By orthogonality, this is impossible. Hence $M/8\backslash 7/9$ is 3-connected. As y_2 and 7 are parallel in $M/8, 9$, we deduce that $M/8, 9\backslash y_2$ is 3-connected.

To complete the proof of the lemma, we establish the following.

10.15.6. *$M/8, 9\backslash y_2$ is internally 4-connected.*

Assume this assertion fails and let (X, Y) be a $(4, 3)$-violator of $M/8, 9\backslash y_2$. Now $M|S \cong M(\mathcal{W}_4)$ and $M|(S \cup \{8, 9, 10, 12\})$ has $8, 9, 10, 12$ as coloops. Therefore $(M|(S \cup \{8, 9, 10, 12\}))/\{8, 9\}\backslash\{10, 12\} = M|S$. We deduce that $M|S$ is a restriction of $M/8, 9\backslash y_2$.

Since the only 3-separations of $M(\mathcal{W}_4)$ have a fan on each side, it follows that $|X \cap S| \leq 2$ or $X \cap S$ is a fan in $M|S$, and $|Y \cap S| \leq 2$ or $Y \cap S$ is a fan in $M|S$. Observe that

10.15.7. *Neither X nor Y contains $\{1, 2, 7, 5, 6\}$.*

If X contains $\{1, 2, 7, 5, 6\}$, then $(X \cup \{8, 9, y_2\}, Y)$ is a 3-separation of M; a contradiction.

10.15.8. *Both $|X \cap S|$ and $|Y \cap S|$ exceed two, so both $X \cap S$ and $Y \cap S$ are fans in $M|S$.*

Assume that $|Y \cap S| \leq 2$. Then X spans S. Hence $(X, Y) \cong (X \cup S, Y - S)$. But $X \cup S$ contains $\{1, 2, 7, 5, 6\}$, so $|Y - S| \leq 3$. If $|Y \cap S| = 2$, then, as both elements of $Y \cap S$ are in the closure of X, we deduce that $|Y - S| = 3$. In that case, $(X, Y) \cong (X \cup s_1, Y - s_1)$ where $Y \cap S = \{s_1, s_2\}$ and $|Y - s_1| \geq 4$.

Thus to complete the proof of (10.15.8), it suffices to eliminate the case when $|Y \cap S| = 1$ and Y is a 4-element fan of $M/8, 9\backslash y_2$ that contains a triangle T where $T \cap S = \{\sigma\}$. By orthogonality, $\sigma \notin \{2, 3, 4, 5\}$ so $\sigma \in \{1, 7, 6, 11\}$. Now T, $T \cup 8$, $T \cup 9$, or $T \cup \{8, 9\}$ is a circuit C of M. But M has no triangle containing 1, 7, 6, or 11 except for those contained in S. Thus $C \neq T$. By orthogonality between C and the cocircuits $\{1, 2, 7, 8\}$ and $\{5, 6, 7, 9\}$, we deduce that either $7 = \sigma$ and $\{8, 9\} \subseteq C$, or $7 \neq \sigma$ and $|C \cap \{8, 9\}| = 1$. In the first case, $C \triangle \{7, 8, 9, y_2\}$ is a triangle of M containing y_2; a contradiction. In the second case, by symmetry, we may assume that $8 \in C$. The cocircuit $\{2, 3, 4, 5\}$ implies that $\sigma = 1$. Then orthogonality implies that $12 \in C$. Hence C is a 4-circuit containing $\{12, 1, 8\}$; a contradiction to (10.15.5). We deduce that (10.15.8) holds.

It follows immediately from (10.15.8) that neither X nor Y contains $\{2, 3, 4, 5\}$. Suppose that X contains exactly three members of $\{2, 3, 4, 5\}$. Then, as $(X \cup \{2, 3, 4, 5\}, Y - \{2, 3, 4, 5\}) \cong (X, Y)$, we deduce that Y is a 4-element fan of $M/8, 9\backslash y_2$. We know that $Y \cap S$ is a fan of $M|S$ having at least three elements. Since X contains three elements of $\{2, 3, 4, 5\}$, by symmetry, Y contains either $\{1, 2, 7\}$ or $\{4, 6, 11\}$. Moreover, Y has a fan ordering (h_1, h_2, h_3, α) in $M/8, 9\backslash y_2$ where α is 2 or 4, and $\{h_2, h_3, \alpha\}$ is a triad of $M\backslash y_2$ containing 1, 7, 6, or 11, so $\{h_2, h_3, \alpha, y_2\}$ is a cocircuit of M. Since this cocircuit avoids $\{8, 9\}$, it must contain 7, so it contains 2. Thus it also contains 1, so $\{y_2, 7, 2, 1\}$ is a cocircuit. This is a contradiction since $\{8, 7, 2, 1\}$ is a cocircuit.

We may now assume that each of X and Y contain exactly two elements of $\{2, 3, 4, 5\}$. Moreover, in each of $X \cap \{2, 3, 4, 5\}$ and $Y \cap \{2, 3, 4, 5\}$, the elements must be consecutive in the cyclic order $(2, 3, 4, 5)$. Suppose that $\{2, 3\} \subseteq X$ and $\{4, 5\} \subseteq Y$. Then, as $X \cap S$ and $Y \cap S$ are both fans of $M|S$, it follows that $1 \in X$ and $6 \in Y$. By symmetry, we may assume that $7 \in X$. Then $(X \cup 5, Y - 5) \cong (X, Y)$ so we reduce to an earlier case unless Y is a fan of $M/8, 9\backslash 7$ with an ordering $(h_1, h_2, h_3, 5)$ where $\{h_2, h_3, 5\}$ is a triangle. In the exceptional case, as Y contains $\{4, 5, 6\}$, the unique triangle of M containing 6, we have $\{h_2, h_3, 5\} = \{4, 6, 5\}$. Thus $\{h_1, h_2, h_3, 7\}$ is a cocircuit of M containing $\{4, 6, 7\}$ but avoiding 2, so we contradict orthogonality.

Finally, suppose that $\{2, 5\} \subseteq X$ and $\{3, 4\} \subseteq Y$. Then $7 \in X$ and $11 \in Y$. If $1 \in X$, then $(X \cup 3, Y - 3) \cong (X, Y)$ and we have reduced to an earlier case unless Y is a 4-element fan of $M/8, 9\backslash y_2$ having $(h_1, h_2, h_3, 3)$ as a fan ordering where $\{h_2, h_3, 3\}$ is a triangle. In the exceptional case, $\{h_2, h_3, 3\} = \{4, 11, 3\}$. Thus $\{4, 11, y_2\}$ is contained in a 4-cocircuit of M that avoids $\{7, 8, 9\}$; a contradiction to orthogonality.

We may now assume that $1 \in Y$. By symmetry, $6 \in Y$. Then $(X - 2, Y \cup 2) \cong (X, Y)$ and we have reduced to an earlier case unless X is a 4-element fan in $M/8, 9\backslash y_2$ having a fan ordering $(h_1, h_2, h_3, 2)$ where $\{h_1, h_2, h_3\}$ is a triad. In the exceptional case, since $\{5, 7\} \subseteq \{h_1, h_2, h_3\}$, we deduce that M has a 4-cocircuit containing $\{5, 7, y_2\}$. This cocircuit avoids $\{4, 6\}$ so

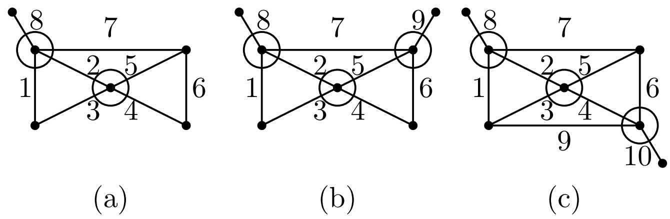

FIGURE 9

we have a contradiction. This completes the proof of (10.15.6) and thereby finishes the proof of the lemma. □

Proof of Theorem 10.1. Assume that M has no proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 3$. Let $(\{1,2,3\}, \{4,5,6\}, \{2,3,4,5\})$ be a bowtie in M such that there is no bowtie $(\{1,2,3\}, T, C^*)$ with $\{1,2,3\} \cap C^* \neq \{2,3\}$. We now apply Lemma 6.3. As $\{1,2,3\}$ is not the central triangle of a quasi rotor, $M\backslash 1$ is $(4,4,S)$-connected. Moreover, by symmetry, we may assume that M has a triangle $\{2,5,7\}$ and a cocircuit $\{1,2,7,8\}$ where $|\{1,2,\dots,8\}| = 8$; that is, M contains the structure in Figure 9(a).

Assume that $M\backslash 1$ has a unique 4-element fan. Then, by Lemma 10.3, M contains either the structure in Figure 9(b) or that in Figure 9(c). Suppose first that M contains the structure in Figure 9(b). Then, by Lemma 10.6, M contains either the structure in Figure 10(a) or that in Figure 10(b). In the latter case, by Lemma 10.8, $M\backslash 11/12$ or $M/12$ is internally 4-connected. We deduce that we may assume that if M contains the structure in Figure 9(b), then it contains the structure in Figure 10(a).

Now assume that M contains the structure in Figure 9(c). Then, by Lemma 10.11, as $M\backslash 1$ has a unique 4-element fan, M contains the structure in Figure 7(a). We deduce that if $M\backslash 1$ has a unique 4-element fan, then M contains either the structure in Figure 10(a) or the structure in Figure 7(a). On the other hand, if $M\backslash 1$ has more than one 4-element fan, then, by Lemma 10.7, M contains the structure in Figure 10(a) with 1 taking the place of 7.

We conclude that we may assume that M contains the structure in Figure 7(a) since, by Lemma 10.9, it contains this structure if it contains the structure in Figure 10(a). Now Lemma 10.13 establishes that, since M contains the structure in Figure 7(a), it contains the structure in Figure 7(b).

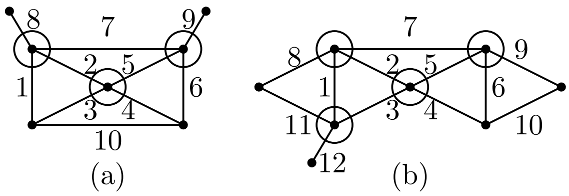

FIGURE 10

Then, by Lemma 10.15, M is the cycle matroid of a terrahawk so the theorem holds. □

11. Rings of Bowties

Recall that a string of bowties is a sequence $T_1, D_1^*, T_2, D_2^*, \dots, D_{n-1}^*, T_n$ such that $T_1, T_2, \dots, T_n$ are pairwise disjoint triangles, each D_i^* is a 4-cocircuit contained in $T_i \cup T_{i+1}$, and $|D_j^* \cap D_{j+1}^*| = 1$ for all j with $1 \leq j \leq n-2$. Theorem 10.1 deals with the case when we have a bowtie (T_1, T_2, D_1^*) that cannot be incorporated into a string $T_0, D_0^*, T_1, D_1^*, T_2$ of bowties. In this section, we deal with the complementary case, proving the following result.

Theorem 11.1. *Let M be an internally 4-connected binary matroid with $|E(M)| \geq 13$. Assume that*

(i) *M contains no quasi rotor;*
(ii) *no restriction of M is isomorphic to $M(K_4)$;*
(iii) *M has a bowtie and every bowtie (T_1, T_2, D_1^*) belongs to a string $T_0, D_0^*, T_1, D_1^*, T_2$ of bowties.*

Then M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 3$ unless M is isomorphic to the cycle matroid of a quartic ladder or a quartic Möbius ladder. In the exceptional cases, M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| = 4$.

Before beginning the proof of this theorem, we prove two lemmas which we shall use in its proof.

Lemma 11.2. *Let $T_0, D_0^*, T_1, D_1^*, T_2$ be a string of bowties in an internally 4-connected binary matroid M. If no triangle of M is the central triangle of a quasi rotor, then M has no triangle that meets each of T_0, T_1, and T_2.*

Proof. Assume that M has a triangle T that meets each of T_0, T_1, and T_2. By orthogonality, T must contain the element of $D_0^* \cap D_1^*$ together with one element of $T_0 \cap D_0^*$ and one element of $T_2 \cap D_1^*$. Then, by Lemma 2.11, T_1 is the central triangle of a quasi rotor. □

Lemma 11.3. *Let M be an internally 4-connected binary matroid having $\{12, 10, 11\}, \{10, 11, 1, 3\}, \{1, 2, 3\}, \{2, 3, 4, 5\}, \{4, 5, 6\}$ and $\{1, 2, 3\}, \{2, 3, 4, 5\}, \{4, 5, 6\}, \{4, 6, 13, 14\}, \{13, 14, 15\}$ as strings of bowties. Assume also that M has $\{2, 5, 7\}$ as a triangle and $\{1, 2, 7, 8\}$ and $\{5, 6, 7, 9\}$ as cocircuits where $\{7, 8, 9\} \cap \big(\{1, 2, \dots, 6\} \cup \{10, 11, \dots, 15\}\big) = \emptyset$. Assume that M has no $M(K_4)$-restriction and that M contains no quasi rotor. Then*

(i) *$M/8$ or $M/9$ is internally 4-connected; or*
(ii) *M has a 4-circuit $\{7, 8, 9, y_2\}$ and triads $\{z_1, y_2, 8\}$ and $\{y_1, y_2, 9\}$ where $\{y_1, y_2, z_2\} \cap \{1, 2, \dots, 15\} = \emptyset$, and $M/8, 9\backslash 7$ is internally 4-connected; or*

(iii) *M has triangles $\{1,8,a\}$ and $\{6,9,b\}$ where a and b are distinct elements that are in $\{10,11\}$ and $\{13,14\}$, respectively.*

Proof. Suppose that neither $M/8$ nor $M/9$ is internally 4-connected. We apply Lemma 8.2, supposing first that (iv) of that lemma holds; that is, M has a circuit $\{x_2,x_3,1,8\}$ and a triad $\{x_1,x_2,x_3\}$. The cocircuit $\{1,3,10,11\}$ implies that $\{x_2,x_3\}$ meets $\{3,10,11\}$; a contradiction since $\{x_1,x_2,x_3\}$ is a triad. Next assume that (i) of Lemma 8.2 holds, that is, M has a triangle T containing $\{1,8\}$. Then, by orthogonality with $\{1,3,10,11\}$, we may assume that $T=\{1,8,10\}$.

Now apply Lemma 8.2 relative to the element 9, rather than relative to 8. From above, (iv) of that lemma does not hold. Moreover, (iii) of Lemma 8.2 does not hold, otherwise 8 is in a triad. Hence (i) of Lemma 8.2 holds, that is, M has a triangle T' containing $\{6,9\}$. The cocircuit $\{4,6,13,14\}$ implies that T' contains 13 or 14. Hence (iii) of Lemma 11.3 holds.

We may now assume that (iii) of Lemma 8.2 holds relative to the element 8 and also relative to 9. Then M has circuits $\{y_2,7,8,9\}$ and $\{z_2,7,8,9\}$, and M has triads $\{y_1,y_2,9\}$ and $\{z_1,z_2,8\}$ where $\{y_1,y_2,z_1,z_2\}\cap\{1,2,\dots,9\}=\emptyset$. Evidently $z_2=y_2$. Moreover, as M is internally 4-connected, none of z_1,y_1, or y_2 is in $\{1,2,\dots,15\}$.

Because 8 is in a triad, $M/8$ is 3-connected. The last matroid has a maximal fan containing $\{7,9,y_2,y_1\}$ that has 7 as an end that is in a triangle. Thus $M/8\backslash 7$ is 3-connected. Since this matroid has a maximal fan containing $\{9,5,6,4\}$ that has 9 as an end that is in a triad, $M/8\backslash 7/9$ is 3-connected.

In $M/8,9$, the elements y_2 and 7 are parallel, so $M/8,9\backslash 7\cong M/8,9\backslash y_2$. Thus, to complete the proof of the lemma, it suffices to show that $M/8,9\backslash y_2$ is internally 4-connected. Assume that it is not, letting (X,Y) be a $(4,3)$-violator of $M/8,9\backslash y_2$. Then $r_M(X\cup\{8,9\})+r_M(Y\cup\{8,9\})=r(M)+4$. Without loss of generality, we may assume that $7\in X$. If $\{z_1,y_1\}\subseteq X$, then $r(Y\cup\{8,9\})=r(Y)+2$ and so $(X\cup\{8,9,y_2\},Y)$ is a 3-separation of M; a contradiction. We deduce that

11.3.1. $\{z_1,y_1\}\cap Y\neq\emptyset$.

Next we observe that if $\{1,2,5,6,7\}\subseteq X$, then $(X\cup\{8,9,y_2\},Y)$ is a 3-separation of M. Thus

11.3.2. $\{1,2,5,6,7\}\not\subseteq X$.

Now suppose that $\{2,7,5\}\subseteq X$. Assume also that $1\in X$. Then, by (11.3.2), $6\in Y$. Suppose that $4\in X$. As $(X\cup 6,Y-6)\cong(X,Y)$, we have that Y is a 4-element fan of $M/8,9\backslash y_2$ containing a triangle $\{g_2,g_3,6\}$ and a triad $\{g_1,g_2,g_3\}$. Then, since $\{y_1,7,5,6\}$ is a cocircuit of $M/8,9\backslash y_2$, it follows that $y_1\in\{g_1,g_2\}$, so we may assume that $y_1=g_3$. Now $\{g_1,g_2,y_1\}$ or $\{g_1,g_2,y_1,y_2\}$ is a cocircuit of M. As $\{7,8,9,y_2\}$ is a circuit of M and $\{g_1,g_2,y_1\}\cap\{7,8,9\}=\emptyset$, we deduce by orthogonality that $\{g_1,g_2,y_1\}$ is a cocircuit of M. Now $\{g_2,y_1,6\}$ is a triangle of $M/8,9$. It follows that $\{g_2, y_1, 6, 9\}$ is a circuit of M. By orthogonality with the cocircuit $\{4, 6, 13, 14\}$, we obtain a contradiction since neither g_2 nor y_1 is in a triangle of M.

We may now assume that $4 \in Y$. We show first that we may suppose that $3 \in X$. If $3 \in Y$, then, as $(X \cup 3, Y - 3) \cong (X, Y)$, we can replace (X, Y) by $(X \cup 3, Y - 3)$ unless $Y - 3$ is a triad of $M\backslash y_2$ containing $\{6, 4\}$. In the exceptional case, $(Y - 3) \cup y_2$ is a cocircuit of M that meets the circuit $\{7, 8, 9, y_2\}$ in one element; a contradiction. We conclude that we may indeed suppose that $3 \in X$. Then $(X \cup 4, Y - 4) \cong (X, Y)$. Thus we have reduced to the case in the previous paragraph unless $|Y| = 4$ and Y contains a triad T^* of $M\backslash y_2$ containing 4. Then $T^* \cup y_2$ is a cocircuit of M meeting $\{7, 8, 9, y_2\}$ in one element; a contradiction.

We have now eliminated the case when $\{2, 7, 5\} \subseteq X$ and $1 \in X$. We assume next that $\{2, 7, 5\} \subseteq X$ and $1 \in Y$. By symmetry, we may also suppose that $6 \in Y$. Assume that $3 \in X$. Then $(X, Y) \cong (X \cup 1, Y - 1)$ and we have reduced to an earlier case unless $Y - 1$ is a triad of $M\backslash y_2$ containing 6. Since neither $Y - 1$ nor $(Y - 1) \cup y_2$ is a cocircuit of M, we conclude that $3 \in Y$. By symmetry, $4 \in Y$. Then $\{2, 5, 7\} \subseteq \mathrm{cl}_{M/8,9\backslash y_2}(Y)$, so $|X| \geq 6$. Then $(X - 2, Y \cup 2) \cong (X, Y)$. But $5 \in \mathrm{cl}^*{}_{M/8,9\backslash y_2}(Y \cup 2)$ and so $(X - 2 - 5, Y \cup 2 \cup 5)$ is a 2-separation of $M/8, 9\backslash y_2$; a contradiction.

We may now assume that X does not contain $\{2, 7, 5\}$. Suppose that $\{2, 7\} \subseteq X$ and $5 \in Y$. Then $(X \cup 5, Y - 5) \cong (X, Y)$ and we reduce to an earlier case unless Y is a 4-element fan in $M/8, 9\backslash y_2$ containing a triangle $\{g_2, g_3, 5\}$. Consider the exceptional case. As $\{2, 3, 4, 5\}$ is a cocircuit of $M/8, 9\backslash y_2$, we deduce that $\{g_2, g_3\}$ meets $\{3, 4\}$. Thus $\{g_1, g_2, g_3, y_2\}$ is a cocircuit of M meeting $\{7, 8, 9, y_2\}$ in a single element; a contradiction. This eliminates the case when $\{2, 7\} \subseteq X$ and $5 \in Y$. By symmetry, this means that we may assume that $7 \in X$ and $\{2, 5\} \subseteq Y$. Then $(X, Y) \cong (X - 7, Y \cup 7)$ and, again, we have reduced to an earlier case unless X is a 4-element fan of $M/8, 9\backslash y_2$ containing a triangle $\{g_2, g_3, 7\}$ and a triad $\{g_1, g_2, g_3\}$. Now $M/8, 9\backslash y_2$ has $\{5, 6, 7, y_1\}$ and $\{1, 2, 7, z_1\}$ as cocircuits. Hence $\{g_2, g_3\}$ meets both $\{5, 6, y_1\}$ and $\{1, 2, z_1\}$. If $\{g_2, g_3\}$ meets $\{1, 2, 5, 6\}$, then $\{g_1, g_2, g_3, y_2\}$ is a cocircuit meeting $\{7, 8, 9, y_2\}$ in just one element. Thus $\{g_2, g_3\} = \{z_1, y_1\}$. Hence $\{z_1, y_1, 7, 8, 9\}$ is a circuit of M. But so is $\{7, 8, 9, y_2\}$. Thus $\{y_2, z_1, y_1\}$ is a triangle of M; a contradiction. □

We are now ready to begin the proof of the main result of this section.

Proof of Theorem 11.1. Assume that the hypotheses of the theorem hold. We show first that we can build a string of bowties that wraps around on itself.

Lemma 11.4. *For some $n \geq 3$, the matroid M has a string $T_1, D_1^*, T_2, D_2^*, \ldots, D_{n-1}^*, T_n$ of bowties and either*

(i) *M has a 4-cocircuit D_n^* contained in $T_n \cup T_1$ such that $|D_{n-1}^* \cap D_n^*| = 1 = |D_n^* \cap D_1^*|$; or*

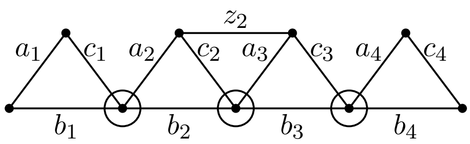

FIGURE 11. The initial configuration in Lemma 11.5.

(ii) *M has a triangle T_{n+1} and a 4-cocircuit D^*_n contained in $T_n \cup T_{n+1}$ such that $|D^*_{n-1} \cap D^*_n| = 1$ and $T_{n+1} \cap (T_1 \cup T_2 \cup \cdots \cup T_n)$ consists of a single element that is in T_1 but not in either D^*_1 or D^*_n.*

Proof. Take a maximum-length string $T_1, D^*_1, T_2, D^*_2, \dots, D^*_{n-1}, T_n$ of bowties in M. By the hypothesis of the theorem, $n \geq 3$. Moreover, the bowtie $(T_{n-1}, T_n, D^*_{n-1})$ belongs to a string $T_{n-1}, D^*_{n-1}, T_n, D^*_n, T_{n+1}$ of bowties. Because $T_1, D^*_1, T_2, D^*_2, \dots, D^*_{n-1}, T_n$ has maximum length, it follows that T_{n+1} meets at least one of $T_1, T_2, \dots, T_{n-2}$.

For all i, let $T_i = \{a_i, b_i, c_i\}$ and $D^*_i = \{b_i, c_i, b_{i+1}, a_{i+1}\}$. Let j be the largest integer in $\{1, 2, \dots, n-2\}$ such that $T_{n+1} \cap T_j \neq \emptyset$. If T_j meets $\{a_{n+1}, b_{n+1}\}$, then, by orthogonality, T_j contains $\{a_{n+1}, b_{n+1}\}$, so $T_j = T_{n+1}$. If T_{n+1} meets $\{b_j, c_j\}$, then T_{n+1} contains $\{b_j, c_j\}$, so $T_j = T_{n+1}$. Thus either

(a) $T_j = T_{n+1}$; or
(b) $T_j \cap T_{n+1} = \{a_j\} = \{c_{n+1}\}$.

In the latter case, suppose that $j > 1$. Then T_{n+1} meets D^*_{j-1}, so T_{n+1} meets $\{b_{j-1}, c_{j-1}\}$. Hence a_{n+1} or b_{n+1} is in T_{j-1}, that is, D^*_n meets T_{j-1}. But $T_n \cap T_{j-1} = \emptyset$, so $\{a_{n+1}, b_{n+1}\} \subseteq T_{j-1}$. Thus $T_{n+1} = T_{j-1}$. Hence $T_{n+1} \cap T_j = \emptyset$; a contradiction. We deduce that if (b) holds, then $j = 1$ and (ii) of the lemma holds.

We may now assume that (a) holds, that is, $T_j = T_{n+1}$. If $\{a_j, b_j\}$ or $\{a_j, c_j\}$ is contained in D^*_n, then (i) of the lemma holds for the string $T_j, D^*_j, T_{j+1}, D^*_{j+1}, \dots, D^*_{n-1}, T_n$ of bowties and the cocircuit D^*_n. The remaining possibility is that $\{b_j, c_j\}$ is contained in D^*_n. Then $D^*_n \triangle D^*_j$ is a cocircuit of M contained in $T_n \cup T_{j+1}$. If $j = n-2$, then $\lambda(T_{n-1} \cup T_n) \leq 2$; a contradiction. Hence $j \leq n-3$. Then (i) of the lemma holds for the string $T_{j+1}, D^*_{j+1}, T_{j+2}, D^*_{j+2}, \dots, D^*_{n-1}, T_n$ of bowties and the cocircuit $D^*_n \triangle D^*_j$ since $(D^*_n \triangle D^*_j) \cap D^*_{j+1} = D^*_j \cap D^*_{j+1}$ and $(D^*_n \triangle D^*_j) \cap D^*_{n-1} = D^*_n \cap D^*_{n-1}$. □

Lemma 11.5. *For each j in $\{1, 2\}$, let $T_j, D^*_j, T_{j+1}, D^*_{j+1}, T_{j+2}$ be a string of bowties. For all i, let $T_i = \{a_i, b_i, c_i\}$ and $D^*_i = \{b_i, c_i, b_{i+1}, a_{i+1}\}$. Suppose that M has a triangle $\{c_2, z_2, a_3\}$. Then either*

(i) *M has cocircuits $\{z_1, a_2, c_2, z_2\}$ and $\{z_2, a_3, c_3, z_3\}$ and triangles $\{e_1, a_2, z_1\}$ and $\{c_3, z_3, e_4\}$ where $e_1 \in \{b_1, c_1\}$ and $e_4 \in \{a_4, b_4\}$, and z_1, z_2, and z_3 are distinct elements that avoid $T_1 \cup T_2 \cup T_3 \cup T_4$; or*
(ii) *M has a proper internally 4-connected minor N such that $|E(M)| - |E(N)| \leq 3$.*

Proof. First we show that

11.5.1. *M has cocircuits $\{z_1, a_2, c_2, z_2\}$ and $\{z_2, a_3, c_3, z_3\}$ where $\{z_1, z_2, z_3\} \cap (T_1 \cup T_2 \cup T_3 \cup T_4) = \emptyset$.*

By Lemma 8.4, M has a 4-cocircuit $\{a_2, c_2, v_1, v_2\}$ where $\{v_1, v_2\}$ avoids $T_1 \cup T_2 \cup T_3$. By orthogonality with the circuit $\{c_2, z_2, a_3\}$, we deduce that $z_2 \in \{v_1, v_2\}$, so we let $(z_1, z_2) = (v_1, v_2)$. By symmetry, M has a 4-cocircuit $\{a_3, c_3, z_2, z_3\}$ and $\{z_2, z_3\} \cap (T_2 \cup T_3 \cup T_4) = \emptyset$. If $\{z_1, z_2\}$ meets T_4, then, by orthogonality, T_4 contains $\{z_1, z_2\}$; a contradiction. It follows that $\{z_1, z_2, z_3\} \cap (T_1 \cup T_2 \cup T_3 \cup T_4) = \emptyset$. Thus (11.5.1) holds.

The lemma now follows immediately by Lemma 11.3. □

If the situation in (i) of Lemma 11.4 arises, we shall say that $T_1, D_1^*, T_2, D_2^*, \dots, D_{n-1}^*, T_n, D_n^*$ is a *ring of bowties*. The next lemma completes the proof of Theorem 11.1 when M has such a ring.

Lemma 11.6. *Let $T_1, D_1^*, T_2, D_2^*, \dots, D_{n-1}^*, T_n, D_n^*$ be a ring of bowties in M. Then either*

(i) *M has a proper internally 4-connected minor N such that $|E(M)| - |E(N)| \leq 3$; or*
(ii) *M is isomorphic to the cycle matroid of a quartic ladder.*

Proof. For all i, let $T_i = \{a_i, b_i, c_i\}$ and $D_i^* = \{b_i, c_i, b_{i+1}, a_{i+1}\}$ where all subscripts are interpreted modulo n. Assume that (i) fails. By Lemma 8.4 and symmetry, we may assume that M has a triangle $\{c_2, a_3, z_2\}$. Then, by Lemma 11.5, M has cocircuits $\{z_1, a_2, c_2, z_2\}$ and $\{z_2, a_3, c_3, z_3\}$ and triangles $\{e_1, a_2, z_1\}$ and $\{c_3, z_3, e_4\}$ where $e_1 \in \{b_1, c_1\}$ and $e_4 \in \{a_4, b_4\}$, and z_1, z_2, and z_3 are distinct elements that avoid $T_1 \cup T_2 \cup T_3 \cup T_4$. By orthogonality between $\{e_1, a_2, z_1\}$ and D_0^*, and using Lemma 11.2, we deduce that $e_1 = c_1$. Likewise, $e_4 = a_4$. By repeating this argument, we get that, for all i in $\{1, 2, \dots, n\}$, the matroid M has $\{c_i, z_i, a_{i+1}\}$ as a triangle and has $\{z_{i-1}, a_i, c_i, z_i\}$ as a cocircuit. Moreover, using orthogonality and induction, we get that $z_1, z_2, \dots, z_n$ are distinct and avoid $T_1 \cup T_2 \cup \dots \cup T_n$.

Now let $A = T_1 \cup T_2 \cup \dots \cup T_n \cup \{z_1, z_2, \dots, z_n\}$. Then $|A| = 4n$. Moreover, A is spanned by $\{a_1, c_1, a_2, c_2, \dots, a_n, c_n\}$, so $r(A) \leq 2n$. Also A contains $2n$ cocircuits of M of one of the forms $\{b_i, c_i, b_{i+1}, a_{i+1}\}$ or $\{z_{i-1}, a_i, c_i, z_i\}$. Since the symmetric difference of any proper collection of these cocircuits is non-empty, we deduce that $r^*(A) \leq |A| - (2n - 1) = 2n + 1$. Thus $\lambda(A) = r(A) + r^*(A) - |A| \leq 2n + (2n + 1) - 4n = 1$. Hence $\lambda(A) = |E(M) - A| \in \{0, 1\}$. In the event that $E(M) - A$ is non-empty, we denote its unique member by α. By orthogonality with the known 4-cocircuits in M, we deduce that $\{a_1, c_1, a_2, c_2, \dots, a_n, c_n\}$ is either a basis or a spanning circuit of M.

Suppose first that $\{a_1, c_1, a_2, c_2, \dots, a_n, c_n\}$ is a spanning circuit. Then $r(A) = 2n - 1$ and α does not exist. The known triangles give that M is represented by the matrix $[I_{2n-1}|D]$ where D is as shown in Figure 12.

	b_1	z_1	b_2	z_2	$\cdots$	b_{n-1}	z_{n-1}	c_n	b_n	z_n
a_1	1	0	0	0	$\cdots$	0	0	1	1	0
c_1	1	1	0	0	$\cdots$	0	0	1	1	1
a_2	0	1	1	0	$\cdots$	0	0	1	1	1
c_2	0	0	1	1	$\cdots$	0	0	1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{n-1}	0	0	0	0	$\cdots$	1	0	1	1	1
c_{n-1}	0	0	0	0	$\cdots$	1	1	1	1	1
a_n	0	0	0	0	$\cdots$	0	1	1	0	1

FIGURE 12. The matrix D.

One easily checks that $M[I_{2n-1}|D]$ is the cycle matroid of a quartic ladder labelled as in Figure 13.

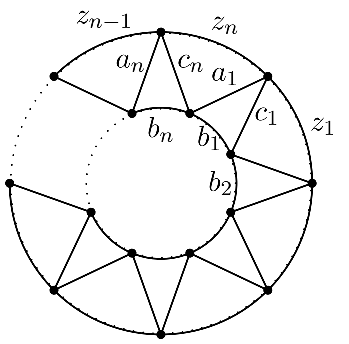

FIGURE 13. A labelled quartic ladder.

We may now suppose that $\{a_1, c_1, a_2, c_2, \ldots, a_n, c_n\}$ is a basis of M. Then $M|A$ is represented by the matrix $[I_{2n}|D']$ where D' is as shown in Figure 14. Thus $M|A \cong M(\mathcal{W}_{2n})$. But the last matroid is not internally 4-connected. Hence, in this case, α exists. Moreover, to prevent M from having any 4-element fans, the column corresponding to α must consist entirely of ones. Now adjoin this column to $[I_{2n}|D']$ and then pivot on the entry in the top right corner of the resulting matrix. After deleting the first row and last column of the resulting matrix and performing a suitable column permutation, we obtain a matrix of the same form as in Figure 12. We deduce that M/α is isomorphic to the cycle matroid of a quartic ladder. Hence M/α is internally 4-connected and the lemma holds. □

The last lemma treated outcome (i) of Lemma 11.4. Next we treat outcome (ii).

Lemma 11.7. *Suppose that, for some $n \geq 3$, the matroid M has a string $T_1, D_1^*, T_2, D_2^*, \ldots, D_{n-1}^*, T_n$ of bowties. Assume, in addition, that M has a triangle T_{n+1} and a 4-cocircuit D_n^* contained in $T_n \cup T_{n+1}$ such that $|D_{n-1}^* \cap$*

$$\begin{array}{c} \\ a_1 \\ c_1 \\ a_2 \\ c_2 \\ \vdots \\ c_{n-1} \\ a_n \\ c_n \end{array}
\begin{array}{c}
\begin{array}{cccccccc} b_1 & z_1 & b_2 & z_2 & \cdots & z_{n-1} & b_n & z_n \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{bmatrix}
\end{array}.$$

FIGURE 14. The matrix D'.

$D_n^*| = 1$ and $T_{n+1} \cap (T_1 \cup T_2 \cup \cdots \cup T_n)$ is a single element that is in T_1 but not in either D_1^* or D_n^*. Then either

(i) *M has a proper internally 4-connected minor N such that $|E(M)| - |E(N)| \leq 3$; or*

(ii) *M is isomorphic to the cycle matroid of a quartic Möbius ladder.*

Proof. Assume that (i) does not hold. For all i, let $T_i = \{a_i, b_i, c_i\}$ and $D_i^* = \{b_i, c_i, b_{i+1}, a_{i+1}\}$. Then $c_{n+1} = a_1$. Now the bowtie (T_1, T_2, D_1^*) can be extended to a string $S_n, E_n^*, T_1, D_1^*, T_2$ of bowties. We may assume that $E_n^* \cap T_1 = \{a_1, c_1\}$. Then E_n^* meets T_{n+1}, so we may suppose that $a_{n+1} \in E_n^*$. Thus $a_{n+1} \in S_n$. Since $b_{n+1} \notin S_n$, it follows that c_n or b_n is in S_n. Now $\{c_{n-1}, b_{n-1}, a_n, b_n\}$ is a cocircuit. Suppose b_n is in S_n. Then S_n contains c_{n-1} or b_{n-1}. Thus S_n meets T_{n+1}, T_n, and T_{n-1}, so we have a contradiction to Lemma 11.2.

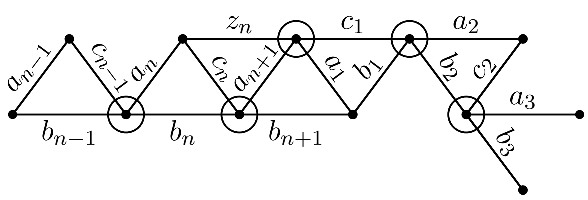

FIGURE 15

We may now assume that $c_n \in S_n$ (see Figure 15). By orthogonality with $D_1^*, D_2^*, \ldots, D_{n-1}^*$, we deduce that the element z_n of $S_n - \{a_{n+1}, c_n\}$ is not in $T_1 \cup T_2 \cup \cdots \cup T_{n+1}$. Moreover, E_n^* contains z_n, so $E_n^* = \{z_n, a_{n+1}, a_1, c_1\}$. Next we extend the bowtie (S_n, T_1, E_n^*) to a string $S_{n-1}, E_{n-1}^*, S_n, E_n^*, T_1$. Now $E_{n-1}^* \cap S_n \neq E_n^* \cap S_n$. Thus $c_n \in E_{n-1}^*$, so a_n or b_n is in E_{n-1}^*. If $b_n \in E_{n-1}^*$, then the cocircuits D_n^* and D_{n-1}^* imply that S_{n-1} contains b_{n+1} and either b_{n-1} or c_{n-1}. This contradicts Lemma 11.2. Hence $a_n \in E_{n-1}^*$. Thus $a_n \in S_{n-1}$, so b_{n-1} or c_{n-1} is in S_{n-1}. But $b_{n-1} \notin S_{n-1}$ otherwise S_{n-1} meets T_{n-2}, T_{n-1}, and T_n contradicting Lemma 11.2. Thus $c_{n-1} \in S_{n-1}$. Let z_{n-1} be the element of $S_{n-1} - \{a_n, c_{n-1}\}$. Then, by orthogonality, $z_{n-1} \notin T_1 \cup T_2 \cup \cdots \cup T_{n+1} \cup z_n$, and $E_{n-1}^* = \{z_{n-1}, a_n, c_n, z_n\}$.

Next we apply Lemma 11.5 to $T_{n+1}, D_n^*, T_n, D_{n-1}^*, T_{n-1}$ and $T_n, D_{n-1}^*, T_{n-1}, D_{n-2}^*, T_{n-2}$ to get a cocircuit $\{z_{n-1}, c_{n-1}, a_{n-1}, z_{n-2}\}$ and a triangle $\{a_{n-1}, z_{n-2}, e_{n-2}\}$ where $e_{n-2} \in \{b_{n-2}, c_{n-2}\}$. Provided $n-2 \geq 2$, the cocircuit D_{n-3}^* implies that $e_{n-2} = c_{n-2}$. By repeating this argument, we obtain, for all i in $\{3, 4, \ldots, n\}$, a cocircuit E_{i-1}^*, which equals $\{z_i, c_i, a_i, z_{i-1}\}$, and a triangle $\{a_i, z_{i-1}, c_{i-1}\}$. Moreover, using orthogonality and induction, we get that $z_2, z_3, \ldots, z_n$ are distinct and avoid $T_1 \cup T_2 \cup \cdots \cup T_{n+1}$.

Now the bowtie (T_n, T_{n+1}, D_n^*) extends to a string $T_n, D_n^*, T_{n+1}, E_0^*, S_1$ of bowties. If $\{z_n, c_1\} \subseteq S_1$, then the 4-cocircuits meeting $\{z_n, c_1\}$ give a contradiction. Thus

11.7.1. $\{z_n, c_1\} \not\subseteq S_1$.

Since $T_{n+1} \cap E_0^* \neq T_{n+1} \cap D_n^*$, we have $a_1 \in E_0^*$, so a_{n+1} or b_{n+1} is in E_0^*. We show next that

11.7.2. $a_{n+1} \notin E_0^*$.

Suppose $a_{n+1} \in E_0^*$. Then, as $c_n \notin E_0^*$, it follows that E_0^* contains $\{a_1, a_{n+1}, z_n\}$, which is contained in E_n^*. Thus $E_0^* = E_n^*$, so S_1 contains $\{z_n, c_1\}$, contradicting (11.7.1).

We now know that $b_{n+1} \in E_0^*$. Since $a_1 \in E_0^*$, either b_1 or c_1 is in E_0^*. If $c_1 \in E_0^*$, then $c_1 \in S_1$, so, by orthogonality with E_n^*, we deduce that $z_n \in S_1$ and we contradict (11.7.1). Thus $b_1 \in E_0^*$, so $b_1 \in S_1$. Thus b_2 or a_2 is in S_1. If $b_2 \in S_1$, then it follows that S_1 meets T_3, T_2, and T_1; a contradiction to Lemma 11.2. Thus $a_2 \in S_1$. Let z_1 be the element of $S_1 - \{b_1, a_2\}$. Then $E_0^* = \{b_{n+1}, a_1, b_1, z_1\}$. By Lemma 8.4, M has a 4-cocircuit E_1^* containing $\{a_2, c_2\}$ but otherwise avoiding $T_1 \cup T_2 \cup T_3$. Orthogonality implies that $E_1^* = \{z_1, a_2, c_2, z_2\}$.

Now let $Z = \{a_1, a_2, \ldots, a_{n+1}\} \cup \{b_1, b_2, \ldots, b_{n+1}\} \cup \{c_1, c_2, \ldots, c_n\} \cup \{z_1, z_2, \ldots, z_n\}$. Then $|Z| = 4n+2$. Let $S = \{a_1, c_1, a_2, c_2, \ldots, a_n, c_n, b_{n+1}\}$. Then S spans Z, so $r(Z) \leq 2n+1$. The cocircuits $D_1^*, D_2^*, \ldots, D_n^*$ and $E_0^*, E_1^*, \ldots, E_n^*$, which are all contained in Z and have symmetric difference equal to the empty set, imply that $r^*(Z) \leq |Z| - ((2n+1) - 1) = 2n+2$. Thus $\lambda(Z) = |E(M) - Z| \in \{0, 1\}$. When $E(M) - Z$ is non-empty, we let its unique element be ζ.

The known 4-cocircuits of M imply that S is either a basis or a spanning circuit. Suppose first that S is a spanning circuit. Then $r(Z) = 2n$ and ζ does not exist. As in the proof of the previous lemma, the known triangles determine a representation $[I_{2n}|D']$ for M, where D' is shown in Figure 17. One easily checks that M is the cycle matroid of a quartic Möbius ladder labelled as in Figure 18.

Now suppose that S is a basis for M. Then $M|S$ has a representation of the form $[I_{2n+1}|D]$ where D is as shown in Figure 16. Since $n \geq 3$ and $M|S$ has many 4-element fans, we deduce that ζ must exist. Indeed, since every row of the indicated matrix, except the first, has two ones, the column

$$
\begin{array}{c}
 \\ a_1 \\ c_1 \\ a_2 \\ c_2 \\ a_3 \\ c_3 \\ \vdots \\ a_{n-1} \\ c_{n-1} \\ a_n \\ c_n \\ b_{n+1}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccc}
b_1 & z_1 & b_2 & z_2 & b_3 & \cdots & b_{n-1} & z_{n-1} & b_n & z_n & a_{n+1}
\end{array} \\
\left[\begin{array}{ccccccccccc}
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1
\end{array}\right]
\end{array}.
$$

FIGURE 16. The matrix D.

corresponding to ζ must have ones in all rows except possibly the first. The first row implies that $\{a_1, b_1, z_1, z_n, a_{n+1}\}$ or $\{a_1, b_1, z_1, z_n, a_{n+1}, \zeta\}$ is a cocircuit C^* of M. By taking the symmetric difference of C^* with the cocircuits $\{a_1, c_1, a_{n+1}, z_n\}$ and $\{a_1, b_1, b_{n+1}, z_1\}$, we get $\{a_1, c_1, b_{n+1}, \zeta\}$ or $\{a_1, c_1, b_{n+1}\}$ depending on whether ζ is or is not in C^*. As a_1 is not in a triad, we deduce that $\zeta \in C^*$, so the column corresponding to ζ consists of all ones. By adjoining this column to $[I_{2n+1}|D]$ and then pivoting on the entry in the top right corner, we get that M/ζ is represented by $[I_{2n}|D']$ where D' is as shown in Figure 17. Thus M/ζ is the cycle matroid of a quartic Möbius ladder labelled as in Figure 18. As the cycle matroid of a quartic Möbius ladder is internally 4-connected, the lemma holds in this case and hence the proof is completed. □

For the reader familiar with the paper of Mayhew, Royle, and Whittle [8], we observe that, at the end of the last proof, the matroid M for which M/ζ is a quartic Möbius ladder is actually the dual of a triadic Möbius matroid.

All but the last sentence of Theorem 11.1 follows immediately by combining Lemmas 11.4, 11.6, and 11.7. With $|E(M)| \geq 13$, if M is the cycle matroid of a quartic ladder or a quartic Möbius ladder, then M has as a minor the next smallest quartic ladder or the next smallest quartic Möbius ladder, so M has an internally 4-connected minor N with $|E(M)| - |E(N)| = 4$. Moreover, it is straightforward to check that M has no proper internally 4-connected minor N' with $|E(M)| - |E(N')| < 4$. □

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$$
\begin{array}{c|cccccccccccc|}
 & b_1 & z_1 & b_2 & z_2 & b_3 & \cdots & b_{n-1} & z_{n-1} & b_n & z_n & a_{n+1} & a_1 \\
\hline
c_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\
a_2 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\
c_2 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\
a_3 & 1 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\
c_3 & 1 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n-1} & 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 1 & 1 & 1 \\
c_{n-1} & 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 1 & 1 \\
a_n & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & 1 \\
c_n & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \\
b_{n+1} & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
$$

FIGURE 17. The matrix D'.

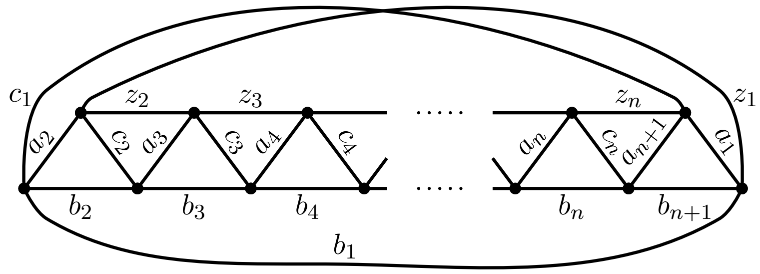

Figure 18. A labelled quartic Möbius ladder.

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