TOWARDS A SPLITTER THEOREM FOR INTERNALLY 4-CONNECTED BINARY MATROIDS IV

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Abstract. In our quest to find a splitter theorem for internally 4-connected binary matroids, we proved in the preceding paper in this series that, except when M or its dual is a cubic Möbius or planar ladder or a certain coextension thereof, an internally 4-connected binary matroid M with an internally 4-connected proper minor N either has a proper internally 4-connected minor M' with an N-minor such that $|E(M) - E(M')| \leq 3$ or has, up to duality, a triangle T and an element e of T such that $M\backslash e$ has an N-minor and has the property that one side of every 3-separation is a fan with at most four elements. This paper proves that, when we cannot find such a proper internally 4-connected minor M' of M, we can incorporate the triangle T into one of two substructures of M: a bowtie or an augmented 4-wheel. In the first of these, M has a triangle T' disjoint from T and a 4-cocircuit D^* that contains e and meets T'. In the second, T is one of the triangles in a 4-wheel restriction of M with helpful additional structure.

1. Introduction

Seymour's Splitter Theorem [10] is a powerful inductive tool for 3-connected matroids. It shows that if such a matroid M has a proper 3-connected minor N, then M has a proper 3-connected minor M' with an N-minor such that $|E(M) - E(M')| = 1$ unless $r(M) \geq 3$ and M is a wheel or a whirl. The current paper is the fourth in a series whose aim is to obtain a splitter theorem for binary internally 4-connected matroids. Specifically, we believe we can prove that if M and N are internally 4-connected binary matroids, and M has a proper N-minor, then M has a proper minor M' such that M' is internally 4-connected with an N-minor, and M' can be produced from M by a bounded number of simple operations.

Johnson and Thomas [6] showed that, even for graphs, a splitter theorem in the internally 4-connected case must take account of some special examples. For $n \geq 3$, let G_{n+2} be the *biwheel* with $n+2$ vertices, that is, G consists of an n-cycle $v_1, v_2, \ldots, v_n, v_1$, the *rim*, and two additional vertices, u and w, both of which are adjacent to every v_i. Thus the dual of G_{n+2} is a cubic planar ladder. Let M be the cycle matroid of G_{2n+2} for some $n \geq 3$ and let N be the cycle matroid of the graph that is obtained by proceeding around the rim of G_{2n+2} and alternately deleting the edges from the rim vertex to u and to w. Both M and N are internally 4-connected but there is no internally 4-connected proper minor of M that has a proper N-minor. We can modify M slightly and still see the same phenomenon.

Date: August 31, 2013.
1991 *Mathematics Subject Classification.* 05B35, 05C40.
Key words and phrases. splitter theorem, binary matroid, internally 4-connected.
The authors were supported by NSF IRFP Grant #OISE0967050, the Marsden Fund, and the National Security Agency, respectively.

Let G^+_{n+2} be obtained from G_{n+2} by adding a new edge a joining the hubs u and w. Let Δ_{n+1} be the binary matroid that is obtained from $M(G^+_{n+2})$ by deleting the edge $v_{n-1}v_n$ and adding the third element on the line spanned by wv_n and uv_{n-1}. This new element is also on the line spanned by uv_n and wv_{n-1}. For $r \geq 3$, Mayhew, Royle, and Whittle [7] call Δ_r the *rank-r triangular Möbius matroid* and note that $\Delta_r\backslash a$ is the dual of the cycle matroid of a cubic Möbius ladder.

In [2], we proved a splitter theorem when M is a 4-connected binary matroid and N is an internally 4-connected proper minor of M. In particular, we showed that, unless M is a certain 16-element non-graphic matroid, we can find an internally 4-connected matroid M' with $|E(M) - E(M')| = 1$ such that M' has an N-minor. This result leaves us to treat the case when M is an internally 4-connected matroid having a triangle or a triad. But we know nothing about how this triangle or triad relates to the N-minors of M. Our second step towards the desired splitter theorem was to consider the case when all the triangles and triads of M are retained in N. In this case, we proved [3, Theorem 1.2] the following result.

Theorem 1.1. *Let M and N be internally 4-connected binary matroids such that $|E(N)| \geq 7$, and N is isomorphic to a proper minor of M. Assume that if T is a triangle of M and $e \in T$, then $M\backslash e$ does not have an N-minor. Dually, assume that if T^* is a triad of M and $f \in T^*$, then M/f does not have an N-minor. Then M has an internally 4-connected minor M' of M such that M' has an N-minor and $1 \leq |E(M) - E(M')| \leq 2$.*

This theorem enables us to assume, by replacing M by its dual if necessary, that M has a triangle T containing an element e for which $M\backslash e$ has an N-minor. In earlier work [1], we found it useful to consider weaker variants of internal 4-connectivity. The only 3-separations allowed in an internally 4-connected matroid have a triangle or a triad on one side. A 3-connected matroid M is $(4,4,S)$-*connected* if, for every 3-separation (X, Y) of M, one of X and Y is a triangle, a triad, or a 4-element fan, that is, a 4-element set $\{x_1, x_2, x_3, x_4\}$ that can be ordered so that $\{x_1, x_2, x_3\}$ is a triangle and $\{x_2, x_3, x_4\}$ is a triad. Somewhat weaker still than $(4,4,S)$-connectivity is the following notion. We call M $(4,5,S,+)$-*connected* if, for every 3-separation (X, Y) of M, one of X and Y is a triangle, a triad, a 4-element fan, or a 5-fan, that is, a 5-element set $\{x_1, x_2, x_3, x_4, x_5\}$ such that $\{x_1, x_2, x_3\}$ and $\{x_3, x_4, x_5\}$ are triangles, while $\{x_2, x_3, x_4\}$ is a triad.

The following is the main result of [4, Theorem 1.2].

Theorem 1.2. *Let M be an internally 4-connected binary matroid with an internally 4-connected proper minor N such that $|E(M)| \geq 15$ and $|E(N)| \geq 6$. Then*

(i) *M has a proper minor M' such that $|E(M) - E(M')| \leq 3$ and M' is internally 4-connected with an N-minor; or*
(ii) *for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 has a triangle T that contains an element e such that $M_0\backslash e$ is $(4,4,S)$-connected having an N-minor; or*
(iii) *M or M^* is isomorphic to $M(G^+_{r+1})$, $M(G_{r+1})$, Δ_r, or $\Delta_r\backslash z$ for some $r \geq 5$.*

To continue the derivation of our desired splitter theorem, this paper will build detailed structure around the triangle T that arises in (ii). Let M be an internally 4-connected binary matroid having disjoint triangles T_1 and T_2 and a 4-cocircuit D^* contained in their union. We call this structure a *bowtie* and denote it by (T_1, T_2, D^*). Now let N be an internally 4-connected proper minor of M, and suppose D^* has an element d such that $M\backslash d$ has an N-minor. If $M\backslash d$ is $(4,4,S)$-connected, then (T_1, T_2, D^*) is a *good bowtie*. If, instead, $M\backslash d$ is $(4,5,S,+)$-connected, then (T_1, T_2, D^*) is a *pretty good bowtie* if it can be labelled $(\{1,2,3\}, \{4,5,6\}, \{2,3,4,5\})$ so that $d = 2$ and there is an element 7 such that $\{3,5,7\}$ is a triangle of M, and $M\backslash 7$ is $(4,4,S)$-connected having an N-minor.

To state our main theorem, we need to define two more structures. A *terrahawk* is the graph that is obtained from a cube by adjoining one new vertex and adding edges from this vertex to each of the four vertices that bound some fixed face of the cube. Now let M be a binary internally 4-connected matroid having a binary internally 4-connected minor N. An *augmented* 4*-wheel* consists of a 4-wheel restriction of M with triangles $\{z_2, x_1, y_2\}, \{y_2, x_3, z_3\}, \{z_3, y_3, x_2\}, \{x_2, y_1, z_2\}$ along with two additional distinct elements z_1 and z_4 such that M has $\{x_1, y_1, z_1, z_2\}, \{x_2, y_2, z_2, z_3\}$, and $\{x_3, y_3, z_3, z_4\}$ as cocircuits. We call an augmented 4-wheel *good* if $M\backslash y_1$ is $(4,4,S)$-connected having an N-minor, while $M\backslash y_2$ has an N-minor. A diagrammatic representation of an augmented 4-wheel is shown in Figure 1. Although the matroid M we are considering need not be graphic, we follow the convention begun in [1] of using a modified graph diagram to keep track of some of the circuits and cocircuits in M. By convention, the cycles in the graph diagram correspond to circuits of the matroid while a circled vertex indicates a known cocircuit of M.

Theorem 1.3. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 6$. Suppose that M has a triangle T containing an element e for which $M\backslash e$ is $(4,4,S)$-connected having an N-minor. Then one of the following holds.*

(i) *M has an internally 4-connected minor M' that has an N-minor such that $1 \leq |E(M) - E(M')| \leq 3$; or*
(ii) *M or M^* has a good bowtie; or*
(iii) *M or M^* has a good augmented 4-wheel; or*
(iv) *$N \cong M(K_4)$ and M is the cycle matroid of a terrahawk.*

We observe here that we can delete outcome (iv) in the last theorem if we allow $|E(M) - E(M')| = 4$ in (i) since the terrahawk has the cube as a minor. In the fifth paper of this series [5], we essentially eliminate the need to consider good augmented 4-wheels by showing that when M contains such a substructure, either it also contains a good bowtie, or, in an easily described way, we can obtain an internally 4-connected minor of M with an N-minor.

As a preliminary step towards proving this theorem, we shall show in Theorem 3.1 that, when (i) does not hold, M or M^* has a good bowtie or a pretty good bowtie. In the next section, we present some preliminary results that will be used in the proof of Theorem 1.3. Then Section 3 outlines the main steps in that proof. Following that, the remaining sections of the paper fill in the details of this outline.

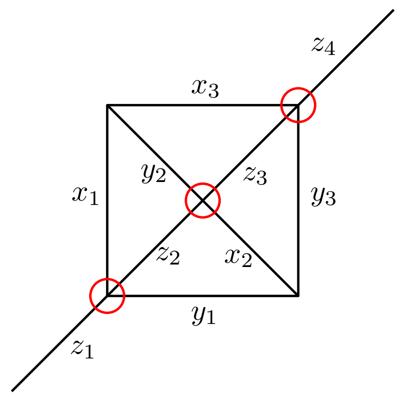

FIGURE 1

2. Preliminaries

The matroid terminology used here will follow Oxley [8]. We shall sometimes write $N \preceq M$ to indicate that M has an N-*minor*, that is, a minor isomorphic to the matroid N. If x is an element of a matroid M and $Y \subseteq E(M)$, we write $x \in \mathrm{cl}^{(*)}(Y)$ to mean that $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$. A *quad* in a matroid is a 4-element set that is both a circuit and a cocircuit. The property that a circuit and a cocircuit in a matroid cannot have exactly one common element will be referred to as *orthogonality*. It is well known ([8, Theorem 9.1.2]) that, in a binary matroid, a circuit and cocircuit must meet in an even number of elements.

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E-X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate λ_M as λ. For a positive integer k, a subset X or a partition $(X, E-X)$ of E is k-*separating* if $\lambda_M(X) \leq k-1$. A k-separating partition $(X, E-X)$ is a k-*separation* if $|X|, |E-X| \geq k$. If n is an integer exceeding one, a matroid is n-*connected* if it has no k-separations for all $k < n$. This definition has the attractive property that a matroid is n-connected if and only if its dual is. Moreover, this matroid definition of n-connectivity is relatively compatible with the graph notion of n-connectivity when n is 2 or 3. For example, if G is a graph with at least four vertices and with no isolated vertices, $M(G)$ is a 3-connected matroid if and only if G is a 3-connected simple graph. But the link between n-connectivity for matroids and graphs breaks down for $n \geq 4$. In particular, a 4-connected matroid with at least six elements cannot have a triangle. Hence, for $r \geq 3$, neither $M(K_{r+1})$ nor $PG(r-1, 2)$ is 4-connected. This motivates the consideration of other types of 4-connectivity in which certain 3-separations are allowed. In particular, a matroid is *internally* 4-*connected* if it is 3-connected and, whenever (X, Y) is a 3-separation, either $|X| = 3$ or $|Y| = 3$. Let n and k be integers with $n \geq 3$ and $k \geq 2$. A matroid M is (n, k)-*connected* if M is $(n-1)$-connected and, whenever (X, Y) is an $(n-1)$-separating partition of $E(M)$, either $|X| \leq k$ or $|Y| \leq k$. In particular, a matroid is $(4, 3)$-connected if and only if it is internally 4-connected. A graph G without isolated vertices is *internally* 4-*connected* if $M(G)$ is internally 4-connected.

A k-separating set X or a k-separation $(X, E-X)$ is *exact* if $\lambda_M(X) = k-1$. A k-separation $(X, E-X)$ is *minimal* if $|X| = k$ or $|E-X| = k$. It is well known (see, for example, [8, Corollary 8.2.2]) that if M is k-connected having $(X, E-X)$ as a k-separation with $|X| = k$, then X is a circuit or a cocircuit of M. The *guts* of a k-separation (U, V) is $\mathrm{cl}(U) \cap \mathrm{cl}(V)$, while the *coguts* is $\mathrm{cl}^*(U) \cap \mathrm{cl}^*(V)$.

A set X in a matroid M is *fully closed* if it is closed in both M and M^*. The intersection of two fully closed sets is fully closed, and the *full closure* $\mathrm{fcl}(X)$ of X is the intersection of all fully closed sets that contain X. Two exactly 3-separating partitions (A_1, B_1) and (A_2, B_2) of a 3-connected matroid M are *equivalent*, written $(A_1, B_1) \cong (A_2, B_2)$, if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$.

Let (X, Y) be an exact 3-separation of a simple binary matroid M. As binary matroids are uniquely representable over $GF(2)$, we can view M as a restriction of $PG(r-1, 2)$, where $r = r(M)$. Let cl_P be the closure operator of $PG(r-1, 2)$. Then $r(X \cup Y) + r(\mathrm{cl}_P(X) \cap \mathrm{cl}_P(Y)) = r(X) + r(Y) = r(M) + 2 = r(X \cup Y) + 2$. Thus $\mathrm{cl}_P(X) \cap \mathrm{cl}_P(Y)$ is a line of $PG(r-1, 2)$, that is, a triangle with some element set $\{a, b, c\}$. We call $\{a, b, c\}$ the *guts line* of the 3-separation (X, Y). Now assume M is cosimple as well as simple and that N is an internally 4-connected minor of M with $|X \cap E(N)| \leq 3$. By [4, Lemma 3.2], N is isomorphic to a minor of either $PG(r-1, 2)|(Y \cup \{a, b, c\})$ or the matroid obtained from $PG(r-1, 2)|(Y \cup \{a, b, c\})$ by performing a Δ-Y exchange on $\{a, b, c\}$. In these cases, we say that N is isomorphic to a minor of the matroid obtained by *replacing X by a triangle or a triad on the guts line of* (X, Y). We also say that we can get an N-minor of the matroid obtained by *putting a triangle or a triad on the guts of* (X, Y).

Let M be a matroid. A subset S of $E(M)$ is a *fan* in M if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of S such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \ldots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads. We call $(s_1, s_2, \ldots, s_n)$ a *fan ordering* of S. We will be mainly concerned with 4-element and 5-element fans. For convenience, we shall always view a fan ordering of a 4-element fan as beginning with a triangle and we shall use the term 4-*fan* to refer to both the 4-element fan and such a fan ordering of it. Moreover, we shall use the terms 5-*fan* and 5-*cofan* to refer to the two different types of 5-element fan where the first contains two triangles and the second two triads. Let $(s_1, s_2, \ldots, s_n)$ be a fan ordering of a fan S. If $n \geq 5$, then the only other fan ordering of S is $(s_n, s_{n-1}, \ldots, s_1)$. We shall view this reversal of the original ordering as being the same fan ordering. The elements s_1 and s_n are called the *ends* of the fan. When (s_1, s_2, s_3, s_4) is a 4-fan, our convention is that $\{s_1, s_2, s_3\}$ is a triangle. We observe that the only other fan ordering for this fan is (s_1, s_3, s_2, s_4). Again, we shall view this ordering as being the same as (s_1, s_2, s_3, s_4). This means that, up to this equivalence, all fans with at least four elements have unique fan orderings. For a 4-fan (s_1, s_2, s_3, s_4), we call s_1 and s_4 the *guts* and *coguts* elements of the fan since $s_1 \in \mathrm{cl}(\{s_2, s_3, s_4\})$ and $s_4 \in \mathrm{cl}^*(\{s_1, s_2, s_3\})$. The elements s_2 and s_3 are the *internal elements* of the 4-fan. If $(s_1, s_2, s_3, s_4, s_5)$ is a 5-fan, then s_1 and s_5 are the *guts elements* of this fan. Dually, if $(s_1, s_2, s_3, s_4, s_5)$ is a 5-cofan, then s_1 and s_5 are its *coguts elements*.

Fans are examples of sequential 3-separating sets in M. A subset X of $E(M)$ is *sequential* if it has a *sequential ordering*, that is, an ordering $(x_1, x_2, \ldots, x_k)$ such that $\{x_1, x_2, \ldots, x_i\}$ is 3-separating for all i in $\{1, 2, \ldots, k\}$. It is straightforward to check that, when M is binary, a sequential set with $3, 4$, or 5 elements is a fan while a 4-element non-sequential 3-separating set is a quad. A 3-separation (X, Y) of a

3-connected matroid M is *sequential* if X or Y is a sequential set. A 3-connected matroid is *sequentially* 4-*connected* if all of its 3-separations are sequential. A 3-connected matroid M is $(4,k,S)$-*connected* if M is both $(4,k)$-connected and sequentially 4-connected.

To motivate one of the other forms of connectivity used here, we return to the example in the introduction letting M be the cycle matroid of the biwheel G_{2n+2} and N be the cycle matroid of the graph that is obtained by proceeding around the rim of G_{2n+2} and alternately deleting the edges from the rim vertex to u and to w. Each triangle of M has an element whose deletion has an N-minor but every such deletion has a 5-fan. Indeed, it is $(4,5,S,+)$-connected because, whenever it has (X,Y) as a 3-separation, one of X and Y is a triangle, a triad, a 4-fan, or a 5-fan.

Let (X,Y) be a 3-separation of a 3-connected binary matroid M. We shall frequently be interested in 3-separations that indicate that M is, for example, not internally 4-connected. We call (X,Y) a $(4,3)$-*violator* if $|X|,|Y| \geq 4$. Similarly, (X,Y) is a $(4,4,S)$-*violator* if, for each Z in $\{X,Y\}$, either $|Z| \geq 5$, or Z is non-sequential. Finally, (X,Y) is a $(4,5,S,+)$-*violator* if, for each Z in $\{X,Y\}$, either $|Z| \geq 6$, or Z is non-sequential, or Z is a 5-cofan.

The next result is an elementary consequence of (the dual of) Tutte's Triangle Lemma [11] (or see [8, Lemma 8.7.7]) so the proof is omitted.

Lemma 2.1. *Let $(s_1, s_2, \ldots, s_n)$ be a fan in a 3-connected matroid M where $n \geq 4$ and $\{s_{n-2}, s_{n-1}, s_n\}$ is a triad. If s_n is not in a triangle, then M/s_n is 3-connected.*

We will frequently need to analyze small 3-separating sets in a 3-connected binary matroid M. The next lemma, which is well-known and easy to verify, catalogues such sets. Before stating the result, we introduce the terminology that we use. A *quad with an element in the guts* is a 5-element set $\{1,2,3,4,5\}$ where $\{1,2,3\}$ and $\{1,4,5\}$ are circuits while $\{2,3,4,5\}$ is both a circuit and a cocircuit. If, in M, the set Q is a quad with an element in the guts, then, in M^*, the set Q is a *quad with an element in the coguts.* A 5-*fan with an element in the guts* consists of a 5-fan (s_1,s_2,s_3,s_4,s_5) and an element s_6 such that $\{s_1,s_5,s_6\}$ is a triangle. Thus $M|\{s_1,s_2,s_3,s_4,s_5,s_6\} \cong M(K_4)$, and $\{s_2,s_3,s_4\}$ is a triad of M. The dual of a 5-fan with an element in the guts is a 5-*cofan with an element in the coguts.* A 5-*cofan with an element in the guts* consists of a 5-cofan (s_1,s_2,s_3,s_4,s_5) and an element s_6 such that $\{s_1,s_3,s_5,s_6\}$ is a circuit. This structure is to be distinguished from a 6-element fan $(s_1,s_2,s_3,s_4,s_5,s_6)$ in which $\{s_4,s_5,s_6\}$ is a triangle, which also has (s_1,s_2,s_3,s_4,s_5) as a 5-cofan with s_6 in the guts of the 3-separation $(\{s_1,s_2,s_3,s_4,s_5\}, E(M)-\{s_1,s_2,s_3,s_4,s_5\})$. The dual of a 5-cofan with an element in the guts is a 5-*fan with an element in the coguts.*

Lemma 2.2. *Let X be a 3-separating set in a 3-connected binary matroid M.*

(i) *If $|X| = 3$, then X is a triangle or a triad.*
(ii) *If $|X| = 4$, then X is a quad or a 4-fan.*
(iii) *If $|X| = 5$, then X is a 5-fan, a 5-cofan, a quad with an element in the guts, or a quad with an element in the coguts.*
(iv) *If $|X| = 6$ and X is sequential, then X is a 6-element fan, a 5-fan with an element in the guts, a 5-fan with an element in the coguts, a 5-cofan with an element in the guts, or a 5-cofan with an element in the coguts.*

The next three lemmas will be used repeatedly throughout the paper. The first is in [9, Lemma 6.1].

Lemma 2.3. *Let M be an internally 4-connected matroid with $|E(M)| \geq 8$. If e is an element of M that is not in a triad, then $M\backslash e$ is 3-connected. In particular, if f is an element of M that is in a triangle, then $M\backslash f$ is 3-connected.*

Lemma 2.4. *Let M be a sequentially 4-connected matroid and M' be a minor of M having a non-sequential 3-separation (X, Y). If (X', Y') is a 3-separation of M' that is equivalent to (X, Y), then (X', Y') does not induce a 3-separation of M.*

Proof. As (X, Y) is non-sequential, so too is (X', Y'). Now assume that (X', Y') induces a 3-separation (X'', Y'') of M. Then, without loss of generality, we may assume that X'' is sequential. Thus there is an ordering $(x_1, x_2, \dots, x_k)$ of X'' such that $\lambda_M(\{x_1, x_2, \dots, x_i\}) \leq 2$ for all i in $\{1, 2, \dots, k\}$. Now $M_1 = M\backslash D/C$ for some C and D. As the connectivity function is monotone on minors (see, for example, [8, Corollary 8.2.5]), $\lambda_{M\backslash D/C}(\{x_1, x_2, \dots, x_i\} - (C \cup D)) \leq 2$ for all i in $\{1, 2, \dots, k\}$. As (X', Y') induces (X'', Y''), it follows that $X'' - (C \cup D) = X'$ and X' is sequential; a contradiction. □

Lemma 2.5. *Let M and N be internally 4-connected binary matroids and $\{e, f, g\}$ be a triangle of M such that $N \preceq M\backslash e$ and $M\backslash e$ is $(4, 4, S)$-connected. Suppose $|E(N)| \geq 7$ and $M\backslash e$ has $(1, 2, 3, 4)$ as a 4-fan. Then either*

(i) *$N \preceq M\backslash e\backslash 1$; or*
(ii) *$N \preceq M\backslash e/4$ and $M\backslash e/4$ is $(4, 4, S)$-connected.*

Proof. First observe that N has no 4-fan. This certainly holds if $|E(N)| \geq 8$; it also holds if $|E(N)| = 7$ since, in this case, $N \cong F_7$ or F_7^*. Since $(1, 2, 3, 4)$ is a 4-fan of $M\backslash e$, either $N \preceq M\backslash e\backslash 1$, or $N \preceq M\backslash e/4$. In the first case, (i) holds. Thus we may assume that $N \preceq M\backslash e/4$. In addition, we may assume that $M\backslash e/4$ is not $(4, 4, S)$-connected, otherwise (ii) holds.

2.5.1. *$M\backslash e/4$ is 3-connected.*

Assume that this statement is false. Then, by Lemma 2.1, $M\backslash e$ has a triangle containing 4, so $M\backslash e$ has a 5-fan F. As $M\backslash e$ is $(4, 4, S)$-connected, it follows that $|E(M\backslash e)| < 10$. Since N is internally 4-connected but $M\backslash e$ has a 5-fan, $7 \leq |E(N)| < |E(M\backslash e)| \leq 9$. As $M\backslash e$ is $(4, 4, S)$-connected, the complement of F in $E(M\backslash e)$ must be a triad or a 4-fan. It follows without difficulty that $M\backslash e$ is the cycle matroid of a 4-wheel or is a single-element deletion of $M(K_5)$. Thus $M\backslash e$ has no 7- or 8-element minor that is internally 4-connected; a contradiction. Hence 2.5.1 holds.

Let (U, V) be a $(4, 4, S)$-violator of $M\backslash e/4$. Suppose that $\{2, 3\} \subseteq U$. Then, as $\{2, 3, 4\}$ is a cocircuit of $M\backslash e$, it follows that $(U \cup 4, V)$ is a $(4, 4, S)$-violator of $M\backslash e$, a contradiction. We deduce that $\{2, 3\} \not\subseteq U$ and, by symmetry, $\{2, 3\} \not\subseteq V$. Hence we may assume that $2 \in U$ and $3 \in V$.

Since $\{1, 2, 3\}$ is a triangle of $M\backslash e$, either $(U \cup 3 \cup 4, V - 3)$ or $(U - 2, V \cup 2 \cup 4)$ is a 3-separating partition of $M\backslash e$ depending on whether 1 is in U or V, respectively. By Lemma 2.4, since $M\backslash e$ is sequentially 4-connected and (U, V) is equivalent to $(U \cup 3, V - 3)$ or $(U - 2, V \cup 2)$, we deduce that (U, V) is sequential. Indeed, as $M\backslash e$ is $(4, 4, S)$-connected, it follows that either $V - 3$ or $U - 2$ is a 4-fan of $M\backslash e$ and hence of $M\backslash e/4$. Thus $M\backslash e/4$ has either V or U as a 5-fan having 3 or 2, respectively, as a guts element t. Then $M\backslash e/4\backslash t$ and hence $M\backslash e\backslash t$ has an N-minor. But $M\backslash e\backslash t$ has $\{2, 3, 4\} - t$ as a cocircuit. Therefore, $M\backslash e\backslash 3/2$ or $M\backslash e\backslash 2/3$ has an

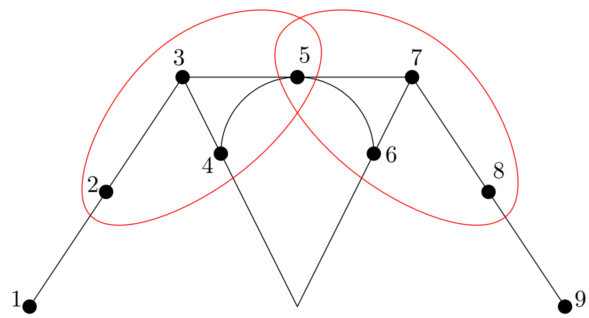

FIGURE 2

N-minor. Thus $M\backslash e/2$ or $M\backslash e/3$ has an N-minor. In both of these matroids, 1 is in a 2-circuit. Hence $M\backslash e\backslash 1$ has an N-minor, so (i) holds. □

The following result is a revision of Lemma 6.3 of [1]. It does no more than extract a stronger statement using a slight modification of the original proof. The result involves another special structure [12]. In an internally 4-connected binary matroid M, we shall call $(\{1,2,3\},\{4,5,6\},\{7,8,9\},\{2,3,4,5\},\{5,6,7,8\},\{3,5,7\})$ a *quasi rotor* with *central triangle* $\{4,5,6\}$ and *central element* 5 if $\{1,2,3\},\{4,5,6\}$, and $\{7,8,9\}$ are disjoint triangles in M such that $\{2,3,4,5\}$ and $\{5,6,7,8\}$ are cocircuits and $\{3,5,7\}$ is a triangle (see Figure 2).

Lemma 2.6. *Let $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ be a bowtie in an internally 4-connected binary matroid M with $|E(M)| \geq 13$. Then $M\backslash 6$ is $(4,4,S)$-connected unless $\{4,5,6\}$ is the central triangle of a quasi rotor whose other triangles are $\{1,2,3\}$, $\{x,y,7\}$, and $\{7,8,9\}$ and whose cocircuits are $\{2,3,4,5\}$ and $\{y,6,7,8\}$, for some x in $\{2,3\}$ and some y in $\{4,5\}$. In addition, when $M\backslash 6$ is $(4,4,S)$-connected, one of the following holds.*

(i) *$M\backslash 6$ is internally 4-connected; or*
(ii) *M has a triangle $\{7,8,9\}$ disjoint from $\{1,2,3,4,5,6\}$ such that $(\{4,5,6\},\{7,8,9\},\{a,6,7,8\})$ is a bowtie for some a in $\{4,5\}$; or*
(iii) *M has a triangle $\{u,v,w\}$ and a cocircuit $\{v,w,x,6\}$ where u and v are in $\{2,3\}$ and $\{4,5\}$, respectively, and $|\{1,2,3,4,5,6,w,x\}| = 8$; or*
(iv) *$M\backslash 1$ is internally 4-connected and M has a triangle $\{1,7,8\}$ and a cocircuit $\{a,6,7,8\}$ where $|\{1,2,3,4,5,6,7,8\}| = 8$ and $a \in \{4,5\}$, so $(\{1,7,8\},\{4,5,6\},\{a,6,7,8\})$ is a good bowtie in M.*

Proof. This is identical to the proof of the original result except that, at the end of the proof of [1, 6.3.2], instead of obtaining a contradiction that $M\backslash 1$ is internally 4-connected, we obtain (iv). □

The next lemma strengthens [1, Lemma 8.2] by eliminating the hypothesis that M has no $M(K_4)$-minor.

Lemma 2.7. *In a binary internally 4-connected matroid M, assume that $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ is a bowtie, and that $\{2,5,7\}$ is a triangle and $\{1,2,7,8\}$ and $\{5,6,7,9\}$ are cocircuits. If $|E(M)| \geq 13$, then $|\{1,2,\dots,9\}| = 9$. Moreover,*

(i) *M has a triangle containing $\{1,8\}$; or*

(ii) *$M/8$ is internally 4-connected; or*
(iii) *M has a circuit $\{y_2, 9, 7, 8\}$ and a triad $\{y_1, y_2, 9\}$ where $|\{1, 2, \dots, 9, y_1, y_2\}| = 11$; or*
(iv) *M has a circuit $\{x_2, x_3, 1, 8\}$ and a triad $\{x_1, x_2, x_3\}$ where $x_1, x_2, x_3, 1, 2, \dots, 9$ are distinct except that, possibly, $x_1 = 9$.*

Proof. This follows the proof of [1, Lemma 8.2]. Because we have omitted the hypothesis that M has no $M(K_4)$-minor, we should add the following sentence between the fifth and sixth sentences of the original proof. If T contains 2, then it also contains 4 and $\lambda(\{1, 2, \dots, 8\}) \le 2$; a contradiction. □

Lemma 2.8. *Let M and N be internally 4-connected matroids with $|E(M)| \ge 11$. Assume that M has a triangle $\{e, f, g\}$ such that $M\backslash e$ is $(4, 4, S)$-connected having $(1, 2, 3, 4)$ as a 4-fan and having an N-minor. Then either*
(i) *M has a good bowtie; or*
(ii) *M has no triangle containing 4, so $\{f, g\}$ meets $\{2, 3\}$; in particular, if $3 \in \{f, g\}$, then the only triangles of M containing 3 are $\{1, 2, 3\}$ and $\{e, f, g\}$, while the only triangles containing 2 or e are $\{1, 2, 3\}$, $\{e, f, g\}$ and possibly one containing $\{2, e\}$.*

Proof. Since M is internally 4-connected, $\{2, 3, 4, e\}$ is a cocircuit of M. Suppose M has a triangle T containing 4. By orthogonality, T meets $\{2, 3, e\}$. If T meets $\{2, 3\}$, then $M\backslash e$ has a 5-fan; a contradiction. Thus $e \in T$, so M has a good bowtie $(T, \{1, 2, 3\}, \{2, 3, 4, e\})$. We may assume that M has no triangle containing 4. Then the rest of (ii) follows by orthogonality because $\{2, 3, 4, e\}$ is a cocircuit. □

Lemma 2.9. *Let M and N be internally 4-connected matroids with $|E(M)| \ge 12$. Assume that M has a triangle $\{e, f, g\}$ such that $M\backslash e$ is $(4, 4, S)$-connected having $(1, 2, 3, 4)$ as a 4-fan. Suppose that each of $M\backslash e$ and $M\backslash 1$ has an N-minor and that $M\backslash 1$ is $(4, 5, S, +)$-connected having a 5-fan. Then M has a good bowtie or a pretty good bowtie.*

Proof. Assume that the lemma fails. Then, by Lemma 2.8, M has no triangle containing 4. Since $\{e, f, g\}$ must meet the cocircuit $\{2, 3, 4, e\}$ of M, by symmetry we may assume that $f = 3$.

Let $(y_1, y_2, y_3, y_4, y_5)$ be a 5-fan in $M\backslash 1$. Then $\{1, y_2, y_3, y_4\}$ is a cocircuit of M. By orthogonality, $\{2, 3\}$ meets $\{y_2, y_3, y_4\}$. As y_3 is in two triangles of $M\backslash 1$, it follows by Lemma 2.8 that $y_3 \notin \{2, 3\}$.

By symmetry, we may now assume that $y_2 \in \{2, 3\}$. Then the triangle $\{y_1, y_2, y_3\}$ must contain e. If $e = y_3$, then M has $\{1, y_2, e, y_4\}$ as a cocircuit so $M\backslash e$ has a 5-cofan containing $\{1, 2, 3, 4\}$; a contradiction. Thus $e \ne y_3$. Then M has $(\{1, 2, 3\}, \{y_3, y_4, y_5\}, \{1, y_2, y_3, y_4\})$ as a bowtie and $\{y_2, y_3, e\}$ as a triangle. Since both $M\backslash e$ and $M\backslash 1$ have N-minors, and $M\backslash e$ and $M\backslash 1$ are $(4, 4, S)$-connected and $(4, 5, S, +)$-connected, respectively, it follows that M has a pretty good bowtie. □

Lemma 2.10. *Let M and N be internally 4-connected matroids with $|E(M)| \ge 10$. Assume that M has a triangle $\{e, f, g\}$ such that $M\backslash e$ is $(4, 4, S)$-connected having an N-minor and having $(1, 2, 3, 4)$ as a 4-fan. Then either*
(i) *M has a good bowtie; or*
(ii) *$|\{2, 3\} \cap \{f, g\}| = 1$ and the common element of $\{2, 3\} \cap \{f, g\}$ is in exactly two triangles of M.*

Moreover, if $3 = f$ *and* $N \preceq M\backslash 3$ *and* $M\backslash 3$ *is* $(4,4,S)$*-connected but not internally* 4*-connected, then either (i) holds, or* M *has an element* t *such that both* $\{1,g,t\}$ *and* $\{2,e,t\}$ *are triangles, so* $M|\{1,2,3,e,g,t\} \cong M(K_4)$*.*

Proof. If M has no good bowtie, then, by Lemma 2.8, (ii) holds. Now suppose that $3 = f$, that $N \preceq M\backslash 3$, and that $M\backslash 3$ is $(4,4,S)$-connected but not internally 4-connected. Suppose also that M has no good bowtie. Then $M\backslash 3$ has a 4-fan (y_1, y_2, y_3, y_4). Lemma 2.8 implies that y_4 is not in a triangle. Thus, by orthogonality, $\{y_2, y_3\}$ meets both $\{1,2\}$ and $\{e,g\}$. As neither $\{1,e\}$ nor $\{2,g\}$ is in a triangle, it follows that $\{y_1,y_2,y_3\}$ is $\{y_1,2,e\}$ or $\{y_1,1,g\}$. But the symmetric difference of one of these with the circuit $\{1,2,e,g\}$ is the other, so M has both $\{y_1,2,e\}$ and $\{y_1,1,g\}$ as triangles. Letting $t = y_1$, it follows that $M|\{1,2,3,e,g,t\} \cong M(K_4)$. □

Lemma 2.11. *Let* M *be an internally* 4*-connected matroid with* $|E(M)| \geq 12$*. Assume that* M *has an element* e *such that* $M\backslash e$ *is* $(4,4,S)$*-connected. If* F_1 *and* F_2 *are distinct* 4*-fans of* $M\backslash e$*, then* $|F_1 \cap F_2| \leq 1$*.*

Proof. Suppose $|F_1 \cap F_2| \geq 2$. Then $\lambda_{M\backslash e}(F_1 \cap F_2) \geq 2$, so $\lambda_{M\backslash e}(F_1 \cup F_2) \leq 2$ as $\lambda_{M\backslash e}(F_1) = 2 = \lambda_{M\backslash e}(F_2)$. But $M\backslash e$ is $(4,4,S)$-connected and $|E(M)| \geq 12$ while $|F_1 \cup F_2| \leq 6$. Hence $|F_1 \cup F_2| = 4$, so $F_1 = F_2$. Moreover, one easily checks that the fan orderings of F_1 and F_2 are equivalent; a contradiction. Hence $|F_1 \cap F_2| \leq 1$. □

Lemma 2.12. *Let* M *be an internally* 4*-connected binary matroid with* $|E(M)| \geq 10$*. If* M *has a triangle* $\{a,b,c\}$ *such that* $M\backslash a$ *has a* 5*-cofan* (x_1,x_2,x_3,x_4,x_5)*, then* $x_3 \in \{b,c\}$

Proof. Since M has no 4-fan, it has $\{x_1,x_2,x_3,a\}$ and $\{x_3,x_4,x_5,a\}$ as cocircuits. Thus, by orthogonality, both $\{x_1,x_2,x_3\}$ and $\{x_3,x_4,x_5\}$ meet $\{b,c\}$. But, at most one element of $\{b,c\}$ is in $\{x_1,x_2,x_3,x_4,x_5\}$ otherwise M has $\{x_1,x_2,x_3,x_4,x_5,a\}$ as a 3-separating set; a contradiction. Thus $x_3 \in \{b,c\}$. □

3. Outline

The proof of Theorem 1.3 occupies the rest of the paper. In this section, we outline the main steps in the argument. Most of the effort in the paper is devoted to proving the following weaker version of the main theorem.

Theorem 3.1. *Let* M *and* N *be internally* 4*-connected binary matroids such that* $|E(M)| \geq 16$ *and* $|E(N)| \geq 7$*. Suppose that* M *has a triangle* T *containing an element* e *for which* $M\backslash e$ *is* $(4,4,S)$*-connected having an* N*-minor. Then one of the following holds.*

(i) *M has an internally* 4*-connected minor* M' *that has an* N*-minor such that* $1 \leq |E(M) - E(M')| \leq 3$*; or*
(ii) *M or M^* has a good bowtie or a pretty good bowtie.*

In the second last section of the paper, we prove, in Lemma 9.2, that, if M has a pretty good bowtie, then M has a good bowtie, a good augmented 4-wheel, or an internally 4-connected minor M' that has an N-minor and satisfies $|E(M)| - |E(M')| = 1$.

Recall that we are assuming that M and N are internally 4-connected binary matroids such that M has a triangle T containing an element e for which $M\backslash e$ is $(4,4,S)$-connected having an N-minor. Since (i) of Theorem 3.1 clearly holds if

$M\backslash e$ is internally 4-connected, we suppose that $M\backslash e$ is not internally 4-connected having $(1,2,3,4)$ as a 4-fan.

The rest of the argument is structured as follows. In the next three sections, we treat the cases when $M\backslash e$ has more than one 4-fan. We show in the next section that, when this occurs, because $M\backslash e$ is $(4,4,S)$-connected, these 4-fans meet in their coguts elements, or they are disjoint, or they meet in their guts elements. These three cases are treated in that order in Sections 4, 5, and 6. Following that, we are able to assume that $M\backslash e$ has $(1,2,3,4)$ as its unique 4-fan. Because N has no 4-fan, either

(i) $N \preceq M\backslash e, 1$; or
(ii) $N \preceq M\backslash e/4$.

We shall refer to these two cases as the *deletion case* and the *contraction case*. We treat these cases in Sections 7 and 8, and thereby complete the proof of Theorem 3.1. In Section 9, we treat the case when M has a pretty good bowtie. The final section of the paper, Section 10, combines that result with Theorem 3.1 to establish Theorem 1.3.

4. Two 4-fans meeting in their coguts elements

We begin this section by showing that, when $M\backslash e$ has two 4-fans, they meet in their coguts elements, they are disjoint, or they meet in their guts elements. We then treat the first of these cases in detail.

Lemma 4.1. *Let M be an internally 4-connected matroid with $|E(M)| \geq 11$. Assume that M has an element e such that $M\backslash e$ is $(4,4,S)$-connected having $(1,2,3,4)$ as a 4-fan. Then $(1,2,3,4)$ is the only 4-fan in $M\backslash e$ having 2 or 3 as an element.*

Proof. As $M\backslash e$ is $(4,4,S)$-connected having at least ten elements, it has no 5-element 3-separating set. Suppose $M\backslash e$ has a 4-fan (a,b,c,d) meeting $\{2,3\}$ where $\{a,b,c,d\} \neq \{1,2,3,4\}$. Suppose first that $\{2,3\}$ meets $\{b,c,d\}$. Then, in $M\backslash e$, the triangle $\{1,2,3\}$ meets the triad $\{b,c,d\}$ and so contains exactly two elements of it. Thus $\{1,2,3,4\} \cup \{b,c,d\}$ and $\{a,b,c,d\} \cup \{1,2,3\}$ are 3-separating sets in $M\backslash e$. Since at least one of these sets has exactly five elements, we have a contradiction. We may now suppose that $\{2,3\}$ meets $\{a\}$. Then the triangle $\{a,b,c\}$ and the cocircuit $\{2,3,4\}$ must meet in two elements, and again we get a 5-element 3-separating set in $M\backslash e$. □

By orthogonality, two different 4-fans in $M\backslash e$ that meet must do so in their coguts elements or in their guts elements. In this section, we show that when we have two 4-fans of $M\backslash e$ meeting in their coguts elements, we have a desirable outcome.

Lemma 4.2. *Let M and N be internally 4-connected matroids with $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Assume that M has a triangle containing an element e such that $M\backslash e$ is $(4,4,S)$-connected having an N-minor and having two 4-fans that meet in their coguts elements. Then either*

(i) *M has a good bowtie; or*
(ii) *M has an internally 4-connected matroid M' such that $1 \leq |E(M) - E(M')| \leq 3$ and M' has an N-minor.*

Proof. Assume that neither (i) nor (ii) holds. Let $(1, 2, 3, 4)$ be a 4-fan of $M\backslash e$ where $\{e, 3, g\}$ is a triangle of M. A second 4-fan of $M\backslash e$ containing 4 must contain g by orthogonality. Thus we may assume this second 4-fan is $(6, 5, g, 4)$. Then the elements $1, 2, 3, 4, e, g, 5$, and 6 are distinct.

Next we show the following.

4.2.1. $N \not\preceq M\backslash e/4$.

Assume $N \preceq M\backslash e/4$. Then, as (ii) does not hold, $M\backslash e/4$ has a $(4, 3)$-violator (U, V). We show first that we may assume that

4.2.2. $|U \cap \{2, 3\}| = 1$.

Suppose $\{2, 3\} \subseteq U$. As $\{2, 3, 4\}$ is a triad of $M\backslash e$, it follows that $(U \cup 4, V)$ is a $(4, 3)$-violator of $M\backslash e$. Thus $g \in V$ otherwise we obtain the contradiction that $(U \cup 4 \cup e, V)$ is a $(4, 3)$-violator of M. Moreover, V must be a 4-fan of $M\backslash e$. As g is an internal element of the fan $(6, 5, g, 4)$, it follows by Lemma 4.1 that $V = \{6, 5, g, 4\}$; a contradiction as $4 \notin V$. Hence 4.2.2 holds.

We may now assume that $u \in U$ and $v \in V$ where $\{u, v\} = \{2, 3\}$. Without loss of generality, we may also assume that $1 \in U$. Then $(U \cup v \cup 4, V - v)$ is a 3-separation of $M\backslash e$. Hence V is a 4-fan (v, s, t, w) of $M\backslash e/4$. Thus $\{4, v, s, t\}$ is a circuit of $M\backslash e$. By orthogonality, $\{s, t\}$ meets $\{g, 5\}$. Then $\{s, t, w\}$ and $\{g, 5, 6\}$ are an intersecting triad and triangle of $M\backslash e$. Hence their union is a 4-fan of $M\backslash e$. But the only 4-fan of $M\backslash e$ containing g or 5 is $(6, 5, g, 4)$. Hence $\{s, t, w\} \cup \{g, 5, 6\} = \{6, 5, g, 4\}$, so $4 \in \{s, t, w\}$; a contradiction. We conclude that 4.2.1 holds.

By 4.2.1, we deduce that $N \preceq M\backslash e\backslash 1$ and $N \preceq M\backslash e\backslash 6$, so $N \preceq M\backslash 1$ and $N \preceq M\backslash 6$. Moreover, M has $(\{1, 2, 3\}, \{g, 5, 6\}, \{2, 3, g, 5\})$ as a bowtie. We now apply [1, Lemma 6.3].

4.2.3. *The triangle $\{g, 5, 6\}$ is not the central triangle of a quasi rotor in M whose other triangles include $\{1, 2, 3\}$ and whose cocircuits include $\{2, 3, 5, g\}$.*

Assume the contrary and suppose first that g is the central element of the quasi rotor. Then the triangle $\{g, 3, e\}$ must be one of the triangles of the quasi rotor. Then, by Lemma 2.8, the quasi rotor contains the only two triangles of M containing 3, one of which is $\{3, 2, 1\}$. Also the quasi rotor contains the only two triangles of M containing e, one of which also contains 2. Thus the element 2 occurs twice in the quasi rotor; a contradiction.

We may now assume, as the quasi rotor contains the cocircuit $\{2, 3, 5, g\}$, that 5 is the central element of the quasi rotor. Then, as the quasi rotor contains the cocircuit $\{2, 3, 5, g\}$, it follows that $\{3, 5\}$ or $\{2, 5\}$ is contained in a triangle T of the quasi rotor. But neither of the two triangles of M that contain 3 also contains 5. Thus $2 \in T$, so $T = \{2, 5, e\}$. Then the symmetric difference of the circuits $\{1, 2, e, g\}$, $\{2, 5, e\}$, and $\{6, 5, g\}$ is $\{1, 6\}$. Thus we have a contradiction that completes the proof of 4.2.3.

We now use Lemma 2.6. This implies that $M\backslash 6$ is $(4, 4, S)$-connected. Moreover, as both $M\backslash 1$ and $M\backslash 6$ have N-minors and M has no good bowties, it follows that (iii) of the lemma holds. Thus M has a triangle T that contains exactly one element of each of $\{2, 3\}$ and $\{5, g\}$ along with another element that is not 1 or 6. Suppose T contains 2. Then it also contains e. But $\{2, e, g\}$ is not a circuit since $\{3, e, g\}$ is. Thus $\{2, e, 5\}$ is a circuit. But $\{3, e, g\}$ is a circuit. Hence $\{2, 5, e, g\}$ is a circuit of M. Since it is also a cocircuit, we have a contradiction. We deduce that $3 \in T$.

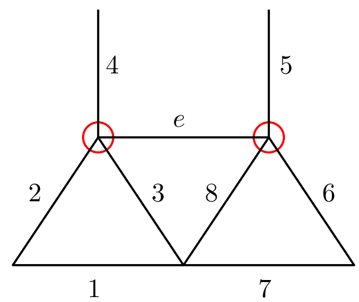

FIGURE 3

Then $T = \{3, g, e\}$. Part (iii) of Lemma 2.6 also gives that M has a 4-cocircuit $\{e, g, 6, z\}$. Thus $M\backslash e$ has $(4, 5, g, 6, e)$ as a 5-cofan; a contradiction. We conclude that Lemma 4.2 holds. □

5. Two disjoint 4-fans

In this section, we treat the case when $M\backslash e$ has two 4-fans that are disjoint. The argument here is long but it will mean that when we come to consider the case when we have two 4-fans meeting in their guts elements, we never have two disjoint 4-fans or two 4-fans meeting in their coguts elements.

We have defined what is meant by M having a good bowtie or a pretty good bowtie. We can think of these structures as being relative to the minor N of M. In the next result, we shall refer to M^* having a good bowtie or a pretty good bowtie. These substructures of M^* exist relative to the minor N^* of M^*.

Theorem 5.1. *Let M and N be internally 4-connected matroids with $|E(M)| \geq 16$ and $|E(N)| \geq 7$. Assume that M has a triangle containing an element e such that $M\backslash e$ is $(4, 4, S)$-connected having an N-minor and having two disjoint 4-fans. Then one of the following holds.*

(i) *M or M^* has a good bowtie;*
(ii) *M or M^* has a pretty good bowtie;*
(iii) *M has a good augmented 4-wheel; or*
(iv) *M has an internally 4-connected matroid M' such that $1 \leq |E(M) - E(M')| \leq 3$ and M' has an N-minor.*

Proof. We assume that $M\backslash e$ has $(1, 2, 3, 4)$ and $(7, 6, 8, 5)$ as disjoint 4-fans. Then $\{2, 3, 4, e\}$ and $\{6, 8, 5, e\}$ are cocircuits of M. Thus, by Lemma 2.8 and orthogonality with the triangle $\{e, f, g\}$, we may assume that $f = 3$ and $g = 8$. Hence M contains the configuration shown in Figure 3 where all the elements shown are distinct.

Lemma 5.2. *One of $M\backslash e\backslash 1$, $M\backslash e\backslash 7$, or $M\backslash e/4, 5$ has an N-minor.*

Proof. As $M\backslash e$ has $(1, 2, 3, 4)$ as a 4-fan and $N \preceq M\backslash e$, it follows that $N \preceq M\backslash e\backslash 1$ or $N \preceq M\backslash e/4$. Now $M\backslash e/4$ has $(7, 8, 6, 5)$ as a 4-fan, so if $N \preceq M\backslash e/4$, then $N \preceq M\backslash e/4\backslash 7$ or $N \preceq M\backslash e/4, 5$. The lemma follows immediately. □

Lemma 5.3. *One of the following holds.*

(i) *$M\backslash 1$ is $(4,4,S)$-connected;*
(ii) *M has a good bowtie;*
(iii) *M has $\{1,3,7,8\}$ as a cocircuit; or*
(iv) *M has $\{1,3,6,7,e\}$ as a cocircuit.*

Proof. Assume the lemma fails. Clearly a $(4,4,S)$-violator of $M\backslash 1$ must have 2 and 3 on opposite sides. We begin by showing the following.

5.3.1. *Suppose (U,V) is a $(4,4,S)$-violator of $M\backslash 1$ with $2 \in U$ and $3 \in V$. Then $\{8,e\} \subseteq V$ and $4 \in U$. Moreover, $|V| \leq 5$ and U meets $\{5,6,7\}$.*

First note that $1 \notin \text{cl}(U)$, so $\{e,8\} \not\subseteq U$. Suppose $e \in U$ and $8 \in V$. If $4 \in U$, then $(U \cup 3 \cup 1, V - 3)$ is a 3-separation of M; a contradiction. Thus $4 \in V$. Then $e \in \text{cl}(V)$ and $2 \in {\text{cl}^*}_{M\backslash 1}(V \cup e)$. Thus $(U - e - 2, V \cup e \cup 2 \cup 1)$ is 3-separating in M. Hence U is a 5-fan $(e,2,a,b,c)$ of $M\backslash 1$ for some elements a, b, and c. Thus $a \in \{5,6,8\}$ by orthogonality, and $\{1,2,a,b\}$ is a cocircuit of M. But $8 \notin \{a,b,c\}$ as $8 \in V$. Thus $a \in \{5,6\}$. Then, by orthogonality, $\{a,b,c\}$ is a triangle containing $\{5,6\}$, so $M\backslash e$ is not $(4,4,S)$-connected; a contradiction.

Next assume that $8 \in U$ and $e \in V$. Then $(U - 8, V \cup 8)$ is a $(4,4,S)$-violator of $M\backslash 1$ unless $U - 8$ is a 4-fan in $M\backslash 1$. In the exceptional case, $M\backslash 1$ has a 5-fan $(8,a,b,c,d)$. If $2 \in \{b,c,d\}$, then, by orthogonality, $4 \in \{b,c,d\}$, a contradiction to Lemma 2.8. Thus we may assume that $2 = a$. But, by orthogonality, $\{2,8\}$ can only be in a triangle with e, which does not occur since $e \in V$. We conclude that $\{8,e\} \subseteq V$.

We observe that $4 \in U$ otherwise $(U - 2, V \cup 2 \cup 1)$ is a $(4,3)$-violator of M; a contradiction. Now $(U, V - e)$ is a 3-separation of $M\backslash 1\backslash e$. Thus $(U \cup 3 \cup 1, V - \{3,e\})$ is 3-separating in $M\backslash e$. As $M\backslash e$ is $(4,4,S)$-connected, we deduce that $|V| \leq 6$. If $|V| \leq 5$, then, as $\{3,8,e\} \subseteq V$, it follows that U meets $\{5,6,7\}$. Thus, to complete the proof of 5.3.1, we need to show that $|V| \neq 6$.

Suppose $|V| = 6$. Then $V - \{e,3\}$ is a 4-element fan in $M\backslash e$ that contains 8, so, by Lemma 4.1, $V = \{e,3,7,6,8,5\}$. Now $\{e,6,8,5\}$ is a cocircuit of M, so it contains no circuit of M. Thus $r(V) \geq 4$. Evidently $r(V) = 4$. Since $2 = \lambda_{M\backslash 1}(V) = r_{M\backslash 1}(V) + r^*_{M\backslash 1}(V) - 6$, we deduce that $r^*_{M\backslash 1}(V) = 4$. Hence V contains a cocircuit C^* in $M\backslash 1$ other than $\{e,6,8,5\}$, and $C^* \cup 1$ is a cocircuit in M. Orthogonality implies that $3 \in C^*$. As 1 is in no triad of M, orthogonality implies that $C^* \cup 1$ is $\{1,3,e,5\}, \{1,3,e,6,7\}, \{1,3,e,5,6,7\}, \{1,3,8,6\}, \{1,3,8,7,5\}, \{1,3,8,7\}$, or $\{1,3,8,6,5\}$. As $(C^* \cup 1) \triangle \{e,6,8,5\}$ is not a triad, the last possibility is excluded. Of the remaining six possibilities, the symmetric difference of the first with $\{e,6,8,5\}$, the second with $\{e,6,8,5\}$, and the third with $\{e,6,8,5\}$ yield the fourth, the fifth, and the sixth. By assumption, $\{1,3,8,7\}$ and $\{1,3,6,7,e\}$ are not cocircuits of M. We conclude that M has $\{1,3,e,5\}$ as a cocircuit. Then $M\backslash e$ has $\{1,3,5\}$ as a cocircuit, so $M\backslash e$ has a 5-cofan. This contradiction completes the proof that 5.3.1 holds.

We may now assume that $M\backslash 1$ has a $(4,4,S)$-violator (U,V) with $\{2,4\} \subseteq U$ and $\{3,8,e\} \subseteq V$. By 5.3.1, U meets $\{5,6,7\}$ and $|V| \leq 5$. Suppose $\{5,6,7\} \subseteq U$. Then $(U \cup 8 \cup e \cup 1, V - 8 - e)$ is 3-separating in M. Thus V is a 5-fan $(8,e,a,b,c)$ in $M\backslash 1$. Hence $a = 3$. Thus 3 is contained in a triangle, $\{a,b,c\}$, contained in V but not containing $2, 4$, or e; a contradiction to orthogonality. We deduce that $\{5,6,7\}$ meets V.

Suppose 6 or 7 is in V. Then the circuit $\{3,e,6,7\}$ implies that $(U-\{6,7\}, V\cup\{6,7\})$ is a 3-separation of $M\backslash 1$. Thus $(U-\{5,6,7\}, V\cup\{5,6,7\})$ is 3-separating in $M\backslash 1$. But $V\cup\{5,6,7\}$ contains $\{3,8,e,5,6,7\}$ and so $|V\cup\{5,6,7\}|\in\{6,7\}$, Then $(U-\{5,6,7\}, V\cup\{5,6,7\})$ is a $(4,4,S)$-violator of $M\backslash 1$ that contradicts 5.3.1. We deduce that we may assume that $\{6,7\}\subseteq U$ and $5\in V$. Then $(U-\{6,7\}, V\cup\{6,7\})$ is a $(4,4,S)$-violator of $M\backslash 1$, which, since $|V\cup\{6,7\}|\in\{6,7\}$, contradicts 5.3.1. □

Lemma 5.4. *Suppose $N\preceq M\backslash 1$ and M has $\{1,3,6,7,e\}$ as a cocircuit. Then $M\backslash 3$ is internally 4-connected having an N-minor or M has a good bowtie.*

Proof. First we observe that $M\backslash 1$ has $\{3,6,7,e\}$ as a quad with 8 in the guts. As N is a minor of the matroid obtained by putting a triangle or a triad on the guts of $(\{3,6,7,e\}, E(M\backslash 1)-\{3,6,7,e\})$, it follows that $N\preceq M\backslash 1\backslash 3$. Thus $N\preceq M\backslash 3$.

As 3 is in a triangle, $M\backslash 3$ is 3-connected. We show next that $M\backslash 3$ is sequentially 4-connected. Let (U,V) be a non-sequential 3-separation of $M\backslash 3$. We may assume that the triad $\{2,4,e\}$ is contained in V. Also we may assume that the triangle $\{6,7,8\}$ is contained in U or V. Now $\{1,8\}\subseteq U$ otherwise $(U, V\cup 3)$ is a non-sequential 3-separation of M; a contradiction. Thus $\{1,8,6,7\}\subseteq U$. The cocircuit $\{1,6,7,e\}$ of $M\backslash 3$ implies that $(U\cup e\cup 3, V-e)$ is a $(4,3)$-violator of M; a contradiction. Thus $M\backslash 3$ is indeed sequentially 4-connected.

We may now assume that $M\backslash 3$ is not internally 4-connected otherwise the lemma holds. Then $M\backslash 3$ has a 4-fan (x_1,x_2,x_3,x_4). Hence $\{x_2,x_3,x_4,3\}$ is a cocircuit C^* of M. Orthogonality of this cocircuit with the circuits $\{1,2,3\},\{e,3,8\}$, and $\{3,6,7,e\}$ implies that $\{x_2,x_3,x_4\}$ meets each of $\{1,2\},\{e,8\}$, and $\{6,7,e\}$. Suppose that $e\notin C^*$. Then C^* is $\{3,8\}$ along with exactly one element from each of $\{1,2\}$ and $\{6,7\}$. Taking the symmetric difference of C^* with $\{1,3,6,7,e\}$ gives a cocircuit containing 8, which cannot be a triad. Thus C^* is $\{2,3,8,6\}$ or $\{2,3,8,7\}$. Therefore $\lambda(\{1,2,3,6,7,8,e\})\le 2$; a contradiction. We deduce that $e\in C^*$. If $e=x_4$, then M has a good bowtie. Thus we may assume that $e=x_3$. Then $\{1,2\}$ meets $\{x_2,x_4\}$. As $M\backslash e$ is $(4,4,S)$-connected, C^*-e cannot be a triad containing $\{3,1\}$. Thus $2\in\{x_2,x_4\}$, so $C^*=\{2,3,4,e\}$. By Lemma 2.8, M has no triangle containing 4. Thus $(x_2,x_4)=(2,4)$. Hence, by orthogonality, $\{x_1,x_2,x_3\}$ is $\{e,2,5\}$. Thus M has $\{1,8,5\}$ as a triangle, so $M\backslash e$ is not $(4,4,S)$-connected; a contradiction. □

Lemma 5.5. *If $N\preceq M\backslash 1$ and $\{1,3,7,8\}$ is a cocircuit of M, then $N\preceq M\backslash 8$.*

Proof. As $M\backslash 1$ has $(e,3,8,7,6)$ as a 5-fan, by [4, Lemma 3.3], either $N\preceq M\backslash 1,e,6$ or $N\preceq M\backslash 1/3\backslash 8$. In the latter case, $N\preceq M\backslash 8$. In the former case, as $\{5,8\}$ is a cocircuit of $M\backslash 1,e,6$, it follows that $N\preceq M\backslash 1,e,6/8$. Thus $N\preceq M\backslash 1/8\backslash 3,7$, so $N\preceq M\backslash 1,3,7,8$. Hence $N\preceq M\backslash 8$. □

Lemma 5.6. *The only triangles of M meeting $\{2,3,4,5,6,8,e\}$ are $\{1,2,3\},\{e,3,8\}$, and $\{6,7,8\}$.*

Proof. Let T be a triangle of M that meets $\{2,3,4,5,6,8,e\}$ but is different from $\{1,2,3\},\{e,3,8\}$, and $\{6,7,8\}$. If $e\in T$, then, by Lemma 2.8, symmetry, and orthogonality with the cocircuits $\{e,2,3,4\}$ and $\{e,5,6,8\}$, we deduce that $T=\{e,2,6\}$, a contradiction as $\{e,2,6,1,7\}$ is a circuit. Thus $e\notin T$. By symmetry, we may suppose that T meets $\{2,3,4\}$. Then $4\in T$, a contradiction to Lemma 2.8. □

Lemma 5.7. *Suppose $N\preceq M\backslash 1$ and $\{1,3,7,8\}$ is a cocircuit of M. Then*

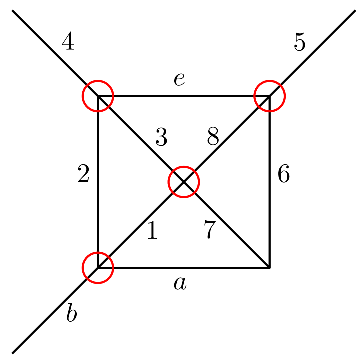

FIGURE 4

(i) *$M\backslash 8$ is $(4,4,S)$-connected and M has a good bowtie; or*
(ii) *M has a pretty good bowtie; or*
(iii) *M has the configuration shown in Figure 4 as a restriction where all the elements shown are distinct.*

Proof. Assume that none of (i)–(iii) holds. Observe first that if M has the configuration shown in Figure 4 as a restriction, then, by assumption, $|\{1,2,3,4,5,6,7,8,e\}| = 9$. Using this and the fact that $|E(M)| \geq 16$, it is straightforward to check that all the elements in the figure are distinct.

Evidently, if $M\backslash 8$ is $(4,4,S)$-connected, then $(\{6,7,8\},\{1,2,3\},\{1,3,7,8\})$ is a good bowtie as $N \preceq M\backslash 8$, by Lemma 5.5. Thus we may assume that $M\backslash 8$ has a $(4,4,S)$-violator (U,V). Without loss of generality, suppose that $|U \cap \{1,3,7\}| \geq 2$.

5.7.1. *(U,V) is equivalent to a $(4,4,S)$-violator (U',V') with $\{1,3,7\} \subseteq U'$ and $\{6,e\} \subseteq V'$.*

Suppose $|U \cap \{1,3,7\}| = 2$. Then $(U \cup \{1,3,7\}, V - \{1,3,7\})$ is a $(4,4,S)$-violator of $M\backslash 8$ that is equivalent to (U,V) unless V is a 5-cofan (v_1,v_2,v_3,v_4,v_5) in $M\backslash 8$ where $v_1 \in \{1,3,7\}$. Consider the exceptional case. Then $\{v_3,v_4,v_5,8\}$ is a cocircuit of M. Thus, by orthogonality with the circuits $\{e,3,8\}$ and $\{6,7,8\}$, it follows that $\{e,6\} \subseteq \{v_3,v_4,v_5\}$, so $\{v_3,v_4,v_5\} = \{e,5,6\}$. But, by Lemma 5.6, $\{e,6\} \cap \{v_2,v_3,v_4\} = \emptyset$, so we have a contradiction. Thus $M\backslash 8$ has a $(4,4,S)$-violator (U',V') that is equivalent to (U,V) and has $\{1,3,7\}$ contained in U'. As $(U' \cup 8, V')$ is not a 3-separation of M, it follows that $\{6,e\} \subseteq V'$. Thus 5.7.1 holds.

5.7.2. *$M\backslash 8$ has a $(4,4,S)$-violator (U',V') that is equivalent to (U,V) such that $\{1,2,3,7\} \subseteq U'$ and $\{4,5,6,e\} \subseteq V'$.*

From 5.7.1, M has a $(4,4,S)$-violator (U',V') that is equivalent to (U,V) such that $\{1,3,7\} \subseteq U'$ and $\{6,e\} \subseteq V'$. Suppose $2 \in V'$. Then $(U' \cup 2, V' - 2)$ is a $(4,4,S)$-violator of $M\backslash 8$ otherwise $M\backslash 8$ has a 5-fan that contains a triangle that contains 2 but is different from $\{1,2,3\}$; a contradiction to Lemma 5.6. Hence we may assume that $\{1,2,3,7\} \subseteq U'$.

Suppose that $4 \in U'$. Then $(U' \cup e \cup 8, V' - e)$ is a $(4,3)$-violator of M; a contradiction. Thus $4 \in V'$. Suppose $5 \in U'$. Then $(U' - 5, V' \cup 5)$ is a $(4,4,S)$-violator of $M\backslash 8$ that is equivalent to (U,V) unless U' is a 5-cofan $(5,u_2,u_3,u_4,u_5)$. In the exceptional case, as $U' - 5$ contains the triangle $\{1,2,3\}$, we deduce that

$\{u_2, u_3, u_4\} = \{1, 2, 3\}$ and $u_5 = 7$. Thus M has a cocircuit that contains $\{5, 8\}$ and is contained in $\{1, 2, 3, 5, 8\}$. This contradicts orthogonality. We deduce that $(U'-5, V'\cup 5)$ is a $(4, 4, S)$-violator of $M\backslash 8$ and $\{1, 2, 3, 7\} \subseteq U'-5$ and $\{e, 4, 5, 6\} \subseteq V' \cup 5$. Thus 5.7.2 holds.

Now let (U', V') be a $(4, 4, S)$-violator of $M\backslash 8$ that is equivalent to (U, V) and has $\{1, 2, 3, 7\} \subseteq U'$ and $\{e, 4, 5, 6\} \subseteq V'$.

5.7.3. $|U'| \le 7$.

Observe that $\lambda_{M\backslash 8}(V' \cup 2) \le 3$. Thus $\lambda_{M\backslash 8}(V' \cup 2 \cup 3 \cup 1) \le 3$. As $7 \in \mathrm{cl}_{M\backslash 8}(V'\cup\{2, 3, 1\})\cap \mathrm{cl}^*{}_{M\backslash 8}(V'\cup\{2, 3, 1\})$, it follows that $\lambda_{M\backslash 8}(V'\cup\{2, 3, 1, 7\}) \le 2$. Thus $\lambda_M(V' \cup \{2, 3, 1, 7, 8\}) \le 2$. Hence $|U' - \{2, 3, 1, 7\}| \le 3$, so $|U'| \le 7$.

5.7.4. $|U'| \ne 7$.

Suppose that $|U'| = 7$, letting $U' = \{1, 2, 3, 7, a, b, c\}$. Then $r(U')+r^*_{M\backslash 8}(U') = 9$. Thus $r(U') \ge 4$ otherwise $M|U' \cong F_7$ and M has a triangle containing 2 other than $\{1, 2, 3\}$; a contradiction to Lemma 5.6.

Assume that $r(U') = 4$. Then $\{1, 2, 3, 7, a, b, c\}$ contains distinct circuits C and C' different from $\{1, 2, 3\}$ such that $C \triangle C' \triangle \{1, 2, 3\} \ne \emptyset$. Clearly C and C' are not both contained in $\{2, a, b, c\}$ so, without loss of generality, C meets $\{1, 3, 7\}$. By orthogonality, $C \cap \{1, 3, 7\}$ is $\{1, 3\}$, $\{1, 7\}$, or $\{3, 7\}$. By orthogonality between C and $\{2, 3, 4, e\}$, these three cases imply, respectively, that C contains $\{1, 2, 3\}$, $\{1, 7\}$, or $\{2, 3, 7\}$. The first case is impossible by assumption. By taking the symmetric difference with $\{1, 2, 3\}$, we may assume that C contains $\{1, 7\}$ and $C \subseteq \{1, 7, a, b, c\}$. Similarly, either $C' \subseteq \{2, a, b, c\}$, or $\{1, 7\} \subseteq C' \subseteq \{1, 7, a, b, c\}$. Suppose $C' \subseteq \{2, a, b, c\}$. Then orthogonality with $\{2, 3, 4, e\}$ implies that $2 \notin C'$. Thus $C' = \{a, b, c\}$. Since $\{1, 7\} \subseteq C \subseteq \{1, 7, a, b, c\}$, without loss of generality, we may take C to be a triangle $\{1, 7, a\}$. On the other hand, if $C' \not\subseteq \{2, a, b, c\}$, then $\{1, 7\} \subseteq C' \subseteq \{1, 7, a, b, c\}$. Thus $C \triangle C'$ must be $\{a, b, c\}$. Since we again get that M has $\{a, b, c\}$ and $\{1, 7, a\}$ as circuits, it follows that we may now assume this.

As $r_{M\backslash 8}(U') = 4$, it follows that $r^*_{M\backslash 8}(U') = 5$. Hence $M\backslash 8$ has a cocircuit C^* that is contained in $\{1, 2, 3, 7, a, b, c\}$ and is not $\{1, 3, 7\}$. By orthogonality with the triangles $\{1, 2, 3\}$, $\{1, 7, a\}$, and $\{a, b, c\}$, it follows by the symmetry between b and c that we may assume that C^* is $\{7, a, b\}$, $\{1, 2, a, b\}$, $\{1, 3, a, b\}$, $\{2, 3, b, c\}$, $\{2, 3, 7, a, b\}$, or $\{1, 2, 7, b, c\}$. The triangle $\{6, 7, 8\}$ implies that C^* is a cocircuit of M when $7 \notin C^*$, while $C^* \cup 8$ is a cocircuit of M when $7 \in C^*$. Then M has, as a cocircuit, one of $\{7, 8, a, b\}$, $\{1, 2, a, b\}$, $\{1, 3, a, b\}$, $\{2, 3, b, c\}$, $\{2, 3, 7, 8, a, b\}$, or $\{1, 2, 7, 8, b, c\}$. Orthogonality with the triangle $\{e, 3, 8\}$ implies that M has $\{1, 2, a, b\}$ or $\{2, 3, 7, 8, a, b\}$ as a cocircuit. As $\{2, 3, 7, 8, a, b\} \triangle \{1, 3, 7, 8\} = \{1, 2, a, b\}$, we deduce that M has $\{1, 2, a, b\}$ as a cocircuit. Thus, if $r(U') = 4$, then M has the structure in Figure 4 as a restriction; a contradiction.

We may now assume that $r(U') \ge 5$, so $r^*_{M\backslash 8}(U') \le 4$. Thus U' contains distinct cocircuits C^*_1 and C^*_2 different from $\{1, 3, 7\}$ such that $C^*_1 \triangle C^*_2 \triangle \{1, 3, 7\} \ne \emptyset$. The circuit $\{3, 6, 7, e\}$ implies that, for each i in $\{1, 2\}$, either $\{3, 7\} \subseteq C^*_i$, or C^*_i avoids $\{3, 7\}$. The triangle $\{1, 2, 3\}$ implies that either C^*_i contains $\{1, 3, 7\}$ or $\{2, 3, 7\}$, or C^*_i avoids $\{3, 7\}$. The first possibility has been excluded. If $\{2, 3, 7\} \subseteq C^*_i$, then $C^*_i \triangle \{1, 3, 7\} \subseteq \{1, 2, a, b, c\}$. Therefore, by taking symmetric differences in this way, we may assume that both C^*_1 and C^*_2 are contained in $\{1, 2, a, b, c\}$ and hence that $\{C_1^*, C_2^*\} = \{\{1,2,a\},\{a,b,c\}\}$. Then $\{1,2,8,a\}$ is a cocircuit of M, contradicting orthogonality with the circuit $\{e,3,8\}$. We conclude that 5.7.4 holds.

5.7.5. $|U'| \neq 6$.

Assume $|U'| = 6$, letting $U' = \{1,2,3,7,a,b\}$. If $r(U') = 3$, then $M|U' \cong M(K_4)$. As $\{1,3,7\}$ is a cocircuit of $M\backslash 8$, it follows that M has a circuit containing $\{3,7\}$ contradicting Lemma 5.6. Thus $r(U') \geq 4$. Suppose $r^*_{M\backslash 8}(U') = 3$. Then, in $M^*/8$, we have $\{1,2,3\}$ as a triad. As $\{2,3,7,8\}$ is not a cocircuit of M, it follows that $\{2,3,a,8\}$ or $\{2,3,b,8\}$ is a cocircuit of M. This contradicts orthogonality with the triangle $\{6,7,8\}$. Hence $r^*_{M\backslash 8}(U') = 4 = r_{M\backslash 8}(U')$.

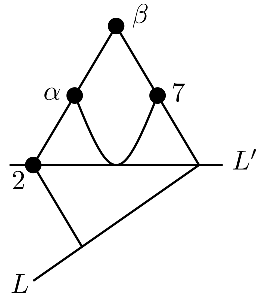

FIGURE 5

In $M\backslash 8$, let L' be the guts line of the 4-fan $(2,1,3,7)$ and let L be the guts line of the 3-separating set U'. Suppose first that L and L' meet in $\{2\}$. Then $2 \in \mathrm{cl}_{M\backslash 8}(V')$. Thus $(U'-2, V' \cup 2)$ is 3-separating in $M\backslash 8$. Hence so is $(U'-2-3, V' \cup 2 \cup 3)$. Then $(U'-2-3, V' \cup 2 \cup 3 \cup 8)$ is a $(4,3)$-violator of M; a contradiction. We deduce that L and L' do not meet in $\{2\}$. However, since $\{1,3,7\}$ is a cocircuit of $M\backslash 8$, we must have L and L' meeting. Thus we have the situation shown in Figure 5 where $\{\alpha,\beta\} = \{1,3\}$. If a or b is on L, then U' contains another triad apart from $\{1,3,7\}$ and it must contain 2. Now $\{2,7\}$ is not contained in a triad of $M\backslash 8$ otherwise $\{2,7,8\}$ is contained in a 4-cocircuit of M; a contradiction. Thus $M\backslash 8$ has a triad containing 2 but not 7, so, by orthogonality, M has a triad containing 2; a contradiction. We conclude that neither a nor b is on L. If $\alpha = 3$, then $\{2,3,a,b\}$ is a cocircuit of $M\backslash 8$ and hence of M. But $\{3,6,7,e\}$ is a circuit of M, so we contradict orthogonality. Thus $\alpha \neq 3$, so $\alpha = 1$ and $\beta = 3$. Now, by orthogonality, $\{3,7,a,b\}$ is not a circuit of M as $\{2,3,4,e\}$ is a cocircuit. Thus, without loss of generality, a is the element on L' that is not on L and is not 2. Thus $M\backslash 8$, and hence M, has $\{1,2,a,b\}$ as a cocircuit and $\{1,7,a\}$ as a circuit. We conclude that the configuration in Figure 4 occurs as a restriction of M; a contradiction. Therefore 5.7.5 holds.

We may now assume that $|U'| = 5$. Let $U' = \{1,2,3,7,a\}$. Then

5.7.6. *U' is a 5-fan* $(2,3,1,7,a)$.

Assume this fails. Then U' is a 5-cofan $(a,u_2,u_3,u_4,7)$ where $u_2 = 2$ and $\{u_3,u_4\} = \{3,1\}$. Hence, by orthogonality with the triangle $\{e,3,8\}$, we deduce that the 5-cofan is $(a,2,3,1,7)$. But, as $a \neq 6$, it follows by orthogonality that $\{a,2,3\}$ is a triad of M; a contradiction. Hence 5.7.6 holds.

We now know that $N \preceq M\backslash a$. Evidently $M\backslash 1$ and $M\backslash 7$ are not $(4,4,S)$-connected. Thus, by [1, Theorem 5.1], either $\{1,7,a\}$ is the central triangle of a quasi rotor, or $M\backslash a$ is $(4,4,S)$-connected.

Next we show the following.

5.7.7. $\{1,7,a\}$ *is not the central triangle of a quasi rotor.*

Assume the contrary. Suppose first that 7 is the central element of this quasi rotor. Then, as $\{1,3,7,8\}$ is a cocircuit, by Lemma 5.6, $\{8,7,6\}$ and $\{1,7,a\}$ are the only triangles of M containing 7. Thus M has a second triangle containing 6, contradicting Lemma 5.6. Similarly, if 1 is the central element of the quasi rotor, then, by Lemma 5.6, the other triangle of the quasi rotor containing 1 is $\{1,2,3\}$. Hence M has a second triangle containing 2, contradicting Lemma 5.6. We conclude that a is the central element of the quasi rotor. This quasi rotor contains a cocircuit $\{1,a,u,v\}$ where $\{u,a\}$ is in a triangle. Now, as neither $\{2,a\}$ nor $\{3,a\}$ is in a triangle, it follows by orthogonality with the triangle $\{1,2,3\}$ that $v \in \{2,3\}$. Hence, by Lemma 5.6, $v=3$ and $u \in \{e,8\}$. This contradiction to Lemma 5.6 implies that 5.7.7 holds.

We now know that $M\backslash a$ is $(4,4,S)$-connected. Then $M\backslash 8$ is not $(4,5,S,+)$-connected otherwise M has a pretty good bowtie. Thus $M\backslash 8$ has a $(4,5,S,+)$-violator (X,Y) where $|X \cap \{1,3,7\}| \geq 2$. Then (X,Y) is a $(4,4,S)$-violator of $M\backslash 8$. Then, by 5.7.2, 5.7.4, 5.7.5, and 5.7.6, it follows that (X,Y) is equivalent to the 3-separation (U',V') where U' is the 5-fan $(2,3,1,7,a)$. Now U' has no element of V' in its closure, otherwise M has a triangle containing $\{3,7\}$; a contradiction to Lemma 5.6. Thus we may assume that V' has an element b that is in the coclosure of U' in $M\backslash 8$. Then $M\backslash 8$ has a cocircuit C^* that contains b and is contained in $U' \cup b$. If $\{b,1,2,a\}$ is a cocircuit of M, then M has the configuration in Figure 4 as a restriction; a contradiction. Thus we may assume that $\{b,1,2,a\}$ is not a cocircuit of M.

5.7.8. $1 \in C^*$.

Suppose $1 \notin C^*$. Then, by orthogonality with the circuits $\{1,2,3\}$ and $\{1,7,a\}$, we deduce that C^* is $\{b,2,3\}$, $\{b,7,a\}$, or $\{b,2,3,7,a\}$. In the first case, the circuit $\{3,7,6,e\}$ implies that $b \in \{6,e\}$. But $\{e,2,3,4\}$ is a cocircuit of $M\backslash 8$, so $b \neq e$. Hence $\{6,2,3\}$ is a cocircuit of $M\backslash 8$. Then $\{6,2,3,8\}$ is a cocircuit of M, so $\lambda(\{1,2,3,6,7,8\}) \leq 2$; a contradiction. If $C^* = \{b,7,a\}$, then the circuit $\{7,1,2,e,6\}$ implies that $b \in \{e,6\}$. Thus $C^* \cup 8$ is a cocircuit of M and $\lambda(\{1,2,3,6,7,8,e,a\}) \leq 2$; a contradiction. If $C^* = \{b,2,3,7,a\}$, then $C^* \triangle \{1,3,7\} = \{1,2,a,b\}$; a contradiction. We conclude that 5.7.8 holds.

Now C^* contains $\{b,1\}$ and exactly one element of each of $\{2,3\}$ and $\{7,a\}$. Thus $|C^*| = 4$. Then, as $C^* \neq \{b,1,2,a\}$ and $C^* \not\supseteq \{1,3,7\}$, it follows that $|C^* \cap \{3,7\}| = 1$. Hence $C^* \triangle \{1,3,7\}$ is a triad that contains b but not 1. Since this case was already eliminated in 5.7.8, we conclude that the lemma holds. □

Lemma 5.8. *Suppose $N \preceq M\backslash 1$ and M has the configuration shown in Figure 4 as a restriction. Then M contains a good bowtie or a good augmented 4-wheel, or M has an internally 4-connected minor M' that has an N-minor and satisfies* $1 \leq |E(M) - E(M')| \leq 2$.

Proof. Assume that the lemma fails. First we observe that

5.8.1. *M has no 4-cocircuit containing* $\{a,6,7\}$.

To see this, note that N is a minor of both $M\backslash 1$ and $M\backslash e$ and the latter is $(4,4,S)$-connected. Thus M has no 4-cocircuit containing $\{a,6,7\}$ otherwise M contains a good augmented 4-wheel.

Next we observe that

5.8.2. $\{1,3,7,8\}$ *is the only* 4*-cocircuit of* M *that contains at least two elements of* $\{1,3,7,8\}$.

If M has another 4-cocircuit C^* containing at least two elements of $\{1,3,7,8\}$, then, by orthogonality, $C^* \subseteq \{1,2,3,6,7,8,e,a\}$. Thus $\lambda(\{1,2,3,6,7,8,e,a\}) \leq 2$; a contradiction.

Since $M^*/1$ has $(e,3,8,7,6)$ as a 5-cofan with 5 in the guts, $M^*/1/8$ has an N^* minor. Thus either $M^*/1,8,7$ or $M^*/1,8\backslash 3,5$ has an N^*-minor. In the first case, $N \preceq M\backslash 7$ and we see that M contains a good augmented 4-wheel. Thus we may assume that

5.8.3. $M\backslash 7$ *has no* N*-minor and* $N \preceq M\backslash 1,8/3,5$.

Next we show the following.

5.8.4. *For each* α *in* $\{a,2,6\}$*, the matroid* $M\backslash\alpha$ *has an* N*-minor and is* $(4,4,S)$*-connected. Moreover, for each* β *in* $\{4,5,b\}$*, the matroid* M/β *has an* N*-minor.*

Now $N \preceq M\backslash 1,8/3,5$. But $M\backslash 1,8/3,5 = M/3\backslash 1,8/5 \cong M/3\backslash 2,e/5 \cong M\backslash 2,e/4,5$. Thus each of $M\backslash 2$, $M/4$, and $M/5$ has an N-minor. Moreover, $M\backslash 1,8/3,5 \cong M\backslash 1,8/7,5 = M/7\backslash 1,8/5 \cong M/7\backslash a,6/5$. Hence both $M\backslash a$ and $M\backslash 6$ have an N-minor. Finally, observe that $M^*/8$ has $(2,3,1,7,a)$ as a 5-cofan with b in the guts. Then $M^*/8\backslash b$ has an N^*-minor, so M/b has an N-minor.

Evidently, $M\backslash\gamma$ is not $(4,4,S)$-connected for all γ in $\{1,3,7,8\}$. Thus one of the triangles $\{2,1,3\},\{6,8,7\}$, and $\{a,7,1\}$ is the central triangle of a quasi rotor otherwise, by [1, Theorem 5.1], $M\backslash\alpha$ is $(4,4,S)$-connected for all α in $\{a,2,6\}$. By Lemma 5.6, each of $2,6$, and e is in a unique triangle of M. Suppose a is in a triangle of M other than $\{1,7,a\}$. By orthogonality and Lemma 2.8, this triangle is $\{a,b,c\}$ for some element c. Then $M\backslash 2$ has $(c,b,a,1,7)$ as a 5-fan, so $N \preceq M\backslash 2\backslash 7$, a contradiction. Thus $\{1,7,a\}$ is the unique triangle of M containing a.

Suppose first that $\{2,1,3\}$ is the central triangle of a quasi rotor. If the central element of this quasi rotor is 3, then, by 5.8.2, one of the 4-cocircuits of the quasi rotor is forced to be $\{2,3,4,e\}$, so M has a triangle containing 4, a contradiction to Lemma 2.8. Thus we may assume that the central element of the quasi rotor is 1. Then $\{1,2,a,b\}$ is one of the cocircuits of the quasi rotor and M has a triangle containing $\{a,b\}$, a contradiction.

Next suppose that $\{6,8,7\}$ is the central triangle of a quasi rotor. Then the central element of this quasi rotor is 8 and not 7 otherwise, by 5.8.2, M has a 4-cocircuit containing $\{a,6,7\}$, a contradiction to 5.8.1. Thus one of the cocircuits of the quasi rotor is forced to be $\{5,6,8,e\}$, so M has a triangle containing 5, a contradiction to Lemma 2.8. Hence $\{6,8,7\}$ is not the central triangle of a quasi rotor.

Finally, suppose $\{a,7,1\}$ is the central triangle of a quasi rotor. Then the central element of this quasi rotor is not 7 otherwise M has a 4-cocircuit containing $\{a,6,7\}$, a contradiction. If the central element of the quasi rotor is 1, then one of the 4-cocircuits of the quasi rotor is $\{1,2,a,b\}$, so M has a triangle containing $\{2,b\}$, a contradiction to Lemma 5.5. We conclude that 5.8.4 holds.

The next assertion is obtained by applying Lemma 2.7 with the element 4 here taking the role of the element 8 in that lemma.

5.8.5. *One of the following occurs.*

(i) *M has a triangle containing $\{2,4\}$; or*
(ii) *$M/4$ is internally 4-connected; or*
(iii) *M has a circuit $\{4,e,5,x_1\}$ and a triad $\{5,x_1,x_2\}$ where $\{x_1,x_2\} \cap \{1,2,3,4,5,6,7,8,e\} = \emptyset$; or*
(iv) *M has a circuit $\{2,4,x_2,x_3\}$ and a triad $\{x_1,x_2,x_3\}$ where $x_1,x_2,x_3,1,2,3,4,5,6,7,8,e$ are distinct except that, possibly, $x_1 = 5$.*

By Lemma 2.8, (i) does not hold. Moreover, as $N \preceq M/4$, (ii) does not hold. Thus (iii) or (iv) holds. Applying Lemma 2.7 again this time relative to the element 5 and then combining this with the information above obtained by considering what happens relative to 4, we find that

5.8.6. *M contains one of the structures shown in Figure 6.*

To see this, first note that if 5.8.5(iii) holds relative to both 4 and 5, then (I) holds. Next we show the following.

5.8.7. *If 5.8.5(iv) holds relative to 4, then M has a circuit $\{2,4,b,x_3\}$ and a cocircuit $\{x_1,b,x_3\}$.*

Certainly M has a circuit $\{2,4,x_2,x_3\}$ and a cocircuit $\{x_1,x_2,x_3\}$. By orthogonality with the cocircuit $\{1,2,a,b\}$, we see that $b \in \{x_2,x_3\}$ so, by symmetry, we may assume that $b = x_2$. Thus 5.8.7 holds.

Returning to the proof of 5.8.6, we note that it follows immediately from 5.8.7 that if 5.8.5(iv) holds relative to both 4 and 5, then (II) holds.

Now suppose that 5.8.5(iii) holds relative to 4, and 5.8.5(iv) holds relative to 5. Then M has circuits $\{4,5,e,x_1\}$ and $\{5,6,y_2,y_3\}$ and has cocircuits $\{5,x_1,x_2\}$ and $\{y_1,y_2,y_3\}$. By orthogonality, $\{x_1,x_2\}$ meets $\{y_2,y_3\}$. Suppose $x_1 \in \{y_2,y_3\}$. Then orthogonality between $\{4,5,e,x_1\}$ and $\{y_1,y_2,y_3\}$ implies that $4 \in \{y_1,y_2,y_3\}$. But $4 \notin \{y_2,y_3\}$ otherwise $\{4,5,e,x_1\} = \{5,6,y_2,y_3\}$; a contradiction. Thus $4 = y_1$, and M has $\{4,x_1\}$ contained in a triad, so (I) holds. We may now assume that $x_2 \in \{y_2,y_3\}$. Then, by symmetry, (III) holds.

Finally, suppose that 5.8.5(iii) holds relative to 5 while 5.8.5(iv) holds relative to 4. Then, by 5.8.7, M has circuits $\{4,5,e,x_1\}$ and $\{2,4,b,y_3\}$ and cocircuits $\{4,x_1,x_2\}$ and $\{y_1,b,y_3\}$. By orthogonality, $\{x_1,x_2\}$ meets $\{b,y_3\}$. If $x_2 = y_3$, then (IV) holds. Next suppose that $x_1 = y_3$. Then, by using symmetric difference, we see that $\{2,5,e,b\}$ is a circuit. Then $\lambda(\{1,2,3,4,5,6,7,8,e,a,b,x_1\}) \le 2$; a contradiction. The same contradiction occurs if $x_1 = b$, so we may assume that $x_2 = b$. Thus M has $\{4,b,x_1\}$ as a cocircuit.

We now apply Lemma 2.7 relative to the element b. By 5.8.4, $N \preceq M/b$ and $M\backslash 2$ is $(4,4,S)$-connected, so neither (i) nor (ii) occurs. Suppose that (iii) occurs. Then M has a triad containing $\{4,y_3\}$. If this triad contains x_1, then (IV) holds. Thus, by orthogonality, $\{4,5,y_3\}$ is a triad of M. Then $\lambda(\{1,2,3,4,5,6,7,8,e,a,b,x_1,y_3\}) \le 1$; a contradiction. We conclude that (iv) of Lemma 2.7 rather than (iii) must hold relative to b. Thus M has a circuit $\{a,b,z_2,z_3\}$ and a triad $\{z_1,z_2,z_3\}$. Since M has $\{y_1,b,y_3\}$ as a cocircuit, by orthogonality with the circuit $\{a,b,z_2,z_3\}$, it follows, by symmetry, that $z_3 \in \{y_1,y_3\}$. Suppose $z_3 = y_1$. Then, by orthogonality between the circuit $\{a,b,z_2,z_3\}$ and the cocircuit $\{4,b,x_1\}$, we deduce that

$\{4, x_1\}$ meets $\{z_2, z_3\}$. Then $\lambda(\{1,2,3,4,5,6,7,8,e,a,b,x_1,y_3,z_2,z_3\}) \leq 1$. But $|\{1,2,3,4,5,6,7,8,e,a,b,x_1,y_3,z_2,z_3\}| \leq 14$, so we have a contradiction. We may now assume that $z_3 = y_3$. Then, as M has $\{2,4,b,y_3\}$ and $\{a,b,z_2,y_3\}$ as circuits, it has $\{2,4,a,z_2\}$ as a circuit. Then, by orthogonality with the cocircuit $\{4,x_1,b\}$, we see that $z_2 = x_1$. Then $\lambda(\{1,2,3,4,5,6,7,8,a,e,x_1\}) \leq 2$; a contradiction. We conclude that 5.8.6 holds.

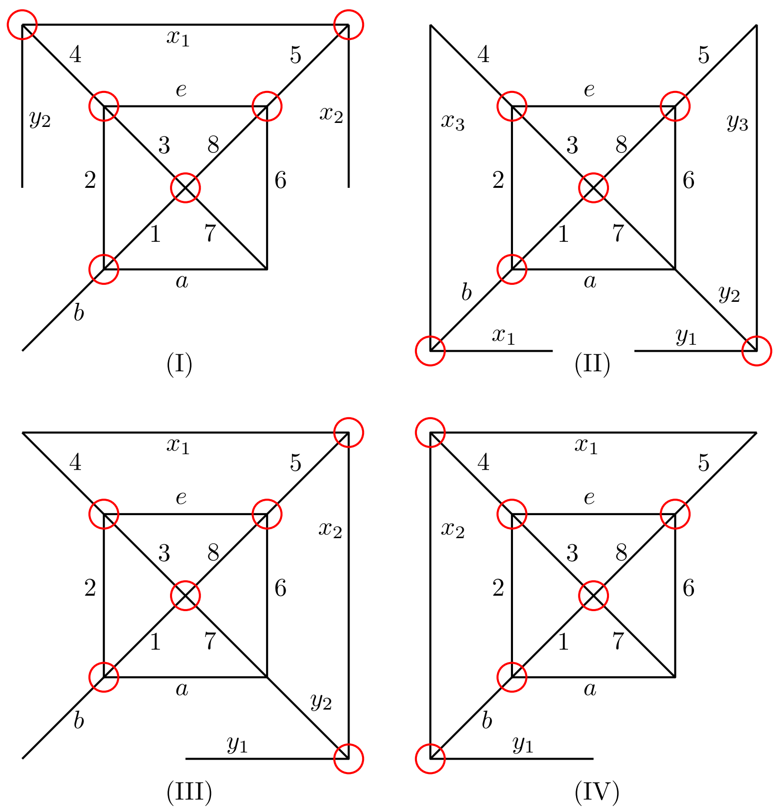

Figure 6

5.8.8. *M does not contain the configuration shown in Figure 7.*

Suppose that M does contain the configuration shown in Figure 7. By Lemma 5.7, the elements $1,2,3,4,5,6,7,8,e,a$, and b are distinct. Let Z be the set of thirteen elements shown. Since $\lambda(Z) \leq 2$, all thirteen elements must be distinct. Moreover, $r(Z) = 7$ otherwise $\lambda(Z) \leq 1$.

Now $M\backslash 1,8/3,5$ has an N-minor and has $(e,x_1,4,x_2)$ and $(7,a,2,b)$ as 4-fans. If $N \preceq M\backslash 1,8/3,5\backslash e$, then $N \preceq M\backslash 8,e/6$, so $N \preceq M\backslash 7$; a contradiction. Thus $N \preceq M\backslash 1,8/3,5/x_2$. As the last matroid has $(7,a,2,b,4)$ as a 5-fan, we deduce that $N \preceq M\backslash 7$; a contradiction. Thus 5.8.8 holds.

Now M does not contain the structure in (IV) in Figure 6 otherwise M contains the structure in Figure 7; a contradiction. Next we show the following.

5.8.9. *If M contains (I) or (III) from Figure 6, then (iv) of Lemma 2.7 holds relative to the element b.*

Suppose that M contains (I) or (III) from Figure 6. If (iv) of Lemma 2.7 does not hold relative to the element b, then (iii) of that lemma holds. Then M has a 4-circuit $\{b, 2, 4, z_1\}$ and a triad $\{4, z_1, z_2\}$. Hence M contains the configuration shown in Figure 8. By 5.8.8, M does not have $\{4, z_1, x_1\}$ as a cocircuit. By orthogonality, $\{z_1, z_2\}$ must meet $\{x_1, 5\}$. We know that $x_1 \neq z_2$. Suppose $z_1 \in \{x_1, 5\}$. Then $\lambda(\{1, 2, 3, 4, 5, 6, 7, 8, e, a, x_1, b\}) \leq 2$; a contradiction. Thus $5 = z_2$, so M has $\{4, 5, z_1\}$ as a cocircuit.

We will now need to distinguish between (I) and (III). Suppose first that (I) holds. Then $\{4, x_1, y_2\}$ is a cocircuit and $\{b, 2, 4, z_1\}$ is a circuit. As $\{4, 5, z_1\}$ is a cocircuit, so is $\{5, z_1, x_1, y_2\}$. By orthogonality, $b \in \{5, x_1, y_2\}$. If $b \in \{5, x_1\}$, then $\lambda(\{1, 2, 3, 4, 5, 6, 7, 8, a, b, e, z_1\}) \leq 2$; a contradiction. Thus $b = y_2$. Then $\lambda(\{1, 2, 3, 4, 5, 6, 7, 8, a, b, e, z_1, x_1\}) \leq 1$; a contradiction. We conclude that (I) does not hold. Next suppose that (III) holds. Then $\{4, 5, z_1\}$, $\{5, x_1, x_2\}$, and $\{y_1, y_2, x_2\}$

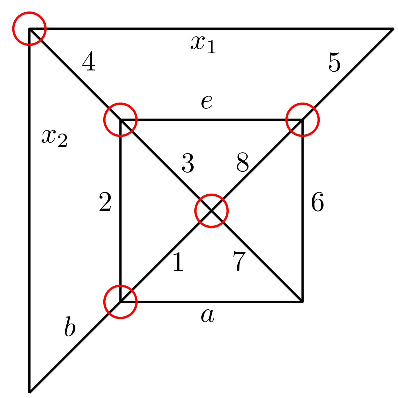

FIGURE 7

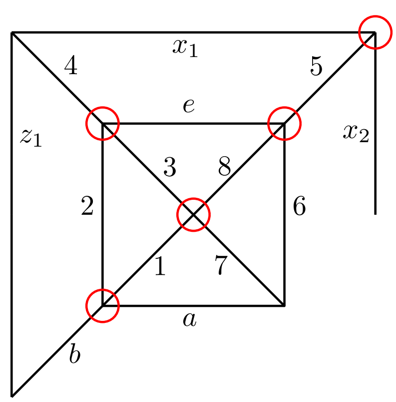

FIGURE 8

are cocircuits of M, and $\{b, 2, 4, z_1\}$ and $\{5, 6, x_2, y_2\}$ are circuits. By orthogonality between $\{5, 6, x_2, y_2\}$ and $\{4, 5, z_1\}$, we deduce that $\{4, z_1\}$ meets $\{x_2, y_2\}$. If $4 \in \{x_2, y_2\}$, then M has a 4-circuit containing $\{4, 5, 6\}$. By orthogonality, the fourth element of this circuit is in $\{2, 3, e\}$. Thus $\lambda(\{1, 2, 3, 4, 5, 6, 7, 8, e, a\}) \leq 2$; a contradiction. Hence $z_1 \in \{x_2, y_2\}$. But $z_1 \neq x_2$ otherwise the triads $\{5, x_1, z_1\}$ and $\{4, 5, z_1\}$ imply the contradiction that $4 = x_1$. We deduce that $z_1 = y_2$.

By orthogonality between $\{b, 2, 4, z_1\}$ and $\{z_1, x_2, y_1\}$, we deduce that $\{b, 4\}$ meets $\{x_2, y_1\}$. If $4 \in \{x_2, y_1\}$, then $\{x_2, y_1, y_2\} = \{4, 5, z_1\}$, so $5 \in \{x_2, y_1\}$; a contradiction. We conclude that $b \in \{x_2, y_1\}$. If $b = x_2$, then the circuit $\{5, 6, z_1, b\}$ and the cocircuit $\{b, 1, 2, a\}$ give a contradiction to orthogonality. Thus $b = y_1$. The symmetric difference of the cocircuits $\{4, 5, z_1\}, \{5, x_1, x_2\}$, and $\{z_1, x_2, b\}$ is $\{4, x_1, b\}$. It follows that $\lambda(\{1, 2, 3, 4, 5, 6, 7, 8, e, a, x_1, b, z_1\}) \leq 1$; a contradiction. We conclude that 5.8.9 holds.

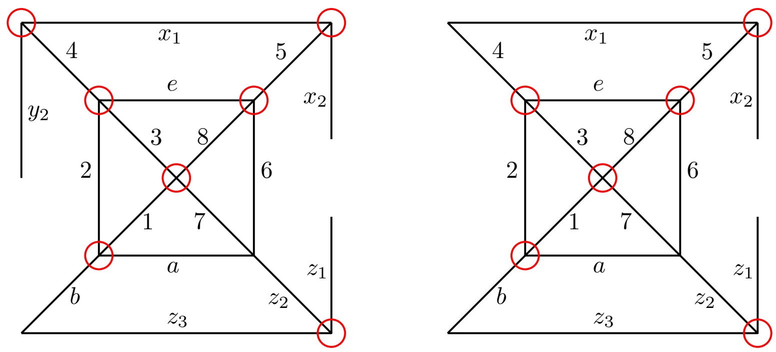

FIGURE 9

By 5.8.9, if (I) or (III) holds, then M contains one of the structures shown in Figure 9 where we have omitted the elements y_1 and y_2 from (III). Comparing these two structures with (II), we observe that, since $M \backslash 2$ is $(4, 4, S)$-connected having an N-minor and having two disjoint 4-fans, the two structures in Figure 9 contain structures symmetric to (II). But there are a number of assumptions that accompany (II). To ensure that we can, indeed, provide a simultaneous treatment of (II) and the two structures in Figure 9, we shall assume that M contains the structure in Figure 10 where the following assumptions hold.

(i) $M \backslash \alpha$ is $(4, 4, S)$-connected having an N-minor for all α in $\{a', 2, e', 6'\}$.
(ii) The only triangles of M meeting $\{1', 2', 3', 4', 5', 6', 7', 8', e', a', b'\}$ are $\{1', 2, 3'\}, \{3', e', 8'\}, \{8', 6', 7'\}$, and $\{7', a', 1'\}$.
(iii) M has no 4-cocircuit containing $\{a', 6', 7'\}$.
(iv) $M \backslash 7'$ has no N-minor.
(v) $M / 3' \backslash 1', 8'$ has an N-minor.

5.8.10. $N \not\preceq M \backslash 3'$.

Assume that $N \preceq M \backslash 3'$. If $M \backslash 3'$ is $(4, 5, S, +)$-connected, then, as $M \backslash a'$ is $(4, 4, S)$-connected having an N-minor, M has a good augmented 4-wheel; a contradiction. We deduce that $M \backslash 3'$ is not $(4, 5, S, +)$-connected. Suppose that $M \backslash 3'$ has a non-sequential 3-separation (U, V). Then we may assume that

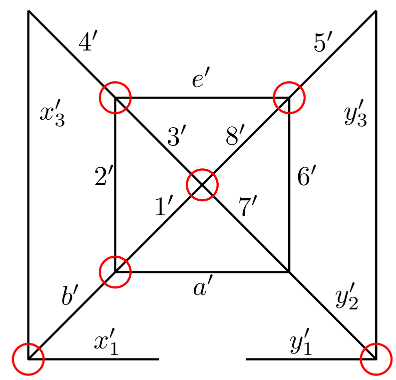

FIGURE 10

$\{1', a', 7', 6', 8'\} \subseteq U$. Since we may also assume that $\{e', 2', 4'\}$ is a subset of U or V, we must have that $\{e', 2', 4'\} \subseteq V$. Thus $5' \in V$ otherwise we can move e', then $2'$, and then $4'$ into U. Now we may assume that $\{y'_1, y'_2, y'_3\}$ is a subset of U or V. If it is contained in U, then we can move $5'$ into U; a contradiction. Thus $\{y'_1, y'_2, y'_3\} \subseteq V$. We can now move $6'$, then $8'$ into V enabling us to add $3'$ to V; a contradiction. We conclude that $M\backslash 3'$ is sequentially 4-connected.

Next suppose that $M\backslash 3'$ has $(u_1, u_2, u_3, u_4, u_5)$ as a 5-cofan. Then M has $\{3', u_1, u_2, u_3\}$ and $\{3', u_3, u_4, u_5\}$ as cocircuits. By orthogonality between these cocircuits and the circuits $\{3', e', 8'\}$ and $\{3', 2', 1'\}$, we deduce that each of $\{u_1, u_2, u_3\}$ and $\{u_3, u_4, u_5\}$ meets each of $\{e', 8'\}$ and $\{2', 1'\}$. Since $\{u_1, u_2, u_3, u_4, u_5\}$ does not contain $\{e', 8'\}$ or $\{2', 1'\}$, we deduce that $u_3 \in \{e', 8'\} \cap \{2', 1'\}$; a contradiction. Hence $M\backslash 3'$ has no 5-cofan.

Now let $(u_1, u_2, u_3, u_4, u_5)$ be a 5-fan of $M\backslash 3'$. Then M has $\{3', u_2, u_3, u_4\}$ as a cocircuit. Thus $\{e', 8'\}$ and $\{2', 1'\}$ both meet $\{u_2, u_3, u_4\}$. The limitations on the possible triangles containing any of $e', 2', 8'$, and $1'$ mean that $u_3 \notin \{e', 2', 1', 8'\}$. Moreover, $e' \notin \{u_2, u_4\}$ otherwise M has a good bowtie. We conclude that we may assume that $u_2 = 8'$. Then $\{u_1, u_2, u_3\} = \{6', 7', 8'\}$. If $u_3 = 6'$, then M has two triangles containing $6'$; a contradiction. Thus $u_3 = 7'$ so $u_4 = 1'$ and $(u_1, u_2, u_3, u_4, u_5) = (6', 8', 7', 1', a')$. We conclude that $(6', 8', 7', 1', a')$ is the unique 5-fan of $M\backslash 3'$. Thus $M\backslash 3'$ has no 5-fan with an element in the guts.

Since $M\backslash 3'$ is not $(4, 4, S)$-connected, it now follows that $M\backslash 3'$ has a 5-fan with an element in the coguts. It follows that M has a 4-cocircuit containing $\{6', 7', a'\}$. This contradiction completes the proof of 5.8.10.

Now suppose $z \in \{6', a'\}$. Then $M\backslash z$ is $(4, 4, S)$-connected having an N-minor. Moreover, $M\backslash z$ has a 4-fan $(3', s_2, s_3, s_4)$ where (z, s_4) is $(6', 5')$ or (a', b'). Thus, by Lemma 2.5, either $N \preceq M\backslash z\backslash 3'$; or $N \preceq M\backslash z/s_4$ and $M\backslash z/s_4$ is $(4, 4, S)$-connected. By 5.8.10, the former does not hold. Hence the latter does. Thus $M\backslash z/s_4$ has a 4-fan (v_1, v_2, v_3, v_4).

5.8.11. *M has $\{v_2, v_3, v_4\}$ as a cocircuit.*

We shall prove this when $(z, s_4) = (6', 5')$ noting that the argument focuses exclusively on the restriction of M to $\{1', 2', 3', 6', 7', 8', e', a', b', 5\}$. The symmetry in that restriction means that this argument also proves 5.8.11 when $(z, s_4) = (a', b')$.

Assume that 5.8.11 fails. Then M has $\{v_2, v_3, v_4, 6'\}$ as a cocircuit C^* of M. By orthogonality, $\{v_2, v_3, v_4\}$ meets $\{7', 8'\}$. Suppose $7' \in C^*$. Then, by orthogonality with the triangle $\{1', 7', a'\}$, we deduce that C^* contains $\{6', 7', a'\}$ or $\{6', 7', 1'\}$. The first possibility has been excluded; the second implies that $C^* \subseteq \{6', 7', 1', 3', 2'\}$, so $\lambda(\{1', 2', 3', 6', 7', 8'\}) \leq 2$; a contradiction. We conclude that $7' \notin C^*$. Hence $8' \in C^*$. Thus C^* contains $\{6', 8', e'\}$ or $\{6', 8', 3'\}$. The first case gives the contradiction that $5' \in C^*$; the second implies that $\lambda(\{1', 2', 3', 6', 7', 8'\}) \leq 2$; a contradiction. We conclude that 5.8.11 holds.

We will now take (v_1, v_2, v_3, v_4) to be a 4-fan in $M\backslash 6'/5'$ and (w_1, w_2, w_3, w_4) to be a 4-fan in $M\backslash a'/b'$. In the 4-fan (v_1, v_2, v_3, v_4), we may clearly interchange v_2 and v_3. In the next assertion, we assume that we have made this interchange if necessary.

5.8.12. *The matroid M has $\{5', e', 4', v_3\}$ as a circuit and $\{4', v_3, v_4\}$ as a cocircuit. Moreover, $v_4 \in \{b', x_3'\}$.*

By 5.8.11, $\{v_1, v_2, v_3, 5'\}$ is a circuit of M. None of $e', 8'$, or $6'$ is in a triad of M. Thus $\{e', 8', 6'\}$ avoids $\{v_2, v_3\}$. Hence $v_1 \in \{e', 8'\}$. Therefore $\{5', 8', v_2, v_3\}$ or $\{5', e', v_2, v_3\}$ is a circuit of M. In the former case, as $\{1', 3', 7', 8'\}$ is a cocircuit of M, it follows that $\{1', 3', 7'\}$ meets $\{v_2, v_3\}$; a contradiction. Thus $\{5', e', v_2, v_3\}$ is a circuit of M. The cocircuit $\{e', 2', 3', 4'\}$ implies that we may assume that $4' = v_2$. Hence M has $\{5', e', 4', v_i\}$ as a circuit and $\{4', v_i, v_4\}$ as a cocircuit. By orthogonality with the circuit $\{2', 4', b', x_3'\}$, we deduce that $\{v_3, v_4\}$ meets $\{b', x_3'\}$. If $v_3 \in \{b', x_3'\}$, then $\lambda(\{1', 2', 3', 4', 5', 6', 7', 8', a', b', e', x_3'\}) \leq 2$; a contradiction. Thus $v_4 \in \{b', x_3'\}$. Hence 5.8.12 holds.

By 5.8.11,

5.8.13. *M has $\{w_2, w_3, w_4\}$ as a cocircuit.*

We now know that M has $\{1', 2', a', b'\}$ and $\{w_2, w_3, w_4\}$ as cocircuits and has $\{w_1, w_2, w_3, b'\}$ as a circuit. By orthogonality and the fact that $a' \notin \{w_1, w_2, w_3, w_4\}$, we see that

5.8.14. *$w_1 \in \{1', 2'\}$.*

Next we show that

5.8.15. *M has $\{4', v_3, x_3'\}$ as a cocircuit.*

Assume that this fails. Suppose $w_1 = 1'$. Then, by orthogonality between $\{1', 3', 7', 8'\}$ and $\{w_1, w_2, w_3, b'\}$, we see that $\{w_2, w_3\}$ meets $\{3', 7', 8'\}$; a contradiction. We deduce that $w_1 = 2'$. Then, by orthogonality, $\{w_2, w_3\}$ meets $\{3', 4', e'\}$. Hence $4' \in \{w_2, w_3\}$. Thus $\{w_1, w_2, w_3, b'\}$ contains $\{2', 4, b'\}$ and so equals $\{2', 4', b', x_3'\}$. Thus $\{w_2, w_3\} = \{4', x_3'\}$. Hence, by orthogonality, $\{w_2, w_3, w_4\}$ is $\{4', x_3', 5'\}$ or $\{4', x_3', v_3\}$. But the second possibility has been excluded. By 5.8.12, $\{4', v_3, b'\}$ is a cocircuit of M. Thus $\lambda(\{1', 2', 3', 4', 5', 6', 7', 8', a', b', e', v_3, x_3'\}) \leq 1$; a contradiction. Thus 5.8.15 holds.

Let $Z = \{1', 2', 3', 4', 5', 6', 7', 8', a', b', e', v_3, x_3'\}$. Then $\lambda(Z) \leq 2$, so $|Z| = 13$. Moreover, $r(Z) = 7$, otherwise $\lambda(Z) \leq 1$.

Now $M/3'\backslash 1', 8'$ has an N-minor. Since $M/3'\backslash 1', 8'$ has $(7', e', 6', 5')$ as a 4-fan but $M\backslash 7'$ has no N-minor, it follows that $M/3'\backslash 1', 8'/5'$ has an N-minor. Now $M/3'\backslash 1', 8'/5'$ has $(7', 2', a', b')$ as a 4-fan. Thus $M/3'\backslash 1', 8'/5'/b'$ has an N-minor and has $(2', x_3', 4', v_3, e')$ as a 5-fan. Thus either $M/3'\backslash 1', 8'/5'/b'\backslash 2', e'$ or

$M/3'\backslash 1', 8'/5'/b'/x'_3\backslash 2'$ has an N-minor. In each case, $M\backslash 1', 2'$ has an N-minor. Since $M\backslash 1', 2'$ has $\{a', b'\}$ as a cocircuit, we deduce that $N \preceq M/a'$. But M/a' has a 2-circuit containing $7'$. Thus $M\backslash 7'$ has an N-minor; a contradiction. We conclude that the lemma holds. □

Lemma 5.9. *If $N \preceq M\backslash 1$, then M has a good bowtie, a pretty good bowtie, or a good augmented 4-wheel, or M has an internally 4-connected minor M' with $1 \leq |E(M) - E(M')| \leq 2$ such that M' has an N-minor.*

Proof. We may assume that $M\backslash 1$ is not internally 4-connected and M does not have a good bowtie. Then, by Lemma 5.3, $M\backslash 1$ is $(4, 4, S)$-connected, or M has $\{1, 3, 7, 8\}$ or $\{1, 3, 6, 7, e\}$ as a cocircuit. If M has $\{1, 3, 6, 7, e\}$ as a cocircuit, then, by Lemma 5.4, $M\backslash 3$ is internally 4-connected having an N-minor and the lemma holds. If M has $\{1, 3, 7, 8\}$ as a cocircuit, then, by Lemma 5.7, $M\backslash 8$ is internally 4-connected having an N-minor, or M has a pretty good bowtie, or M has the configuration in Figure 4 as a restriction. In the last case, by Lemma 5.8, M contains a good bowtie or a good augmented 4-wheel, or M has an internally 4-connected minor M' with $1 \leq |E(M) - E(M')| \leq 2$ such that M' has an N-minor.

It remains to consider the case when $M\backslash 1$ is $(4, 4, S)$-connected having a 4-fan (a, b, c, d). Then $\{1, b, c, d\}$ is a cocircuit of M. By orthogonality with the triangle $\{1, 2, 3\}$, either 2 or 3 is in $\{b, c, d\}$.

5.9.1. $3 \notin \{b, c, d\}$.

Suppose $3 \in \{b, c, d\}$. Then 8 or e is in $\{b, c, d\}$ by orthogonality. Suppose $8 \in \{b, c, d\}$. Then $\{1, b, c, d\}$ contains $\{1, 3, 8\}$. By orthogonality, it must also contain 7 or 6. In the first case, $(\{1, 3, 2\}, \{7, 8, 6\}, \{1, 3, 7, 8\})$ is a good bowtie. Thus $\{1, b, c, d\} = \{1, 3, 8, 6\}$. Hence, by taking symmetric differences, we see that $\{1, 3, e, 5\}$ is a cocircuit of M. Thus $M\backslash e$ has $\{1, 3, 5\}$ as a cocircuit and so has $(4, 2, 3, 1, 5)$ as a 5-cofan; a contradiction.

We may now assume that $\{3, e\} \subseteq \{b, c, d\}$. Then $\{1, b, c, d\}$ is $\{1, 3, e, \alpha\}$ for some element α. Thus $M\backslash e$ has $\{1, 2, 3, \alpha\}$ as a 3-separating set. As $\{1, 2, 3, 4\}$ is also 3-separating and $M\backslash e$ is $(4, 4, S)$-connected, we deduce that $\alpha = 4$. But then $\{1, 3, e, 4\}$ and $\{2, 3, e, 4\}$ are cocircuits; a contradiction. Hence 5.9.1 holds.

It follows by 5.9.1 that $2 \in \{b, c, d\}$. As M does not have a good bowtie, $2 \neq d$, so $2 \in \{b, c\}$. Then, by Lemma 2.8, $\{2, e\} \subseteq \{a, b, c\}$. Thus, by orthogonality, $\{a, b, c\}$ is $\{2, e, 5\}$ or $\{2, e, 6\}$. The first possibility gives the contradiction that $(\{2, e, 5\}, \{6, 7, 8\}, \{e, 5, 6, 8\})$ is a good bowtie. The second contradicts the fact that $\{2, e, 6, 1, 7\}$ is a circuit. Thus Lemma 5.9 holds. □

The next result follows without difficulty by combining the last lemma with Lemmas 2.5 and 5.2 and by using symmetry.

Corollary 5.10. *Neither $M\backslash 1$ nor $M\backslash 7$ has an N-minor. But $M\backslash e/4, 5$ has an N-minor and both $M\backslash e/4$ and $M\backslash e/5$ are $(4, 4, S)$-connected.*

Lemma 5.11. *Either $M/5$ is $(4, 4, S)$-connected, or M has a triad $\{4, \gamma, \delta\}$ and has circuits $\{4, e, 6, \delta\}$, $\{4, e, 5, \gamma\}$, and $\{5, 6, \gamma, \delta\}$.*

Proof. Let (U, V) be a $(4, 4, S)$-violator of $M/5$. Then neither U nor V contains $\{e, 8, 6\}$. Without loss of generality, we may assume that U contains two or three elements of $\{e, 3, 8\}$. If $6 \in U$, then $(U \cup \{e, 3, 8\} \cup 5, V - \{e, 3, 8\})$ is a $(4, 3)$-violator of M; a contradiction. Thus we may assume that $6 \in V$.

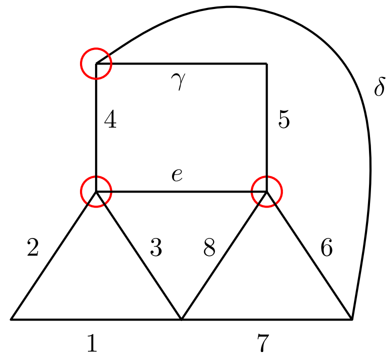

FIGURE 11. M has $\{4, e, 6, \delta\}$, $\{4, e, 5, \gamma\}$, and $\{5, 6, \gamma, \delta\}$ as circuits.

5.11.1. $\{e, 3, 8\} \not\subseteq U$.

Assume that $\{e, 3, 8\} \subseteq U$. If $7 \in U$, then $(U \cup 6 \cup 5, V - 6)$ is a $(4, 3)$-violator of M; a contradiction. Thus $7 \in V$. As $M/5\backslash e$ is $(4, 4, S)$-connected, $(U - e, V)$ is not a $(4, 4, S)$-violator of this matroid. Thus $U - e$ is a triangle, a triad, or a 4-fan of $M/5\backslash e$. Hence $|U| \leq 5$. As $8 \in \mathrm{cl}_{M/5}(V)$, it follows that $|U| = 5$ otherwise (U, V) is not a $(4, 4, S)$-violator of $M/5$.

We now have that, in $M/5$, the set U is a quad $\{3, e, a, b\}$ with the element 8 in its closure, or U is a 5-fan. But no element of $\{e, 8, 3\}$ is in a triad of M, so U is not a 5-fan. Thus $\{3, e, a, b\}$ is a cocircuit of M. By orthogonality with the circuit $\{1, 2, 3\}$, we deduce that $\{a, b\}$ meets $\{1, 2\}$ in a single element. Then $(U \cup \{1, 2\}, V - \{1, 2\})$ is a $(4, 4, S)$-violator of $M/5$ with $|U \cup \{1, 2\}| = 6$ and $\{e, 3, 8\} \subseteq U \cup \{1, 2\}$. This contradiction to the last paragraph completes the proof of 5.11.1.

By 5.11.1, $|\{e, 3, 8\} \cap U| = 2$. Then $(U \cup \{e, 3, 8\}, V - \{e, 3, 8\})$ is a 3-separation of $M/5$. By 5.11.1, this 3-separation is not a $(4, 4, S)$-violator of $M/5$. If $e \notin V$, then $(U - e, V)$ is a $(4, 4, S)$-violator of $M/5\backslash e$; a contradiction. Thus $e \in V$ and $\{3, 8\} \subseteq U$ so V is a 5-fan $(e, \beta, \gamma, \delta, \varepsilon)$ in $M/5$ that contains 6. As 6 is not in a triad of M, we deduce that $\varepsilon = 6$. Then $\{e, \beta, \gamma, 5\}$ and $\{\gamma, \delta, 6, 5\}$ are circuits of M. Hence so is $\{e, \beta, \delta, 6\}$. Now $\{\beta, \gamma, \delta\}$ is a triad of M, so $\{\beta, \gamma, \delta\} \cap \{1, 2, 3, e, 6, 7, 8\} = \emptyset$. Also $5 \notin \{\beta, \gamma, \delta\}$. By orthogonality between the cocircuit $\{2, 3, 4, e\}$ and the cocircuits $\{e, \beta, \gamma, 5\}$ and $\{e, \beta, \delta, 6\}$, we deduce that $4 \in \{\beta, \gamma\} \cap \{\beta, \delta\}$, so $\beta = 4$. Thus $\{e, 4, \gamma, 5\}$ and $\{e, 4, \delta, 6\}$ are circuits of M, and the lemma holds. □

The structure that arises in the last lemma when $M/5$ is not $(4, 4, S)$-connected is depicted in Figure 11. The next result is an immediate consequence of Lemma 5.11.

Corollary 5.12. *If neither $M/4$ nor $M/5$ is $(4, 4, S)$-connected, then M contains the configuration shown in Figure 12 where $\{4, \gamma, \delta\}$ and $\{5, \gamma, \varepsilon\}$ are triads of M and all of $\{4, e, 6, \delta\}, \{5, 6, \gamma, \delta\}, \{5, e, 2, \varepsilon\}$, and $\{2, 4, \gamma, \varepsilon\}$ are circuits of M.*

Lemma 5.13. *The configuration in Figure 12 does not arise in M.*

Proof. Assume that the configuration shown in Figure 12 does arise. Then, letting $Z = \{1, 2, 3, e, 4, 5, 6, 7, 8, \gamma, \delta, \varepsilon\}$, we have $r(Z) \leq 6$ and $|Z| - r^*(Z) \geq 4$, so $\lambda(Z) \leq 2$. This is a contradiction as $|E(M)| \geq 16$.

□

By combining Corollary 5.12 and Lemma 5.13, we immediately obtain the following.

Corollary 5.14. *At least one of $M/4$ and $M/5$ is $(4,4,S)$-connected.*

Lemma 5.15. *Suppose that $M/4$ is $(4,4,S)$-connected but not internally 4-connected, and $M/5$ is not $(4,4,S)$-connected. Then M contains the configuration in Figure 13 where $\{e,2,5,\varepsilon\}$ is not a circuit of M.*

Proof. By Lemma 5.11, since $M/5$ is not $(4,4,S)$-connected, M contains the configuration shown in Figure 11. Clearly $M/4$ has a 4-fan (a,b,c,d). Thus $\{4,a,b,c\}$ is a circuit of M and $\{b,c,d\}$ is a cocircuit of M. Thus, by orthogonality, $\{a,b,c\}$ meets $\{2,3,e\}$. But $\{2,3,e\}$ avoids the triad $\{b,c,d\}$ otherwise M has a 4-fan. Thus $a \in \{2,3,e\}$. Also $\{4,\gamma,\delta\}$ is a cocircuit of M, so $\{b,c\}$ meets $\{\gamma,\delta\}$. Note that $\{b,c,d\}$ avoids $\{1,2,3,4,e,6,7,8\}$.

Suppose first that $\delta \in \{b,c\}$. Then, without loss of generality, $\delta = b$. Thus $\{\delta,c,d\}$ is a triad. As $\{\delta,4,e,6\}$ is a circuit, it follows by orthogonality that $\{c,d\}$ meets $\{4,e,6\}$; a contradiction. We deduce that $\delta \notin \{b,c\}$. Thus $\gamma \in \{b,c\}$, so we may assume that $\gamma = b$. Then $\{\gamma,c,d\}$ is a triad. As $\{4,e,5,\gamma\}$ ia a circuit, it follows that $5 \in \{c,d\}$, and $\{\gamma,5\}$ is contained in a triad of M.

Suppose $5 = c$. Then $\{b,c,4\} = \{\gamma,5,4\}$. As $\{a,b,c,4\}$ and $\{e,\gamma,5,4\}$ are circuits of M, it follows that $e = a$. Thus M contains the configuration in Figure 13. To see that $\{e,2,5,\varepsilon\}$ is not a circuit of M, we observe that otherwise M contains the configuration in Figure 12, which contradicts Lemma 5.13.

We may now assume that $5 = d$. Then $(a,b,c,d) = (a,\gamma,c,5)$ and $a \in \{2,3,e\}$. Consider the circuit $\{a,\gamma,c,4\}$. As $c \neq 5$, it follows that $a \neq e$. If $a = 3$, then the symmetric difference of the circuits $\{3,4,c,\gamma\}$ and $\{3,4,5,\gamma,8\}$ is $\{c,5,8\}$, which must be a triangle of M. Since 5 is in the triad $\{b,c,d\}$, we have a contradiction. We conclude that $a \neq 3$, so $a = 2$. Thus $\{2,\gamma,c,4\}$ is a circuit of M and $\{\gamma,c,5\}$ is a cocircuit. Thus M contains the structure shown in Figure 12; a contradiction. □

Lemma 5.16. *Assume that M contains the configuration in Figure 13. Then the only triads of M meeting $\{4,5,e,\gamma\}$ are $\{4,\delta,\gamma\}$ and $\{5,\gamma,\varepsilon\}$. Moreover, either*

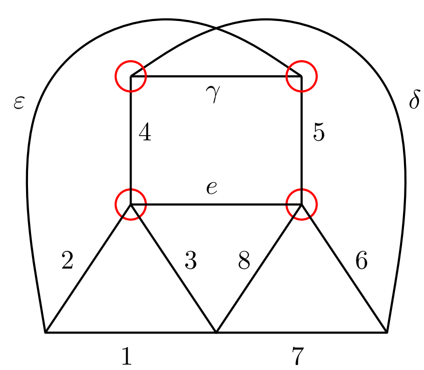

FIGURE 12

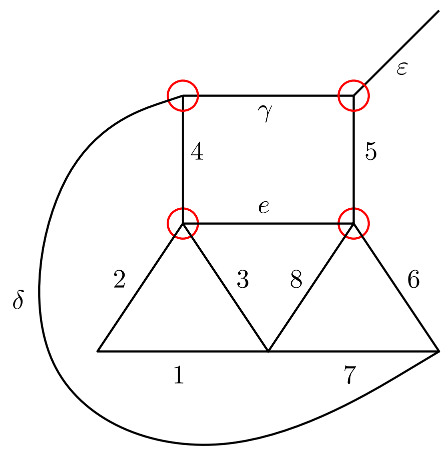

FIGURE 13. M does not have $\{e, 2, 5, \varepsilon\}$ as a circuit.

M/ε is internally 4-connected, or M/ε is $(4, 4, S)$-connected having $(x, \delta, \gamma, 4)$ as its unique 4-fan for some new element x that is not in $\{1, 2, 3, 4, 5, 6, 7, 8, e, \delta, \gamma, \varepsilon\}$.

Proof. First, observe that, by orthogonality, a triad of M meeting $\{4, 5, e, \gamma\}$ must either be $\{4, \delta, \gamma\}$ or $\{5, \gamma, \varepsilon\}$, or must contain $\{4, 5\}$. But, by orthogonality with the circuit $\{4, e, 6, \delta\}$, a triad containing $\{4, 5\}$ must be $\{4, 5, \delta\}$ and so cannot exist as $\{4, \delta, \gamma\}$ is a triad. We conclude that the first assertion of the lemma holds.

Next we show the following.

5.16.1. *M/ε is sequentially 4-connected.*

Let (U, V) be a non-sequential 3-separation of M/ε. Then we may assume that $\{4, \gamma, \delta\} \subseteq U$. It follows that $5 \in V$, so $e \in V$ otherwise we can move 5 into U and then add ε to U to get a non-sequential 3-separation of M. The circuit $\{\gamma, \delta, 5, 6\}$ implies that $6 \in V$. Thus we may assume that $8, 7$, and 3 are in V as $\{8, 7, 3\} \subseteq \mathrm{fcl}_{M/\varepsilon}(V)$. If 1 or 2 is in V, then $\mathrm{fcl}_{M/\varepsilon}(V)$ contains $\{1, 2, 4, \gamma, \delta\}$ so we may assume that these elements are in V; a contradiction. Thus $\{1, 2\} \subseteq U$. We can now move 3, then e, then 8, then 6, and then 5 into U; a contradiction. Hence 5.16.1 holds.

5.16.2. *M has no 4-circuit containing $\{4, \gamma, \varepsilon\}$.*

Suppose M has $\{4, \gamma, \varepsilon, z\}$ as a circuit. Then, by orthogonality with the cocircuit $\{2, 3, 4, e\}$, it follows that $z \in \{2, 3, e\}$. If $z = 2$, then the configuration in Figure 12 arises; a contradiction to Lemma 5.13. If $z = 3$, then, as $\{4, \gamma, 3, 8, 5\}$ is also a circuit, $\{\varepsilon, 8, 5\}$ is a triangle, so M has a 4-fan; a contradiction. Finally, $z \neq e$ as $\{4, e, \gamma, 5\}$ is also a circuit. Thus 5.16.2 holds.

5.16.3. *Let (t_1, t_2, t_3, t_4) be a 4-fan of M/ε. Then $\{t_2, t_3\} = \{\delta, \gamma\}$, and $(t_1, t_4) = (x, 4)$ for some element x that is not in $\{1, 2, 3, 4, 5, 6, 7, 8, e, \delta, \gamma, \varepsilon\}$.*

Clearly $\{t_1, t_2, t_3, \varepsilon\}$ is a circuit of M and $\{t_2, t_3, t_4\}$ is a triad of M. By orthogonality, $\{t_1, t_2, t_3\}$ meets $\{5, \gamma\}$. Suppose first that $5 \in \{t_2, t_3\}$. Then, by the first asertion of the lemma, $\{t_2, t_3, t_4\} = \{5, \gamma, \varepsilon\}$; a contradiction. Next suppose that $\gamma \in \{t_2, t_3\}$. Then $\{t_2, t_3, t_4\} = \{4, \delta, \gamma\}$. If $t_4 = \delta$, then $\{t_1, t_2, t_3, \varepsilon\}$ contains

$\{4, \gamma, \varepsilon\}$; a contradiction to 5.16.2. If $t_4 = 4$, then the required result holds by letting $t_1 = x$ where we note that if x is in the set Z of twelve elements shown in Figure 13, then $\lambda(Z) \leq 2$ so we get a contradiction since $|E(M)| \geq 16$. We may now assume that $\{5, \gamma\} \cap \{t_2, t_3\}$.

Now suppose that $5 = t_1$. Then $\{\varepsilon, 5, t_2, t_3\}$ is a circuit and $\{5, 6, e, 8\}$ is a cocircuit so $\{6, e, 8\}$ meets $\{t_2, t_3\}$, so M has a 4-fan; a contradiction.

Finally, suppose that $\gamma = t_1$. Then $\{\varepsilon, \gamma, t_2, t_3\}$ is a circuit and $\{4, \delta, \gamma\}$ is a cocircuit so, by orthogonality, 4 or δ meets $\{t_2, t_3\}$. The first possibility was eliminated by 5.16.2. The second gives a contradiction since $\{\delta, \gamma, 5, 6\}$ is a circuit and $\{t_2, t_3, t_4\}$ is a triad containing δ but avoiding γ, 5, and 6.

An immediate consequence of 5.16.3 is that M/ε has no 5-fan and has no 5-cofan since both such structures contain two 4-fans, which do not have the same ends. The lemma now follows by applying 5.16.1. □

Lemma 5.17. *Assume that M contains the configuration in Figure 13. Then either*

(i) *M^* has a good bowtie; or*
(ii) *M/ε or $M/4, 5\backslash e$ is internally 4-connected having an N-minor.*

Proof. Recall that, by Corollary 5.10, $N \preceq M\backslash e/4, 5$. Now $M\backslash e/4, 5 \cong M/4, 5\backslash\gamma \cong M\backslash\gamma/\delta, \varepsilon$. Thus $N \preceq M/\varepsilon$. If M/ε is internally 4-connected, the lemma holds. Thus, by the last lemma, we may assume that M/ε is $(4, 4, S)$-connected having $(x, \delta, \gamma, 4)$ as its unique 4-fan. Thus M has $\{\varepsilon, x, \delta, \gamma\}$ as a circuit.

Now $M\backslash e/4$ is $(4, 4, S)$-connected having $(7, 8, 6, 5)$ as a 4-fan. Thus $M\backslash e/4/5$ is 3-connected, so $M\backslash\gamma/\delta, \varepsilon$ is 3-connected. Next we show that

5.17.1. *$M\backslash\gamma/\delta, \varepsilon$ is sequentially 4-connected.*

To see this, first note that the restriction of $M\backslash\gamma/\delta, \varepsilon$ to $\{3, 4, 6, 7, 8, e\}$ is isomorphic to $M(K_4)$. Since $M(K_4)$ is not the union of two lines, for every partition (X, Y) of its ground set, either X or Y contains a basis of $M(K_4)$. This observation means that if (U, V) is a non-sequential 3-separation of $M\backslash\gamma/\delta, \varepsilon$, then we may assume that $\{3, 4, 6, 7, 8, e\} \subseteq U$. It follows that we may assume that 5 is in U. Now we can add γ and then δ and ε to U to obtain a non-sequential 3-separation of M; a contradiction. Thus 5.17.1 holds.

If $M\backslash\gamma/\delta, \varepsilon$ is internally 4-connected, then the lemma holds. Thus we may assume that $M\backslash\gamma/\delta, \varepsilon$ has a 4-fan (t_1, t_2, t_3, t_4). Then $\{t_2, t_3, t_4\}$ or $\{t_2, t_3, t_4, \gamma\}$ is a cocircuit of M.

Assume first that $\{t_2, t_3, t_4\}$ is a cocircuit. Then M has a circuit C that is contained in $\{t_1, t_2, t_3, \delta, \varepsilon\}$ and properly contains $\{t_1, t_2, t_3\}$. Suppose that $\delta \in C$. Then, by orthogonality, $4 \in \{t_1, t_2, t_3\}$. As $\{4, 2, 3, e\}$ is a cocircuit, it follows that $\{2, 3, e\}$ meets $\{t_1, t_2, t_3\}$. But $\{2, 3, e\}$ avoids the triad $\{t_2, t_3, t_4\}$. Thus $t_1 \in \{2, 3, e\}$, so 4 is in the triad $\{t_2, t_3, t_4\}$. As this triad is not $\{4, \gamma, \delta\}$, we contradict the first assertion of Lemma 5.16. We conclude that $\delta \notin C$. Then $C = \{t_1, t_2, t_3, \varepsilon\}$. Thus, by orthogonality, $5 \in \{t_1, t_2, t_3\}$. Hence $\{e, 6, 8\}$ meets $\{t_1, t_2, t_3\}$. Thus $t_1 \in \{e, 6, 8\}$. Therefore $5 \in \{t_2, t_3\}$ so, by the first assertion of Lemma 5.16 again, $\{t_2, t_3, t_4\} = \{5, \gamma, \varepsilon\}$; a contradiction. We deduce that $\{t_2, t_3, t_4\}$ is not a cocircuit of M.

We may now suppose that $\{t_2, t_3, t_4, \gamma\}$ is a cocircuit of M. Then, it follows by orthogonality with the circuit $\{\varepsilon, x, \delta, \gamma\}$ that $\{\delta, \varepsilon, x\}$ meets $\{t_2, t_3, t_4\}$. Hence $x \in \{t_2, t_3, t_4\}$. The cocircuit $\{t_2, t_3, t_4, \gamma\}$ contains $\{x, \gamma\}$ and so meets each of

$\{4,5,e\}$ and $\{5,6\}$. By orthogonality with the circuit $\{6,7,8\}$, it follows that $\{t_2,t_3,t_4,\gamma\}$ does not contain 6 and so contains 5. Thus $\{5,x,\gamma,y\}$ is a cocircuit of M for some element y. But $\{\gamma,5,\varepsilon\}$ is a cocircuit too, so $\{\varepsilon,x,y\}$ is a cocircuit of M. Since $M^*\backslash\varepsilon$ is $(4,4,S)$-connected having an N^*-minor, it follows that $(\{\varepsilon,x,y\},\{\gamma,4,\delta\},\{\varepsilon,\gamma,\delta,x\})$ is a good bowtie in M^*. We conclude that the lemma holds. □

Recall that M contains the configuration shown in Figure 3. By combining 5.12–5.17, we may now assume that both $M/4$ and $M/5$ have N-minors and are $(4,4,S)$-connected but not internally 4-connected.

Lemma 5.18. *Let (a,x,y,z) be a 4-fan in $M/5$. Then $\{5,a,x,y\}$ is a circuit of M and $\{x,y,z\}$ is a cocircuit that is disjoint from $\{1,2,3,5,6,7,8,e\}$. Moreover, one of the following holds.*

- (I) *$a=e$ and $4\in\{x,y\}$; or*
- (II) *$a=6$ and 4 avoids $\{x,y\}$; or*
- (III) *$a=8$ and 4 avoids $\{x,y\}$.*

Proof. Since $\{5,a,x,y\}$ is a circuit, it follows by orthogonality that $\{a,x,y\}$ meets $\{e,6,8\}$. But $\{e,6,8\}$ avoids $\{x,y,z\}$, so $a\in\{e,6,8\}$. If $a=e$, then, by orthogonality, $\{x,y\}$ meets $\{2,3,4\}$. Hence $4\in\{x,y\}$ and (I) holds. We may now assume that $a\in\{6,8\}$. Then (II) or (III) holds unless $4\in\{x,y\}$. But, in the exceptional case, orthogonality implies that $\{2,3,e\}$ meets $\{x,y\}$; a contradiction. □

Lemma 5.19. *Assume that M has a circuit $\{5,x,y,\alpha\}$ for some α in $\{6,8\}$ where $\{x,y,z\}$ is a cocircuit that is disjoint from $\{1,2,3,5,6,7,8,e\}$. Suppose that $z\neq 4$. Then*

- (i) *M/z or $M/5,z$ is internally 4-connected having an N-minor; or*
- (ii) *M^* has a good bowtie; or*
- (iii) *$M^*\backslash z$ is $(4,5,S,+)$-connected having an N-minor and M^* has a pretty good bowtie; or*
- (iv) *M has an internally 4-connected matroid M' such that $1\leq|E(M)-E(M')|\leq 3$ and M' has an N-minor.*

Proof. Assume that the lemma fails. By Corollary 5.10, neither $M\backslash e\backslash 1$ and $M\backslash e\backslash 7$ has an N-minor, and $N\preceq M\backslash e/4,5$. Now $M\backslash e/5$ has (α,y,x,z) as a 4-fan. Thus $N\preceq M\backslash e/5\backslash\alpha$ or $N\preceq M\backslash e/5/z$. In the former case, letting $\{\alpha,\beta\}=\{6,8\}$, we have that $M\backslash e\backslash\alpha$ has $\{5,\beta\}$ as a cocircuit. Thus $N\preceq M\backslash e/\beta$. But the last matroid has $\{\alpha,7\}$ as a circuit. Hence $N\preceq M\backslash e\backslash 7$; a contradiction. We deduce that $N\preceq M\backslash e/5/z$.

5.19.1. $4\notin\{x,y\}$.

If $4\in\{x,y\}$, then, by orthogonality between $\{5,x,y,\alpha\}$ and $\{2,3,4,e\}$, we deduce that $\{x,y\}$ meets $\{2,3,e\}$; a contradiction. Thus 5.19.1 holds.

5.19.2. *M/z is sequentially 4-connected.*

Let (U,V) be a non-sequential 3-separation of M/z. Then we may assume that $\{6,7,8\}\subseteq U$ and that $x\in U$ and $y\in V$. Thus $5\in V$ otherwise we can move y into U and then adjoin z to U to get a non-sequential 3-separation of M; a contradiction. Hence $e\in V$ otherwise we can move 5 into U. Similarly, $3\in V$ otherwise we can move e into U. We can now move 8, then 6, and then x into V. Now adjoining z to V gives a non-sequential 3-separation of M; a contradiction. Thus 5.19.2 holds.

5.19.3. *Let (t_1, t_2, t_3, t_4) be a 4-fan in M/z. Then $\{t_2, t_3\} \not\subseteq \{5, x, y\}$.*

Clearly $\{t_2, t_3\} \neq \{x, y\}$ otherwise $\{t_2, t_3, t_4\} = \{x, y, z\}$; a contradiction. Now assume that $(t_2, t_3) = (5, x)$. By orthogonality between the cocircuit $\{5, e, 6, 8\}$ and the circuit $\{t_1, x, 5, z\}$, we deduce that $t_1 \in \{e, 6, 8\}$. If $t_1 = e$, then it follows by orthogonality between $\{t_1, x, 5, z\}$ and the cocircuit $\{2, 3, 4, e\}$ that $4 \in \{x, z\}$, which contradicts 5.19.1 or the lemma hypothesis. Thus we may assume that $t_1 \in \{6, 8\}$. The circuits $\{t_1, x, 5, z\}$ and $\{x, y, 5, \alpha\}$ imply that $t_1 \neq \alpha$. Then, by taking the symmetric difference of the last two circuits and $\{6, 7, 8\}$, we deduce that $\{y, z, 7\}$ is a circuit; a contradiction. We conclude, by symmetry, that 5.19.3 holds.

5.19.4. *M/z has no 5-fan.*

Suppose that M/z has $(t_1, t_2, t_3, t_4, t_5)$ as a 5-fan. Then, by the dual of Lemma 2.12 and symmetry, we may assume that $t_3 = x$. Then $\{t_2, x, t_4\}$ is a triad of M. By orthogonality with the circuit $\{5, x, y, \alpha\}$, we deduce by symmetry that $t_2 = 5$. This gives us a contradiction to 5.19.3. We conclude that 5.19.4 holds.

5.19.5. *If $(t_1, t_2, t_3, t_4, t_5)$ is a 5-cofan of M/z, then $5 \in \{t_1, t_5\}$. In particular, if $t_1 = 5$, then $t_2 \in \{x, y\}$.*

Clearly $\{t_2, t_3, t_4, z\}$ is a circuit of M so, by orthogonality, $\{t_2, t_3, t_4\}$ meets $\{x, y\}$. Because $\{t_1, t_2, t_3, t_4, t_5\}$ is 3-separating in M/z, it contains exactly one of x and y. Moreover, $\alpha \notin \{t_1, t_2, t_3, t_4, t_5\}$, otherwise M has a 4-fan. Thus, by orthogonality between the circuit $\{5, x, y, \alpha\}$ and the cocircuits $\{t_1, t_2, t_3\}$ and $\{t_3, t_4, t_5\}$, it follows that $t_3 \notin \{x, y\}$. Now suppose that $t_2 = x$. Then, by orthogonality, $5 \in \{t_1, t_3\}$. Thus, by 5.19.3, $5 = t_1$. We deduce, by symmetry, that 5.19.5 holds.

5.19.6. *M/z has no 5-cofan with an element in the coguts.*

Let $(t_1, t_2, t_3, t_4, t_5)$ be a 5-cofan of M/z with an element, t_6, in the coguts. Then M/z also has $(t_1, t_3, t_2, t_4, t_6)$ and $(t_5, t_3, t_4, t_2, t_6)$ as 5-cofans. By 5.19.5, 5 must be an end of all three of these 5-cofans; a contradiction.

5.19.7. *M/z has no 5-cofan with an element in the guts.*

Let $(t_1, t_2, t_3, t_4, t_5)$ be a 5-cofan of M/z with an element, t_6, in the guts. Then, by 5.19.5 and symmetry, we may assume that $(t_1, t_2) = (5, x)$. Then $N \preceq M/z/5$. Moreover, as M/z has no 5-fan, $\{5, t_3, t_5, t_6\}$ is a circuit of M/z. By orthogonality with the cocircuit $\{5, e, 6, 8\}$, we deduce, as $\{t_3, t_5\}$ avoids $\{e, 6, 8\}$, that $t_6 \in \{e, 6, 8\}$.

Now $\{5, t_3, t_5, t_6, z\}$ or $\{5, t_3, t_5, t_6\}$ is a circuit of M. In the first case, $\{x, y\}$ meets $\{t_3, t_5\}$; a contradiction. Thus $\{5, t_3, t_5, t_6\}$ is a circuit of M. Hence $M/5$ has (t_6, t_5, t_3, t_4) and (α, x, y, z) as 4-fans. Suppose these 4-fans are disjoint. Then, by applying Corollary 5.10 and duality, we obtain the contradiction that $M/5/z$ has no N-minor. We deduce that the two 4-fans meet, so $t_6 = \alpha$. As these two 4-fans in $M/5$ meet in their guts elements, the corresponding two fans in $M^*\backslash 5$ meet in their coguts elements and, by Lemma 4.2, the lemma holds. Thus 5.19.7 holds.

On combining 5.19.4, 5.19.6, and 5.19.7, we immediately obtain the following.

5.19.8. *$M^*\backslash z$ is $(4, 5, S, +)$-connected.*

Evidently if M/z is internally 4-connected, then the lemma holds.

5.19.9. *If M/z is $(4, 4, S)$-connected but not internally 4-connected, then either M^* has a good bowtie, or M/z has a 4-fan $(t_1, t_2, t_3, 5)$ where $|\{x, y\} \cap \{t_2, t_3\}| = 1$.*

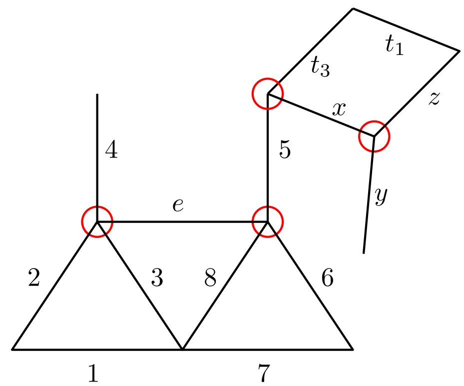

Figure 14

Let (t_1, t_2, t_3, t_4) be a 4-fan in M/z. Then $\{t_1, t_2, t_3, z\}$ is a circuit so, by orthogonality and symmetry, we may assume that $x \in \{t_1, t_2\}$. Suppose $x = t_1$. Then, as $M^*\backslash z$ is $(4, 4, S)$-connected, M^* has $(\{t_2, t_3, t_4\}, \{x, y, z\}, \{x, z, t_2, t_3\})$ as a good bowtie. We may now assume that $x = t_2$. Then, by orthogonality between the cocircuit $\{t_2, t_3, t_4\}$ and the circuit $\{5, x, y, \alpha\}$, we deduce that $5 \in \{t_3, t_4\}$. If $5 = t_3$, we contradict 5.19.3. Thus $5 = t_4$ and 5.19.9 holds.

5.19.10. *If M/z is not $(4, 4, S)$-connected, then M^* has a pretty good bowtie.*

As M/z is not $(4, 4, S)$-connected, 5.19.8 and 5.19.5 imply that we may assume that $(5, x, t_3, t_4, t_5)$ is a 5-cofan in M/z. Then $(\{t_3, t_4, t_5\}, \{x, y, z\}, \{x, z, t_3, t_4\})$ is a bowtie in M^*. Since $\{5, x, t_3\}$ is a triangle of M^* and $M^*\backslash 5$ is $(4, 4, S)$-connected having an N^*-minor, while $M^*\backslash z$ is $(4, 5, S, +)$-connected having an N^*-minor, it follows that the specified bowtie is a pretty good bowtie in M^*. Thus 5.19.10 holds.

To complete the proof of Lemma 5.19, it follows by 5.19.9 that we may assume that M/z is $(4, 4, S)$-connected having a 4-fan $(t_1, t_2, t_3, 5)$ where, by symmetry, we may assume that $x = t_2$. Then M has $\{x, z, t_1, t_3\}$ as a circuit. Thus M contains the structure shown in Figure 14 where there is a circuit $\{5, x, y, \alpha\}$ for some $\alpha \in \{6, 8\}$.

5.19.11. *$M/5, z$ is sequentially 4-connected.*

Let (U, V) be a non-sequential 3-separation of $M/5, z$. Then we may assume that $\{x, t_1, t_3\} \subseteq U$. Then we can add 5 to U to get a non-sequential 3-separation of M/z; a contradiction. Hence 5.19.11 holds.

5.19.12. *$M/5, z$ is internally 4-connected.*

It suffices to show that $M/5, z$ has no 4-fan. Assume to the contrary that (s_1, s_2, s_3, s_4) is a 4-fan of $M/5, z$. Then $\{s_1, s_2, s_3, 5\}$, $\{s_1, s_2, s_3, z\}$, or $\{s_1, s_2, s_3, 5, z\}$ is a circuit C of M. Assume first that $5 \in C$. Then, by orthogonality, $\{e, 6, 8\}$ meets $\{s_1, s_2, s_3\}$. But $\{e, 6, 8\}$ avoids $\{s_2, s_3\}$. Hence $s_1 \in \{e, 6, 8\}$. Now the cocircuit $\{5, x, t_3\}$ meets C. If $t_3 \in \{s_2, s_3\}$, then t_3 is in a triad of M other than $\{5, t_3, x\}$ and not containing z. Thus M/z has a 5-cofan; a contradiction. We deduce that $x \in \{s_2, s_3\}$. By orthogonality between the cocircuit $\{s_2, s_3, s_4\}$ and the circuit $\{5, x, y, \alpha\}$, we obtain a contradiction as $\{s_2, s_3, s_4\} \neq \{x, y, z\}$ and α is not in a triad. We conclude that $5 \notin C$.

We now know that $C = \{s_1, s_2, s_3, z\}$. Then (s_1, s_2, s_3, s_4) and $(t_1, x, t_3, 5)$ are 4-fans of M/z. Since $s_4 \neq 5$, these 4-fans are either disjoint or meet in their guts elements. In the first case, since $N \preceq M/z/5$, we obtain a contradiction to Corollary 5.10. In the second case, the lemma follows by Lemma 4.2. We conclude that 5.19.12 holds and, therefore, so does Lemma 5.19. □

Lemma 5.19 means that if (II) or (III) of Lemma 5.18 holds, then we may assume that $z = 4$ otherwise Theorem 5.1 certainly holds. Now, by symmetry, we can apply Lemma 5.18 to $M/4$ resulting in the analogue of one of options (I), (II), and (III) holding for that matroid. We begin by eliminating the first of these options.

Lemma 5.20. *Assume that M has a circuit $\{5, x, y, \alpha\}$ for some α in $\{6, 8\}$ where $\{x, y, 4\}$ is a cocircuit. Then M has no 4-circuit containing $\{4, 5, e\}$.*

Proof. Assume that M has a 4-circuit $\{4, 5, e, h\}$. Then $M/5$ has $\{4, e, h\}$ as a circuit and has $(\alpha, x, y, 4)$ as a 4-fan. Hence $M/5$ has a 5-fan; a contradiction □

The next lemma treats the case when option (II) or (III) of Lemma 5.18 holds for each of $M/5$ and $M/4$. Proving this will enable us to assume that option (I) of Lemma 5.18 holds for each of $M/5$ and $M/4$.

Lemma 5.21. *Assume that M has a circuit $\{5, x, y, \alpha\}$ for some α in $\{6, 8\}$ where $\{x, y, 4\}$ is a cocircuit. Suppose, in addition, that M has a circuit $\{4, x', y', \beta\}$ for some β in $\{2, 3\}$ where $\{x', y', 5\}$ is a cocircuit. Then $M/4, 5$ is internally 4-connected having an N-minor.*

Proof. By Lemma 5.20, M has no circuit containing $\{4, 5, e\}$. Now $M/5$ is 3-connected having $(\alpha, x, y, 4)$ as a 4-fan. As 4 is not in a triangle of $M/5$, it follows that $M/5, 4$ is 3-connected. Next we observe that, as $N \preceq M\backslash e/4, 5$, we certainly have $N \preceq M/4, 5$.

By orthogonality between the cocircuit $\{x, y, 4\}$ and the circuit $\{\beta, 4, x', y'\}$, since β is not in a triad, we may assume that $x = x'$.

5.21.1. *$M/4, 5$ is sequentially 4-connected.*

Suppose $M/4, 5$ has a non-sequential 3-separation (U, V). Then we may assume that $\{1, 2, 3\} \subseteq U$. If $e \in U$, then we can add 4 to U to get a non-sequential 3-separation of $M/5$; a contradiction. Thus $e \in V$. We may also assume that $\{6, 7, 8\}$ is a subset of U or V. If it is a subset of U, we can move e into U; a contradiction. Thus $\{6, 7, 8\} \subseteq V$. As $e \in V$, we can add 5 to V to get a non-sequential 3-separation of $M/4$; a contradiction. Hence 5.21.1 holds.

5.21.2. *$M/4, 5$ has no 4-fan.*

Suppose $M/4, 5$ has a 4-fan (t_1, t_2, t_3, t_4). Then M has a circuit C such that $\{t_1, t_2, t_3\} \subsetneqq C \subseteq \{t_1, t_2, t_3, 4, 5\}$. By symmetry, we may assume that $4 \in C$. As $\{4, 2, 3, e\}$ is a cocircuit, it follows by orthogonality with C that $\{2, 3, e\}$ meets $\{t_1, t_2, t_3\}$. But $\{t_2, t_3\}$ avoids $\{2, 3, e\}$, so $t_1 \in \{2, 3, e\}$. As M has $\{4, x, y\}$ as a cocircuit, $\{x, y\}$ meets $\{t_2, t_3\}$. By orthogonality between $\{t_2, t_3, t_4\}$ and $\{5, x, y, \alpha\}$, we deduce that $\{t_2, t_3, t_4\}$ contains $\{x, y\}$ and so is $\{x, y, 4\}$; a contradiction. We conclude that 5.21.2 holds and Lemma 5.21 follows immediately. □

We may now assume that option (I) of Lemma 5.18 holds with respect to each of $M/5$ and $M/4$. Hence M contains the structure shown in Figure 15 where all of the elements shown are distinct.

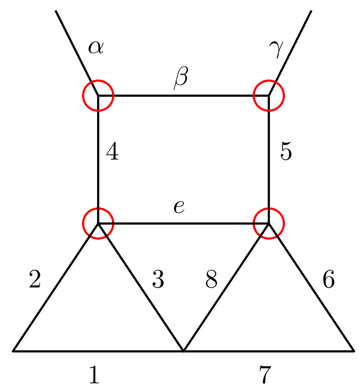

FIGURE 15

Lemma 5.22. *$M/4,5\backslash e$ is internally 4-connected having an N-minor.*

Proof. We know, by Corollary 5.10, that $M/4\backslash e$ is $(4,4,S)$-connected having $(7,8,6,5)$ as a 4-fan and having an N-minor. Thus $M/4,5\backslash e$ is 3-connected.

5.22.1. *$M/4,5\backslash e$ is sequentially 4-connected.*

Let (U,V) be a non-sequential 3-separation of $M/4,5\backslash e$. Then we may assume that $\{6,7,8\}$ is contained in U, so we can add 5 to U to get a non-sequential 3-separation of $M/4\backslash e$; a contradiction. We deduce that 5.22.1 holds.

5.22.2. *$M/4,5\backslash e$ has no 4-fan.*

Let (t_1,t_2,t_3,t_4) be a 4-fan of $M/4,5\backslash e$. Then either $\{t_2,t_3,t_4\}$ or $\{t_2,t_3,t_4,e\}$ is a cocircuit C^* of M. Moreover, M has a circuit C that contains $\{t_1,t_2,t_3\}$ and is contained in $\{t_1,t_2,t_3,4,5\}$. Suppose first that $C^*=\{t_2,t_3,t_4\}$. Then $C\neq\{t_1,t_2,t_3\}$. By symmetry, we may suppose that C contains $\{t_1,t_2,t_3,5\}$. Then, by orthogonality with the cocircuit $\{5,6,8,e\}$, it follows that C meets $\{6,8\}$. Thus $t_1\in\{6,8\}$. Moreover, by orthogonality with the cocircuit $\{e,2,3,4\}$, it follows that $4\notin C$. Thus $C=\{t_1,t_2,t_3,5\}$ where $t_1\in\{6,8\}$. Hence (II) or (III) of Lemma 5.18 holds; a contradiction.

We may now assume that $C^*=\{t_2,t_3,t_4,e\}$. Then, by orthogonality with the circuits $\{e,3,8\}$ and $\{e,4,5,\beta\}$ and using the fact that $\{4,5\}$ is disjoint from $\{t_1,t_2,t_3,t_4\}$, we deduce that C^* contains β and meets $\{3,8\}$. Then C^* contains $\{e,\beta,3\}$ or $\{e,\beta,8\}$. Thus $M\backslash e$ has C^*-e as a triad avoiding $\{4,5\}$ and meeting one of the triangles $\{1,2,3\}$ or $\{6,7,8\}$. Thus $M\backslash e$ has a 5-cofan; a contradiction. We conclude that 5.22.2 holds, and Lemma 5.22 follows immediately. □

The lemmas that preceded Lemma 5.22 told us that the theorem holds unless option (I) of Lemma 5.18 holds with respect to each of $M/5$ and $M/4$. But, in the exceptional case, we are forced to have the structure shown in Figure 15, and Lemma 5.22 shows that the theorem also holds when that occurs. □

6. Two 4-fans meeting in their guts elements

The purpose of this section is to prove the following result.

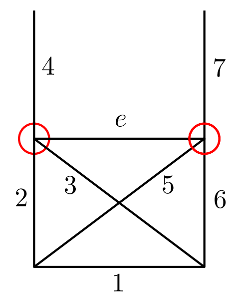

FIGURE 16

Theorem 6.1. *Let M and N be internally 4-connected matroids with $|E(M)| \geq 16$ and $|E(N)| \geq 7$. Assume that M has a triangle containing an element e such that $M\backslash e$ is $(4,4,S)$-connected having an N-minor and having two 4-fans that meet in their guts elements. Then one of the following holds.*

(i) *M has a good bowtie; or*
(ii) *M^* has a good bowtie or a pretty good bowtie; or*
(ii) *M has an internally 4-connected matroid M' such that $1 \leq |E(M) - E(M')| \leq 3$ and M' has an N-minor.*

Proof. By the main results of the last two sections, we may assume that, for each (M_1, N_1) in $\{(M,N),(M^*,N^*)\}$, the matroid M_1 has no element that is in a triangle such that the deletion of this element from M_1 has an N_1-minor, is $(4,4,S)$-connected, and has two 4-fans that are either disjoint or meet in their coguts elements. Let $(1,2,3,4)$ and $(1,5,6,7)$ be 4-fans of $M\backslash e$. Then the elements $1,2,3,4,5,6,7,e$ are distinct. Using orthogonality and symmetry, we may assume that the triangle containing e is $\{e,2,5\}$ (see Figure 16). First we show the following.

Lemma 6.2. *$M\backslash e\backslash 1$ is internally 4-connected.*

Proof. Assume the contrary, letting (U,V) be a $(4,3)$-violator of $M\backslash e\backslash 1$.

6.2.1. $\{2,3\} \not\subseteq U$.

Assume $\{2,3\} \subseteq U$. Then $(U \cup 1, V)$ is a $(4,3)$-violator of $M\backslash e$. Thus V is a 4-fan (y_1,y_2,y_3,y_4) in $M\backslash e$. Then $\{y_2,y_3,y_4,e\}$ is a cocircuit of M. Moreover, $\{y_2,y_3,y_4\}$ contains $\{5,6\}$ by orthogonality with the triangles $\{2,5,e\}$ and $\{3,6,e\}$. Thus the 4-fans $\{y_1,y_2,y_3,y_4\}$ and $\{1,5,6,7\}$ in $M\backslash e$ meet in at least two elements but are distinct; a contradiction to Lemma 2.11. We deduce that 6.2.1 holds.

By symmetry, neither U nor V contains $\{2,3\}$ or $\{5,6\}$. Without loss of generality, we may assume that $2 \in U$ and $3 \in V$.

6.2.2. $6 \in U$ *and* $5 \in V$.

Assume this fails. Then $6 \in V$ and $5 \in U$. Thus $(U \cup e, V)$ and $(U, V \cup e)$ are 3-separations of $M\backslash 1$. By symmetry, we may assume that $4 \in U$. Then $3 \in \mathrm{cl}^*{}_{M\backslash 1}(U \cup e)$, so $(U \cup e \cup 3 \cup 1, V - 3)$ is a 3-separation of M. Thus $|V - 3| = 3$. Hence V is a 4-fan $(v_1,v_2,v_3,3)$ in $M\backslash 1$ where 6 is in the circuit $\{v_1,v_2,v_3\}$ of M. Orthogonality with $\{5,6,7,e\}$ implies that $7 \in \{v_1,v_2,v_3\}$, so $M\backslash e$ has a 5-fan; a contradiction. Thus 6.2.2 holds.

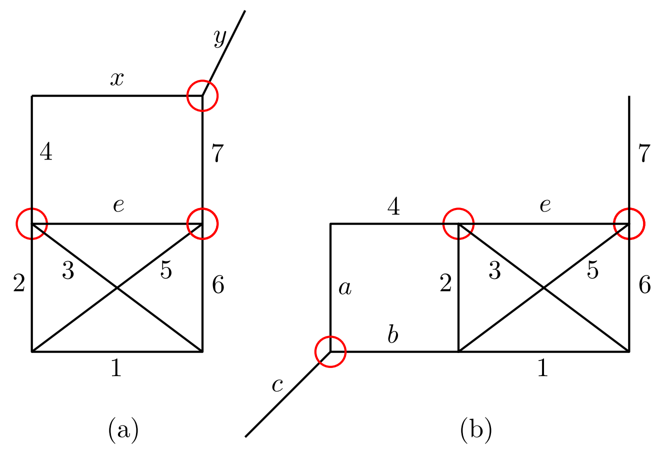

FIGURE 17

We now know that $\{2,6\} \subseteq U$ and $\{3,5\} \subseteq V$. By symmetry, we may assume that $4 \in U$. Then $(U \cup 3 \cup 1, V - 3)$ is a 3-separation of $M\backslash e$. But $V - 3$ is not the 4-fan $\{1,5,6,7\}$, so $|V - 3| = 3$. Thus V is a 4-fan $(v_1, v_2, v_3, 3)$ in $M\backslash e\backslash 1$. Therefore $\{v_1, v_2, v_3\}$ is a triangle containing 5 but avoiding $\{6, e\}$. Thus the triangle contains 7 and so $M\backslash e$ has a 5-fan. This contradiction completes the proof of Lemma 6.2. □

Lemma 6.3. *Either both $M\backslash e/4$ and $M\backslash e/7$ are $(4,4,S)$-connected and $N \preceq M\backslash e/4/7$, or M has an internally 4-connected minor M' that has an N-minor such that $|E(M) - E(M')| = 2$.*

Proof. As $(1,2,3,4)$ is a 4-fan of $M\backslash e$, either $N \preceq M\backslash e\backslash 1$, or $N \preceq M\backslash e/4$. By the last lemma, $M\backslash e\backslash 1$ is internally 4-connected. Thus we may assume that $N \not\preceq M\backslash e\backslash 1$ otherwise the lemma holds. Then, by Lemma 2.5, $M\backslash e/4$ is $(4,4,S)$-connected having an N-minor. Since $M\backslash e/4$ has $(1,5,6,7)$ as a 4-fan and $M\backslash e\backslash 1$ does not have an N-minor, it follows that $N \preceq M\backslash e/4/7$. Moreover, by symmetry, since $M\backslash e/4$ is $(4,4,S)$-connected, so is $M\backslash e/7$. □

By the last lemma, in our proof of Theorem 6.1, we may assume from now on that both $M\backslash e/4$ and $M\backslash e/7$ are $(4,4,S)$-connected and $N \preceq M\backslash e/4/7$. We may also assume that neither $M\backslash e/4$ nor $M\backslash e/7$ is internally 4-connected otherwise the theorem holds.

Lemma 6.4. *Either $M/4$ is 3-connected, or M has a good bowtie.*

Proof. As $M\backslash e/4$ is 3-connected, if $M/4$ is not, then M has a triangle containing $\{e, 4\}$. Thus, by Lemma 2.8, M has a good bowtie. □

The configurations that arise in the next lemma are shown in Figure 17.

Lemma 6.5. *If $M/4$ is not internally 4-connected, then it is $(4,4,S)$-connected and either*

(i) *M has a circuit $\{4, e, 7, x\}$ and a triad $\{7, x, y\}$ where $\{x, y\}$ avoids $\{1,2,3,4,5,6,7,e\}$; or*
(ii) *M has a circuit $\{4, x, a, b\}$ and a triad $\{a, b, c\}$ where $x \in \{2,3\}$ and if $\{a, b, c\}$ meets $\{1,2,3,4,5,6,7,e\}$, then $7 = c$.*

Proof. Let (U, V) be a $(4,3)$-violator of $M/4$. Since $\{2,3,4,e\}$ is a cocircuit of M, neither U nor V contains $\{2,3,e\}$. By symmetry between the ordered pairs $(2,5)$ and $(3,6)$, we may assume that either

(I) $\{2,e\} \subseteq U$ and $3 \in V$; or
(II) $\{2,3\} \subseteq U$ and $e \in V$.

Assume that (I) holds. Suppose that 1 or 6 is in U. Then $(U \cup 3 \cup 4, V - 3)$ is a 3-separation of M. Thus V is a 4-fan $(3, a, b, c)$ of $M/4$, so $\{4, 3, a, b\}$ is a circuit of M and $\{a, b, c\}$ is a triad of M, that is, by the symmetry between 2 and 3, the first part of (ii) holds. Now suppose that $\{a, b, c\}$ meets $\{1, 2, 3, 4, 5, 6, 7, e\}$. Then, as M is internally 4-connected, $7 \in \{a, b, c\}$. If $7 \in \{a, b\}$, then orthogonality with the cocircuit $\{5, 6, 7, e\}$ implies that $\{a, b\} \subseteq \{5, 6, 7\}$, so $\lambda_M(\{1, 2, 3, 4, 5, 6, 7, e\}) \leq 2$; a contradiction. This leaves the possibility that $7 = c$, so the last part of (ii) holds.

We may now assume that $\{1,6\} \subseteq V$. Then $\{2,e\} \subseteq \mathrm{cl}_{M/4}(U)$. Thus $|U| \notin \{3,4\}$. As $(U - \{2,e\}, V \cup \{2,e,4\})$ is a 3-separation of M, it follows that U is a 5-fan $(2, u_2, u_3, u_4, e)$ in $M/4$. Thus $\{2, u_2, u_3, 4\}$ and $\{4, u_3, u_4, e\}$ are circuits of M, and $\{u_2, u_3, u_4\}$ is a triad of M. Neither 5 nor 6 is in a triad of M, so orthogonality between $\{4, u_3, u_4, e\}$ and $\{5, 6, 7, e\}$ implies that $7 \in \{u_3, u_4\}$. But $7 \neq u_3$ otherwise $\{2, u_2, u_3, 4\}$ and $\{e, 5, 6, 7\}$ violate orthogonality. Hence $u_4 = 7$. Then $\{2, u_2, u_3, 4\} \triangle \{4, u_3, 7, e\} \triangle \{2, e, 5\} = \{7, u_2, 5\}$. Thus M has a 4-fan; a contradiction.

We may now assume that (II) holds. Suppose 5 or 6 is in U. Then $(U \cup e \cup 4, V - e)$ is a 3-separation of M, so V is a 4-fan (e, a, b, c) in $M/4$. Then $\{4, e, a, b\}$ is a circuit of M. By orthogonality with the cocircuit $\{5, 6, 7, e\}$, we deduce that $\{a, b\}$ meets $\{5, 6, 7\}$. Neither 5 nor 6 is in a triad of M, so $7 \in \{a, b\}$, as $\{a, b, c\}$ is a triad. Then, without loss of generality, $7 = a$. Thus (i) holds.

We may now suppose that $\{5,6\} \subseteq V$. Then $\{2,3\} \subseteq \mathrm{cl}_{M/4}(V)$, so $(U - \{2,3\}, V \cup \{2,3\} \cup 4)$ is a 3-separating partition in M. Hence $|U| = 5$. As $M/4\backslash e$ is $(4,4,S)$-connected, it follows that $|V - e| \leq 4$. Thus $|E(M)| \leq 11$. This contradiction completes the proof of 6.5. □

If $M/4$ or $M/7$ is internally 4-connected, then the theorem holds. Thus we may assume that neither of these matroids is internally 4-connected. Then, by the last lemma, both are $(4,4,S)$-connected. Then either part (i) of the last lemma holds both when we consider $M/4$ and $M/7$, or part (ii) of the lemma holds with respect to $M/4$. We begin by considering the first possibility. Then M contains the structure shown in Figure 18 and all the elements shown there are distinct. Since $N \preceq M/4, 7\backslash e$, we see that $N \preceq M/4, 7\backslash x$, so $N \preceq M\backslash x/z, y$. Because y is in a triad, M/y is 3-connected.

Lemma 6.6. *Suppose that M contains the structure shown in Figure 18 where all the elements shown are distinct. Then*

(i) *M/y is internally 4-connected having an N-minor; or*
(ii) *M/y is $(4,4,S)$-connected and M^* has a good bowtie; or*
(iii) *$M^*\backslash y$ is $(4,5,S,+)$-connected and M^* has a pretty good bowtie; or*
(iv) *M has an internally 4-connected minor M' having an N-minor such that $1 \leq |E(M)| - |E(M')| \leq 2$.*

Proof. Assume that M/y is not internally 4-connected. First we note that

6.6.1. *M/y is sequentially 4-connected.*

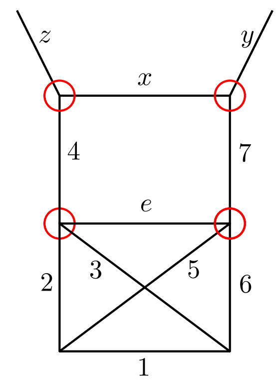

FIGURE 18

Suppose that M/y has a non-sequential 3-separation (U, V). Then we may assume that $\{2, 5, e\} \subseteq U$. If $6 \in U$, then we may assume that U contains $7, 1, 3, 4$ and x. Then we add y to U to get a non-sequential 3-separation of M; a contradiction. Thus $6 \notin U$. Also $7 \in V$ otherwise we can move 6 into U. Similarly, $1 \in V$. But now we can move 5, then e, and then 2 into V; a contradiction. Thus 6.6.1 holds.

It is an easy consequence of orthogonality that

6.6.2. *x is in exactly two triads of M, namely $\{x, z, 4\}$ and $\{x, y, 7\}$.*

Next we show the following.

6.6.3. *If (t_1, t_2, t_3, t_4) is a 4-fan in M/y, then $\{t_1, t_2, t_3\} \cap \{4, x, z\} = \{x, z\}$.*

Clearly M has $\{t_1, t_2, t_3, y\}$ as a circuit and $\{t_2, t_3, t_4\}$ as a triad. As $\{x, y, 7\}$ is a triad of M, exactly one of x and 7 is in $\{t_1, t_2, t_3\}$.

Assume first that $7 \in \{t_1, t_2, t_3\}$. Suppose $t_1 = 7$. Then $\{7, t_2, t_3, y\}$ is a circuit. Thus, by orthogonality, $\{t_2, t_3\}$ meets $\{e, 5, 6\}$ so M has a 4-fan; a contradiction. We may now assume that $7 \in \{t_2, t_3\}$. Then, by orthogonality with the circuit $\{4, x, 7, e\}$, we deduce that $4 \in \{t_2, t_3, t_4\}$. Suppose $4 \in \{t_2, t_3\}$. Then $\{t_1, 4, 7, y\}$ is a circuit. Thus, by orthogonality with the cocircuit $\{4, x, z\}$, we deduce that $t_1 = z$ so M has $\{z, 4, 7, y\}$ as a quad; a contradiction. Thus $t_4 = 4$. Hence we may assume that M has $\{7, t_3, 4\}$ as a cocircuit and $\{t_1, 7, t_3, y\}$ as a circuit. By orthogonality between $\{t_1, 7, t_3, y\}$ and $\{7, e, 5, 6\}$, we deduce that $t_1 \in \{e, 5, 6\}$. Letting $Z = \{1, 2, 3, 4, 5, 6, 7, e, x, y, t_3\}$, we see that $r(Z) \leq 6$ and $|Z| - r^*(Z) \geq 4$, so $\lambda(Z) \leq 2$. This is a contradiction as $|Z| \leq 11$ yet $|E(M)| \geq 16$.

We may now assume that $7 \notin \{t_1, t_2, t_3\}$, so $x \in \{t_1, t_2, t_3\}$. Orthogonality with $\{4, x, z\}$ implies that $\{t_1, t_2, t_3\}$ also meets $\{4, z\}$. Suppose $4 \in \{t_1, t_2, t_3\}$. Then orthogonality with $\{2, 3, 4, e\}$ implies that $\{2, 3, e\}$ meets $\{t_1, t_2, t_3\}$, so $t_1 \in \{2, 3, e\}$ and, without loss of generality, $x = t_2$ and $4 = t_3$. Thus $t_4 = z$. In this case, $M/4$ has $(e, 7, x, y, t_1)$ as a 5-fan; a contradiction to Lemma 6.5. We conclude that $4 \notin \{t_1, t_2, t_3\}$, so $z \in \{t_1, t_2, t_3\}$ and hence $\{x, z\} \subseteq \{t_1, t_2, t_3\}$. Thus 6.6.3 holds.

Because $\{x, z\}$ is contained in the triangle of every 4-fan in M/y, it follows that

6.6.4. *$M^*\backslash y$ has no 5-cofans.*

We show next that

6.6.5. *$M^*\backslash y$ has no 5-fan with an element in the guts.*

To see this, suppose that $M^*\backslash y$ has a 5-fan $(t_1, t_2, t_3, t_4, t_5)$ with an element t_6 in the guts. Then, by 6.6.3, $\{x, z\} \subseteq \{t_2, t_3, t_4\}$. Because $M^*|\{t_1, t_2, t_3, t_4, t_5, t_6\} \cong M(K_4)$, this restriction contains at least two triangles containing x. As $y \notin \{t_1, t_2, t_3, t_4, t_5, t_6\}$, this contradicts 6.6.2. Thus 6.6.5 holds.

It is slightly more complicated to show that

6.6.6. *$M^*\backslash y$ has no 5-fan with an element in the coguts.*

Assume that M/y has a 5-cofan $(t_0, t_1, t_2, t_3, t_4)$ with an element t_5 in the guts. Then, by 6.6.3, $\{x, z\} \subseteq \{t_1, t_2, t_3\}$. Hence, by 6.6.2 and symmetry, we may assume that $(t_0, t_1, t_2) = (4, x, z)$. Now $\{4, z, t_4, t_5\}$ or $\{4, z, t_4, t_5, y\}$ is a circuit of M. Suppose the latter holds. Then, by orthogonality with the cocircuit $\{y, x, 7\}$, we deduce that $7 \in \{t_4, t_5\}$. If $7 = t_4$, then the triad $\{z, t_3, 7\}$ has only a single element in common with the circuit $\{e, 4, 7, x\}$; a contradiction. If $7 = t_5$, then the cocircuit $\{2, 3, 4, e\}$ implies that $t_4 \in \{2, 3, e\}$; a contradiction. We conclude that $\{4, z, t_4, t_5\}$ is a circuit of M. Then, by orthogonality with the cocircuit $\{2, 3, 4, e\}$, we deduce that $\{t_4, t_5\}$ meets $\{2, 3, e\}$. As t_4 is not in a triangle, it follows that $t_5 \in \{2, 3, e\}$. If $t_5 = e$, then, by orthogonality, $t_4 = 7$. Thus $\{4, z, 7, e\}$ is a triangle of M, so $\lambda(\{1, 2, 3, 4, 5, 6, 7, e, x, z\}) \leq 2$; a contradiction. Hence $t_5 \in \{2, 3\}$.

Now consider $M/4$. By Lemma 6.5, it is $(4, 4, S)$-connected. Moreover, it has an N-minor and has $(e, 7, x, y)$ and (t_5, t_4, z, t_3) as disjoint 4-fans. Thus we have a contradiction that completes the proof of 6.6.5.

On combining 6.6.4, 6.6.5, and 6.6.6, we deduce that

6.6.7. *$M^*\backslash y$ is $(4, 5, S, +)$-connected.*

Next we extend 6.6.3 by showing the following.

6.6.8. *If (t_1, t_2, t_3, t_4) is a 4-fan in M/y, then $\{t_2, t_3\} = \{x, z\}$ and $t_4 = 4$.*

Suppose that $t_1 \in \{x, z\}$. Then, by 6.6.3 and symmetry, we may assume that $t_2 \in \{x, z\}$. Suppose $(t_1, t_2) = (x, z)$. Then $(\{z, t_3, t_4\}, \{7, x, y\}, \{x, y, z, t_3\})$ is a bowtie in M^*, and $M^*\backslash y$ is $(4, 5, S, +)$-connected with an N-minor, while $M^*\backslash 4$ is $(4, 4, S)$-connected with an N-minor. Thus M^* has a pretty good bowtie and the lemma holds. Therefore we may assume that $(t_1, t_2) = (z, x)$. Then $\{x, t_3, t_4\}$ is a triad that avoids 4 and 7; a contradiction to 6.6.2. We conclude that $\{t_2, t_3\} = \{x, z\}$, so $t_4 = 4$.

By 6.6.8, M/y has no 5-fan. Thus

6.6.9. *M/y is $(4, 4, S)$-connected.*

We now know that M contains the configuration shown in Figure 19. To complete the proof of the lemma, we show the following.

6.6.10. *$M/4/y$ is internally 4-connected having an N-minor.*

Since $M\backslash x$ has an N-minor, we see that $M/4/y$ has an N-minor. Assume that 6.6.10 fails. Let (U, V) be a non-sequential 3-separation of $M/4/y$. Then we may assume that $\{7, e, x\} \subseteq U$, so $(U \cup y, V)$ is a non-sequential 3-separation of $M/4$; a contradiction to Lemma 6.5. Thus we may assume that $M/4/y$ has a 4-fan (u_1, u_2, u_3, u_4). By 6.6.8, it is not a 4-fan of M/y, so $\{4, u_1, u_2, u_3\}$ or $\{4, y, u_1, u_2, u_3\}$ is a circuit of M. By orthogonality with $\{2, 3, 4, e\}$, we deduce

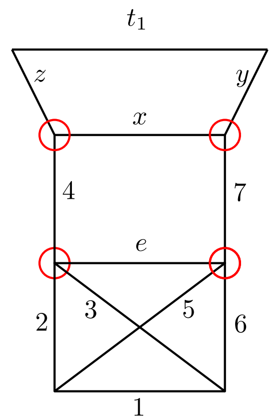

FIGURE 19

that $\{2,3,e\}$ meets $\{u_1,u_2,u_3\}$. Thus $u_1 \in \{2,3,e\}$. By orthogonality with the cocircuit $\{4,x,z\}$, we see that $\{u_1,u_2,u_3\}$ meets $\{x,z\}$. Thus we may assume that $u_2 \in \{x,z\}$. Hence $\{u_2,u_3,u_4\}$ is a triad of M that contains x or z but avoids $\{4,y\}$. By 6.6.2, $u_2 = z$. Hence M/y has $\{t_1,x,z\}$ as a triangle that meets the triads $\{4,x,z\}$ and $\{z,u_2,u_3\}$, so M/y has a 5-cofan. This contradiction to 6.6.9 completes the proof of 6.6.10 and thereby finishes the proof of Lemma 6.6. □

It follows by the last lemma that we may assume that part (i) of Lemma 6.5 does not hold with respect to either $M/4$ or $M/7$. Thus part (ii) holds with respect to both $M/4$ and $M/7$. Since part (ii) holds with respect to $M/4$, it follows that M contains the configuration shown in Figure 17(b). Recall that $N \preceq M/4/7\backslash e$. Suppose first that $c \neq 7$. Then $M/4/7\backslash e$ has $(2,a,b,c)$ as a 4-fan, so $N \preceq M/4/7\backslash e\backslash 2$ or $N \preceq M/4/7\backslash e/c$. If $N \preceq M/4/7\backslash e\backslash 2$, then $N \preceq M\backslash e\backslash 2/3$, so $N \preceq M\backslash e\backslash 1$ and the required result holds by Lemma 6.2. We conclude that either

(A) $c \neq 7$ and $N \preceq M/4/7\backslash e/c$; or
(B) $c = 7$.

In case (A), Lemma 6.7 will complete the proof of the theorem. Now consider case (B). Since part (ii) of Lemma 6.5 holds with respect to $M/7$, we deduce that M contains a circuit $\{7,x',a',b'\}$ and a triad $\{a',b',c'\}$ where $x' \in \{5,6\}$ and if $\{a',b',c'\}$ meets $\{1,2,3,4,5,6,7,e\}$, then $c' = 4$. If $c' \neq 4$, then we reduce to a case symmetric to case (A). Thus, assuming we can deal with case (A), we may assume that $c' = 4$. Then, by orthogonality between $\{a',b',4\}$ and $\{4,a,b,2\}$, it follows by symmetry that we may assume that $a' = a$. Then $\lambda(\{1,2,3,4,5,6,7,e,a,b,b'\}) \leq 2$; a contradiction. It remains to treat case (A).

Lemma 6.7. *Suppose that $c \neq 7$ and $N \preceq M/4/7\backslash e/c$. Then one of the following holds.*

(i) *M/c or $M/4,c$ is internally 4-connected having an N-minor; or*
(ii) *M^* has a good bowtie or a pretty good bowtie.*

Proof. We begin by showing the following.

6.7.1. *M/c is sequentially 4-connected.*

Let (U, V) be a non-sequential 3-separation of M/c. Then we may assume that $a \in U$ and $b \in V$. In addition, we may assume that $\{2, e, 5\}$ is in U. Suppose U meets $\{6, 3, 1\}$. Then we may assume that it contains all of this set. Hence we may assume that, in addition, it contains 4. Thus we can move b into U and then add c to it; a contradiction. We may now assume that $\{6, 3, 1\} \subseteq V$. Then V spans $\{2, e, 5\}$ so all of these elements can be added to V. We now obtain a contradiction as before. Thus 6.7.1 holds.

6.7.2. *M/c has no 5-fan.*

Suppose that M/c has a 5-fan $(t_1, t_2, t_3, t_4, t_5)$. Then, by the dual of Lemma 2.12, $t_3 \in \{a, b\}$, so we may assume that $t_3 = a$. Then the triad $\{t_2, t_3, t_4\}$ meets the circuit $\{4, 2, a, b\}$. We know that 2 is not in a triad, and $b \notin \{t_2, t_3, t_4\}$ since $\{t_2, t_3, t_4\} \neq \{a, b, c\}$. Thus $4 \in \{t_2, t_4\}$, so we may assume that $4 = t_2$. Then $\{t_1, 4, a, c\}$ is a circuit. Thus $M/4$ has $(t_1, c, a, b, 2)$ as a 5-fan; a contradiction since $M/4$ is $(4, 4, S)$-connected. We deduce that 6.7.2 holds.

6.7.3. *If $(t_1, t_2, t_3, t_4, t_5)$ is a 5-cofan in M/c, then $|\{a, b\} \cap \{t_1, t_2, t_3, t_4, t_5\}| = |\{a, b\} \cap \{t_2, t_4\}| = 1$.*

First we note that $|\{a, b\} \cap \{t_1, t_2, t_3, t_4, t_5\}| \leq 1$ otherwise $\{t_1, t_2, t_3, t_4, t_5, c\}$ is 3-separating in M. Clearly $\{t_2, t_3, t_4, c\}$ is a circuit of M. Thus $\{t_2, t_3, t_4\}$ meets $\{a, b\}$. By symmetry, we may assume that $a \in \{t_2, t_3\}$. If $a = t_3$, then $\{t_1, t_2, a\}$ and $\{a, t_4, t_5\}$ are cocircuits of M. Thus, by orthogonality, both $\{t_1, t_2\}$ and $\{t_4, t_5\}$ meet $\{b, 2, 4\}$. But $2, b \notin \{t_1, t_2, t_4, t_5\}$. Thus $4 \in \{t_1, t_2\} \cap \{t_4, t_5\}$; a contradiction. We conclude that $a \neq t_3$. Thus 6.7.3 holds.

6.7.4. *If (t_1, a, t_3, t_4, t_5) is a 5-cofan in M/c, then $t_1 = 4$.*

Evidently $\{t_1, a, t_3\}$ and $\{4, 2, 3, e\}$ are cocircuits of M while $\{c, a, t_3, t_4\}$ and $\{4, a, b, 2\}$ are circuits. Thus, by orthogonality, $4 \in \{t_1, t_3\}$. But if $t_3 = 4$, then $t_4 \in \{2, 3, e\}$; a contradiction. Hence 6.7.4 holds.

6.7.5. *M/c has no 5-cofan with an element in the coguts.*

Assume that M/c has a 5-cofan $(t_1, t_2, t_3, t_4, t_5)$ with an element t_6 in the coguts. Then M has $\{t_1, t_2, t_3\}, \{t_3, t_4, t_5\}$, and $\{t_1, t_5, t_6\}$ as triads. By the last two assertions, we may assume that $t_2 = a$ and $t_1 = 4$. Then $\{4, t_5, t_6\}$ is a cocircuit. As $\{4, a, b, 2\}$ is a circuit, by orthogonality, b or 2 is in $\{t_5, t_6\}$ But 2 is not in a triad, so $b \in \{t_5, t_6\}$. Hence $c \in \mathrm{cl}^*(\{t_1, t_2, \ldots, t_6\})$; a contradiction. Thus 6.7.5 holds.

6.7.6. *M/c has no 5-cofan with an element in the guts.*

Assume that M/c has a 5-cofan $(t_1, t_2, t_3, t_4, t_5)$ with an element t_6 in the guts. Then, by 6.7.3 and 6.7.4, we may assume that $t_1 = 4$ and $t_2 = a$. Now $\{t_1, t_3, t_5, t_6\}$ is a circuit of M/c, so $\{t_1, t_3, t_5, t_6\}$ or $\{t_1, t_3, t_5, t_6, c\}$ is a circuit of M. But $b \notin \{t_1, t_2, \ldots, t_6\}$. Thus, by orthogonality, $\{t_1, t_3, t_5, t_6\}$ is a circuit of M, that is, $\{4, t_3, t_5, t_6\}$ is a circuit of M. Hence, by orthogonality, $\{2, 3, e\}$ meets $\{t_3, t_5, t_6\}$. But $\{2, 3, e\}$ avoids $\{t_3, t_5\}$. Therefore $t_6 \in \{2, 3, e\}$.

Suppose $t_6 = 2$. Then $\{4, t_3, t_5, 2\}$ and $\{4, a, b, 2\}$ are circuits of M. Thus $M/4$ has $(2, t_3, t_5, t_4)$ and $(2, a, b, c)$ as 4-fans. Hence $M^*\backslash 4$ has $(t_4, t_5, t_3, 2)$ and $(c, b, a, 2)$ as 4-fans meeting in their coguts elements; a contradiction.

Next suppose that $t_6 \in \{3, e\}$. Then $\{4, t_3, t_5, t_6\}$ is a circuit of M so $M^*\backslash 4$ has (t_4, t_5, t_3, t_6) and $(c, b, a, 2)$ as distinct 4-fans. Moreover, as $t_6 \neq 2$ and $t_4 \neq c$, these 4-fans are disjoint; a contradiction. We conclude that 6.7.6 holds.

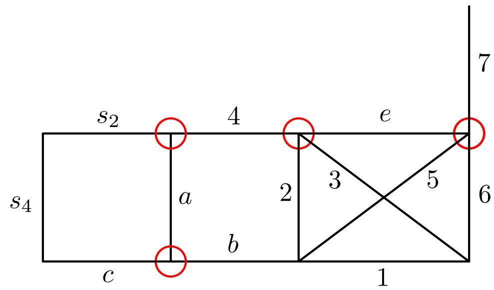

FIGURE 20

By combining 6.7.1, 6.7.2, 6.7.5, and 6.7.6, we deduce that $M^*\backslash c$ is $(4,5,S,+)$-connected. If $M^*\backslash c$ is internally 4-connected, then the theorem holds. Suppose next that $M^*\backslash c$ is not $(4,4,S)$-connected. Then $M^*\backslash c$ has a 5-fan (t_1,t_2,t_3,t_4,t_5) and, by 6.7.3 and 6.7.4, we may assume that this 5-fan is $(4,a,t_3,t_4,t_5)$. Then $(\{b,a,c\},\{t_3,t_4,t_5\},\{a,c,t_3,t_4\})$ is a pretty good bowtie in M^* because $M^*\backslash c$ is $(4,5,S,+)$-connected having an N^*-minor and $\{a,4,t_3\}$ is a triangle of M^* and $M^*\backslash 4$ is $(4,4,S)$-connected having an N^*-minor. We may now assume that $M^*\backslash c$ is $(4,4,S)$-connected having a 4-fan (s_1,s_2,s_3,s_4). Then $\{s_2,s_3,s_4,c\}$ is a cocircuit of M^* so, by orthogonality, we may assume that $a \in \{s_3,s_4\}$. If $a = s_4$, then $(\{s_1,s_2,s_3\},\{a,c,b\},\{s_2,s_3,a,c\})$ is a good bowtie in M^*. Thus, we may assume that $a = s_3$. The cocircuit $\{a,b,2,4\}$ of M^* implies that $4 \in \{s_1,s_2\}$. If $4 = s_2$, then M^* has $\{4,a,c,s_4\}$ as a cocircuit, so $M^*\backslash 4$ has $(2,b,a,c,s_4)$ as a 5-cofan; a contradiction. We deduce that $4 = s_1$. We conclude that

6.7.7. *M contains the structure shown in Figure 20 and every 4-fan of M/c has 4 as its coguts element.*

Next we establish the following.

6.7.8. *$M/4,c$ is sequentially 4-connected having an N-minor.*

Since $M/4$ is $(4,4,S)$-connected having $(2,a,b,c)$ as a 4-fan, $M/4/c$ is 3-connected. Now recall from 6.7.1 that M/c is sequentially 4-connected. Suppose (U,V) is a non-sequential 3-separation of $M/4,c$. Then we may assume that $\{a,s_2,s_4\} \subseteq U$. Then we can add 4 to U to get a non-sequential 3-separation of M/c; a contradiction.

6.7.9. *Every 4-fan of $M/4$ has c as its coguts element.*

To see this, recall that $M/4$ has $(2,a,b,c)$ as a 4-fan. If $M/4$ has another 4-fan, then, by assumption, it is not disjoint from $(2,a,b,c)$ and it does not have the same guts element. Thus it has the same coguts element.

To complete the proof of Lemma 6.7, we now show that

6.7.10. *$M/4/c$ is internally 4-connected.*

Assume the contrary. Then $M/4/c$ has a 4-fan (p_1,p_2,p_3,p_4). Then $\{p_1,p_2,p_3,4\}$, $\{p_1,p_2,p_3,c\}$, or $\{p_1,p_2,p_3,4,c\}$ is a circuit C of M. In the first two cases, (p_1,p_2,p_3,p_4) is a 4-fan in $M/4$ or M/c, respectively. But the only 4-fans of $M/4$ and M/c have c and 4, respectively, as their coguts elements.

Since $\{p_1, p_2, p_3, p_4\}$ avoids $\{4, c\}$, this gives a contradiction. We conclude that $\{p_1, p_2, p_3, 4, c\}$ is a circuit of M. We also know that $\{p_2, p_3, p_4\}$ is a triad of M. Since $\{4, 2, 3, e\}$ is a cocircuit of M, we deduce by orthogonality that $\{p_1, p_2, p_3\}$ meets $\{2, 3, e\}$. But M has no 4-fan, so $\{2, 3, e\}$ avoids $\{p_2, p_3\}$. Hence $p_1 \in \{2, 3, e\}$. As $\{a, b, c\}$ is a triad, $\{a, b\}$ meets $\{p_1, p_2, p_3\}$ and therefore meets $\{p_2, p_3\}$. Since $\{a, b, 2, 4\}$ is a circuit meeting $\{p_2, p_3, p_4\}$, it must contain at least two elements of this triad. But $2 \notin \{p_2, p_3, p_4\}$ and $4 \notin \{p_2, p_3, p_4\}$. Hence $\{a, b\} \subseteq \{p_2, p_3, p_4\}$. But $\{a, b, c\}$ is a triad so $c \in \{p_2, p_3, p_4\}$; a contradiction. We conclude that 6.7.8 holds. This completes the proof of the lemma. □

The theorem follows immediately by combining the various lemmas. □

7. The deletion case with just one 4-fan

In view of the results established in the last three sections, throughout the rest of the proof of Theorem 3.1, we may assume that the following condition holds.

Hypothesis F. *Whenever a triangle of M contains an element t such that $M\backslash t$ is $(4, 4, S)$-connected having an N-minor, $M\backslash t$ has at most one 4-fan; and, whenever a triad of M contains an element t such that M/t is $(4, 4, S)$-connected having an N-minor, M/t has at most one 4-fan.*

We maintain our assumption that $(1, 2, 3, 4)$ is a 4-fan in $M\backslash e$. In this section, we assume that $N \preceq M\backslash e, 1$.

Lemma 7.1. *Let M and N be internally 4-connected matroids with $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Assume that M has a triangle $\{e, f, g\}$ such that $M\backslash e$ is $(4, 4, S)$-connected having $(1, 2, 3, 4)$ as a 4-fan. Suppose that $M\backslash e, 1$ has an N-minor and that Hypothesis F holds. Then one of the following holds.*

- (i) *M has a good bowtie or a pretty good bowtie;*
- (ii) *M has an internally 4-connected minor M' such that $1 \leq |E(M) - E(M')| \leq 2$ and M' has an N-minor;*
- (iii) *$M\backslash 1$ and $M\backslash 1, e$ are $(4, 4, S)$-connected but not internally 4-connected. Furthermore, when $3 = f$, the matroid M has as a substructure one of the configurations shown in Figure 21 and*
  - (a) *in part (a) of the figure, M also has a triangle $\{v_1, v_2, v_3\}$ and a cocircuit $\{v_2, v_3, 3, 1, e\}$, and the elements $1, 2, 3, 4, q_4, g, e, v_1, v_2, v_3$ are distinct; and*
  - (b) *in part (b) of the figure, M also has a triangle $\{u_1, g, u_3\}$ and a cocircuit $\{2, g, u_3, 1, e\}$ and the elements $1, 2, 3, 4, q_2, g, e, u_1, u_3, q_4$ are distinct.*

Proof. Assume that the lemma fails. Then, by Lemma 2.8, M has no triangle containing 4. Thus $\{2, 3\}$ meets $\{f, g\}$. By symmetry, we may assume that $f = 3$. By Lemma 2.10,

7.1.1. $\{1, 2, 3\}$ *and* $\{3, e, g\}$ *are the only triangles of M containing* 3.

Now $M\backslash 1$ is 3-connected since 1 is in a triangle of M. If $M\backslash 1$ is internally 4-connected, then (ii) holds. If $M\backslash 1$ is $(4, 5, S, +)$-connected having a 5-fan, then, by Lemma 2.9, (i) holds. The rest of the proof will treat the two cases when $M\backslash 1$ is

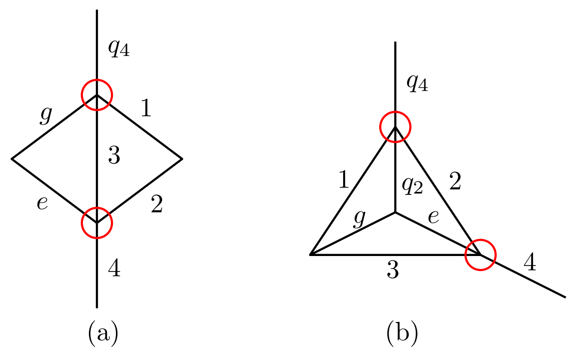

FIGURE 21

$(4,4,S)$-connected but not internally 4-connected and when $M\backslash 1$ is not $(4,5,S,+)$-connected. We begin by showing that the second of these cases does not occur. Thus assume that $M\backslash 1$ is not $(4,5,S,+)$-connected.

7.1.2. *Let (U,V) be a $(4,5,S,+)$-violator of $M\backslash 1$.*
- (i) *If $2 \in U$, then $3 \in V$.*
- (ii) *If $\{2,e\} \subseteq U$, then $\{3,4\} \not\subseteq V$.*

Part (i) is immediate. Now suppose that $\{2,e\} \subseteq U$ and $\{3,4\} \subseteq V$. If $g \in V$, then $e \in \mathrm{cl}(V)$ and $2 \in {\mathrm{cl}^*}_{M\backslash 1}(V \cup e)$. Thus $(U-e-2, V \cup e \cup 2 \cup 1)$ is a 3-separating partition of M. Thus U is a 4-fan or a 5-fan in $M\backslash 1$; a contradiction. We may now suppose that $g \in U$. Then $3 \in \mathrm{cl}(U)$, so $(U \cup 3 \cup 1, V-3)$ is a 3-separation of M. Thus V is a 4-fan in $M\backslash 1$; a contradiction. Hence 7.1.2 holds.

7.1.3. *Let (U,V) be a $(4,5,S,+)$-violator of $M\backslash 1$. Then $|U \cap \{2,3,4,e\}| = 2$.*

By symmetry and 7.1.2(i), we may assume that $|U \cap \{2,3,4,e\}| \in \{2,3\}$. If $|U \cap \{2,3,4,e\}| = 3$, then $(U \cup \{2,3,4,e\} \cup 1, V - \{2,3,4,e\})$ is a 3-separation of M. Thus V is a 4-fan in $M\backslash 1$; a contradiction. Hence 7.1.3 holds.

Next we show the following.

7.1.4. *$M\backslash 1$ has a $(4,5,S,+)$-violator (U,V) with $\{2,4\} \subseteq U$ and $\{3,e,g\} \subseteq V$.*

On combining 7.1.2 and 7.1.3, we get that $M\backslash 1$ has a $(4,5,S,+)$-violator (U,V) with $\{2,4\} \subseteq U$ and $\{3,e\} \subseteq V$. Thus 7.1.4 holds unless $g \in U$. Consider the exceptional case. Then $(U, V-e)$ is a 3-separation of $M\backslash 1, e$. Thus $(U \cup 3 \cup 1 \cup e, V-e-3)$ is 3-separating in M so $|V| \in \{4,5\}$. But $(U-g, V \cup g)$ is also a 3-separation of $M\backslash 1$ and $|U-g| \geq 6$ while either $|V \cup g| \geq 6$ or $V \cup g$ is non-sequential. Hence $(U-g, V \cup g)$ is a $(4,5,S,+)$-violator of $M\backslash 1$. Thus 7.1.3 holds.

7.1.5. *Let (U,V) be a $(4,5,S,+)$-violator of $M\backslash 1$ in which $\{2,4\} \subseteq U$ and $\{3,e,g\} \subseteq V$. Then $V-g$ is a quad in $M\backslash 1$ having g in its guts, so V is the union of two triangles $\{3,e,g\}$ and, say, $\{z_1,z_2,g\}$. Moreover, $\{z_1,z_2\} \cap \{1,2,3,4,e,g\} = \emptyset$.*

Evidently $(U \cup 3 \cup 1, V-e-3)$ is a 3-separating partition of $M\backslash e$. Since $V-e-3$ is not the unique 4-fan of $M\backslash e$, we deduce that $|V-e-3| \leq 3$. Hence $|V| = 4$ or $|V| = 5$. In the first case, V must be a quad in $M\backslash 1$ but this cannot be as V contains the triangle $\{3,e,g\}$. Thus $|V| = 5$ and $V-e$ is 3-separating in $M\backslash 1, e$. Since $(1,2,3,4)$ is a 4-fan of the $(4,4,S)$-connected matroid $M\backslash e$, it follows that

$M\backslash e\backslash 1$ is 3-connected. Thus $V-e$ is a 4-fan or a quad of $M\backslash 1, e$. Since $e \in \mathrm{cl}(V-e)$, in the latter case, e is in a triangle with the two elements of $V-e-3-g$. By orthogonality, 2 or 4 is one of these elements; a contradiction as $\{2,4\} \subseteq U$. Thus $V-e$ is a 4-fan (z_3, z_2, z_1, z_0) of $M\backslash 1, e$. Moreover, since the only triangles of M containing 3 are $\{3, e, g\}$ and $\{3, 2, 1\}$, we deduce that $3 = z_0$. Now $g \in \{z_1, z_2, z_3\}$. If $g \in \{z_2, z_1\}$, then V is a 5-fan in $M\backslash 1$; a contradiction. Thus $g = z_3$ and $V - g$ is a quad in $M\backslash 1$ having g in its guts. We conclude that 7.1.5 holds.

7.1.6. *When $M\backslash 1$ is not $(4,5,S,+)$-connected, either M has a good bowtie, or M has an internally 4-connected minor M' that has an N-minor such that $|E(M) - E(M')| = 1$.*

By 7.1.5, $M\backslash 1$ has a quad $\{e, 3, z_1, z_2\}$. Then $M\backslash 1\backslash e \cong M\backslash 1\backslash 3$. Hence $M\backslash 3$ has an N-minor. Since 3 is in no triad of M, we know that $M\backslash 3$ is 3-connected. If $M\backslash 3$ is internally 4-connected, then 7.1.6 holds with $M' = M\backslash 3$. Thus we may assume that $M\backslash 3$ has a $(4,3)$-violator (X, Y). Then $|X \cap \{1,2\}| = 1$. Suppose (X, Y) is non-sequential. Without loss of generality, we may assume that the cocircuit $\{2, 4, e\}$ is contained in X. Then $\{1, g\} \subseteq Y$. If $\{z_1, z_2\} \subseteq X$, then $(X \cup 3, Y)$ is a $(4,3)$-violator of M; a contradiction. Thus z_1 or z_2 is in Y, so they are both in the closure of Y and we may assume that $\{z_1, z_2\} \subseteq Y$. The cocircuit $\{1, e, z_1, z_2\}$ in $M\backslash 3$ means that we can move e into V and then add 3 to V to get a non-sequential 3-separation of M; a contradiction. We conclude that (X, Y) is sequential.

We now know that $M\backslash 3$ has a 4-fan $(\alpha, \beta, \gamma, \delta)$. Then $\{3, \beta, \gamma, \delta\}$ is a cocircuit in M. Hence, by orthogonality, $\{e, g\}$ meets $\{\beta, \gamma, \delta\}$. If $g \in \{\beta, \gamma, \delta\}$, then orthogonality implies that $\{3, \beta, \gamma, \delta\} \subseteq \{1, 2, 3, e, g, z_1, z_2\}$. Hence $\lambda(\{1, 2, 3, 4, e, g, z_1, z_2\}) \leq 2$; a contradiction. Thus $g \notin \{\beta, \gamma, \delta\}$, so $e \in \{\beta, \gamma, \delta\}$. Suppose that $e \in \{\beta, \gamma\}$. Then orthogonality with $\{2, 3, 4, e\}$ implies, since 4 is in no triangle by Lemma 2.8, that $2 \in \{\alpha, \beta, \gamma\}$. Now orthogonality with the cocircuit $\{1, 3, e, z_1, z_2\}$ implies that $\{\alpha, \beta, \gamma\} \subseteq \{1, 2, e, z_1, z_2\}$. Then $\lambda(\{1, 2, 3, 4, e, g, z_1, z_2\}) \leq 2$; a contradiction. We conclude that $e = \delta$. Thus $(\{\alpha, \beta, \gamma\}, \{3, e, g\}, \{\beta, \gamma, 3, e\})$ is a bowtie and $M\backslash e$ is $(4,4,S)$-connected. Hence M contains a good bowtie and 7.1.6 holds.

We may now assume that $M\backslash 1$ is $(4,4,S)$-connected but not internally 4-connected. Then $M\backslash 1$ has a unique 4-fan (q_1, q_2, q_3, q_4). Thus M has $\{q_2, q_3, q_4, 1\}$ as a cocircuit and, by Lemma 2.8, q_4 is in no triangles of M. As $M\backslash 1, e$ is 3-connected, it has no 2-cocircuits so $e \notin \{q_2, q_3, q_4\}$. By Lemma 2.10 and symmetry, we may assume that either $q_3 = 3$ or $q_3 = 2$. Moreover, 3 is in exactly two triangles of M. Thus, if $q_3 = 3$, then $\{q_1, q_2\} = \{e, g\}$ and, as $M\backslash e$ has a unique 4-fan, $(q_1, q_2) = (e, g)$. Similarly, if $q_3 = 2$, then, by orthogonality, $\{q_3, q_2, q_1\}$ must contain e, so $q_1 = e$.

7.1.7. *Let (U, V) be a $(4,3)$-violator of $M\backslash 1, e$ with 3 in V. Then $\{2, g\} \subseteq U$.*

To see this, suppose first that $2 \in V$. Then $(U, V \cup 1)$ is a $(4,3)$-violator of $M\backslash e$ in which U is not the unique 4-fan and so is not a 4-element 3-separating set; a contradiction. Thus $2 \in U$.

Now suppose that $g \in V$. Then $(U, V \cup e)$ is a $(4,3)$-violator of $M\backslash 1$. Thus U is a 4-fan of $M\backslash 1$ containing 2. But 2 is in at most two triangles of M, namely $\{1, 2, 3\}$ and possibly one containing $\{2, e\}$. As neither of these triangles is contained in U, we deduce that 2 is the coguts element of the 4-fan U. Because 2 is in a triangle of M, we have a contradiction to Lemma 2.8. We conclude that 7.1.7 holds.

7.1.8. *The matroid $M\backslash 1, e$ is $(4,4,S)$-connected. Moreover, (iii)(a) or (iii)(b) of the lemma holds.*

Let (U, V) be a $(4,3)$-violator of $M\backslash 1, e$. By 7.1.7, we may assume that $\{2, g\} \subseteq U$ and $3 \in V$.

Suppose first that $4 \in V$. Clearly $(U - 2, V \cup 2 \cup 1)$ is a 3-separation of $M\backslash e$ in which $U - 2$ is not its unique 4-element 3-separating set, $\{1,2,3,4\}$. Thus $|U - 2| = 3$. Hence U is a 4-fan $(u_1, u_2, u_3, 2)$ of $M\backslash 1, e$ where $g \in \{u_1, u_2, u_3\}$. Now $\{u_2, u_3, 2, 1\}$ is a cocircuit of $M\backslash e$ by orthogonality. If $u_1 = g$, then $\{u_2, u_3, 1, 2\}$ is a cocircuit of M so M has a good bowtie; a contradiction. Hence we may assume, by symmetry, that $u_2 = g$. Moreover, $\{g, u_3, 1, 2, e\}$ is a cocircuit of M.

Suppose $q_3 = 3$. Then $(q_1, q_2) = (e, g)$. Moreover, $\{u_1, u_2, u_3\}$ contains $\{g, q_4\}$ as it must contain a second element of the cocircuit $\{1, 3, g, q_4\}$. Thus $M\backslash 1$ has a 5-fan; a contradiction. We deduce that $q_3 = 2$. Then M contains the structure in Figure 21(b) and has $\{u_1, g, u_3\}$ as a circuit and $\{g, u_3, 1, 2, e\}$ as a cocircuit.

We now show that $1, 2, 3, 4, q_2, g, e, u_1, u_3, q_4$ are distinct. To see this, first observe that, as $(\{u_1, u_3, 2, g\}, V)$ is a partition of $E(M\backslash 1, e)$ with $\{3, 4\}$ contained in V, the elements $1, 2, 3, 4, g, e, u_1, u_3$ are distinct. Possibly q_2 or q_4 is one of these elements. But, as $(e, q_2, 2, q_4)$ is a 4-fan of $M\backslash 1$, the elements $1, 2, 3, e, g, q_2, q_4$ are distinct. It remains to consider whether $\{q_2, q_4\}$ meets $\{4, u_1, u_3\}$. By Lemma 2.8, neither 4 nor q_4 is in a triangle. This leaves only the possibilities that $4 = q_4$ or $q_2 \in \{u_1, u_3\}$. In the first case, as M has $\{1, q_2, 2, q_4\}$ and $\{2, 3, 4, e\}$ as cocircuits, their symmetric difference, $\{1, q_2, 3, e\}$ is a cocircuit. Since it is also a circuit, we have a contradiction. Thus $4 \neq q_4$. If $q_2 \in \{u_1, u_3\}$, then, as $\{q_2, g, 1\}$ is a triangle, it must equal $\{u_1, g, u_3\}$. Thus $1 \in \{u_1, u_3\}$; a contradiction. Thus $1, 2, 3, 4, q_2, g, e, u_1, u_3, q_4$ are indeed distinct. Hence (iii)(b) of the lemma holds.

We may now assume that $4 \in U$. Then $(U \cup 3 \cup 1 \cup e, V - 3)$ is a 3-separation of M. Hence $|V| = 4$ so V is a 4-fan $(v_1, v_2, v_3, 3)$ of $M\backslash 1, e$. It follows that $M\backslash 1, e$ is $(4,4,S)$-connected. As q_4 is not in a triangle, it is not in V so the elements $1, 2, 3, 4, q_4, g, e, v_1, v_2, v_3$ are distinct provided $q_4 \neq 4$. Moreover, by orthogonality, $\{v_2, v_3, 3, 1, e\}$ is a cocircuit of M. Now, $q_4 \neq 4$ otherwise $\lambda(\{1, 2, 3, 4, e, g\}) \leq 2$; a contradiction. Thus (iii)(a) of the lemma holds. We conclude that 7.1.8 holds and this completes the proof of the lemma. □

Lemma 7.2. *If, in Lemma 7.1, part (iii)(a) holds, then either M has an internally 4-connected matroid M' such that $1 \leq |E(M) - E(M')| \leq 2$ and M' has an N-minor, or M has a good bowtie.*

Proof. Assume that part (iii)(a) of Lemma 7.1 holds but that the lemma is false. Then, in addition to the structure shown in Figure 21(a), we know that M has $\{v_1, v_2, v_3\}$ as a triangle and has $\{v_2, v_3, 3, 1, e\}$ as a cocircuit. We shall apply [1, Lemma 8.1]. We note first that (i) of that lemma does not hold, otherwise M has a triangle containing 4, a contradiction to Lemma 2.8. The same lemma also implies that M has no triangle containing q_4 since $(e, g, 3, q_4)$ is a 4-fan of $M\backslash 1$.

Now $M\backslash 1, e$ has an N-minor and has $(v_1, v_2, v_3, 3)$ as a 4-fan. Thus either $M\backslash 1, e/3$, or $M\backslash 1, e, v_1$ has an N-minor.

7.2.1. *$M\backslash 1, e, v_1$ has no N-minor.*

Assume the contrary. We may also assume that $M\backslash v_1$ is not $(4,3)$-connected, otherwise the lemma holds. We show next that

7.2.2. *$M\backslash v_1$ is sequentially 4-connected.*

Since $\{v_1, v_2, v_3\}$ is a triangle, $M\backslash v_1$ is 3-connected. Suppose $M\backslash v_1$ has a non-sequential 3-separation (U, V). Then we may assume that $\{1, 2, 3\} \subseteq U$. Since $|U \cap \{v_2, v_3\}| = 1$, we may also suppose that $v_2 \in U$ and $v_3 \in V$. Now $e \in V$ otherwise we can move v_3 into U and then add v_1 to U to get a non-sequential 3-separation of M, which cannot exist. Similarly, the triangle $\{e, g, 3\}$ implies that $g \in V$ or we can move e into U. The cocircuit $\{1, 3, g, q_4\}$ now implies that $q_4 \in V$ otherwise we can move g into U, contrary to what we already know. Therefore we have $\{v_2, 1, 2, 3\} \subseteq U$ and $\{v_3, e, g, q_4\} \subseteq V$. Thus we can move 3, then 1, and then v_2 into V; a contradiction. We conclude that 7.2.2 holds.

Next we establish a useful property of 4-fans in $M\backslash v_1$.

7.2.3. *In every 4-fan of $M\backslash v_1$, the coguts element is in $\{v_2, v_3\}$ but no internal element is in this set.*

Let $(\alpha, \beta, \gamma, \delta)$ be a 4-fan in $M\backslash v_1$. Then $\{v_1, \beta, \gamma, \delta\}$ is a cocircuit of M, so $\{v_2, v_3\}$ meets $\{\beta, \gamma, \delta\}$. Suppose v_2 or v_3 is an internal element in $(\alpha, \beta, \gamma, \delta)$. Without loss of generality, we may suppose that $v_2 = \gamma$. Orthogonality with the cocircuit $\{1, 3, e, v_2, v_3\}$ implies that $\{\alpha, \beta\}$ meets $\{1, 3, e\}$.

The next step in the proof of 7.2.3 is to show the following.

7.2.4. *Both $\{1, g, v_2\}$ and $\{2, e, v_2\}$ are triangles of M.*

First observe that, since $\{1, 2, e, g\}$ is a circuit of M, it suffices to prove that at least one of $\{1, g, v_2\}$ and $\{2, e, v_2\}$ is a triangle. By Lemma 2.8, $3 \notin \{\alpha, \beta, \gamma\}$ and if $e \in \{\alpha, \beta, \gamma\}$, then $\{\alpha, \beta, \gamma\} = \{2, e, v_2\}$ and 7.2.4 holds. Finally, if $1 \in \{\alpha, \beta, \gamma\}$, then, by orthogonality with $\{1, 3, g, q_4\}$, it follows, since q_4 is not in a triangle, that $\{\alpha, \beta, \gamma\} = \{1, g, v_2\}$. We conclude that 7.2.4 holds.

By orthogonality between the cocircuit $\{\beta, \delta, v_1, v_2\}$ and the triangles $\{1, g, v_2\}$ and $\{2, e, v_2\}$, we deduce that $\{\beta, \delta\}$ meets $\{1, g\}$ and $\{2, e\}$. Orthogonality implies that $\{\beta, \delta, v_1, v_2\}$ is $\{1, 2, v_1, v_2\}$ or $\{e, g, v_1, v_2\}$, and, as $M\backslash e$ and $M\backslash 1$ are $(4, 4, S)$-connected, M contains a good bowtie; a contradiction. We conclude that every 4-fan of $M\backslash v_1$ has v_2 or v_3 as its coguts element, but not as an internal element, that is, 7.2.3 holds.

7.2.5. *$M\backslash v_1$ is $(4, 4, S)$-connected.*

This holds by 7.2.3 unless $M\backslash v_1$ has a 5-cofan $(v_2, \alpha, \beta, \gamma, v_3)$ In the exceptional case, $\{v_1, v_2, v_3, \alpha, \beta, \gamma\}$ is a $(4, 3)$-violator of M; a contradiction.

As we have assumed that $M\backslash v_1$ is not internally 4-connected, we know that $M\backslash v_1$ has a 4-fan. By 7.2.3, this 4-fan has v_2 or v_3 as its coguts element. Thus M has a good bowtie; a contradiction. We conclude that 7.2.1 holds.

Next we establish the following.

7.2.6. *$N \preceq M\backslash 1, e/3$, so $N \preceq M\backslash 2, e/4$ and $N \preceq M\backslash 2, g/3$.*

The fact that $N \preceq M\backslash 1, e/3$ is an immediate consequence of 7.2.1. Now observe that $M\backslash 1, e/3 \cong M/3\backslash 2, e \cong M\backslash 2, e/4$. Since $M/3\backslash 2, e \cong M\backslash 2, g/3$, 7.2.6 holds.

Next we show that

7.2.7. *$M/4$ is not internally 4-connected.*

Suppose $M/4$ is internally 4-connected. By 7.2.6, $N \preceq M/4$, so we obtain the contradiction that the lemma holds. Thus 7.2.7 holds.

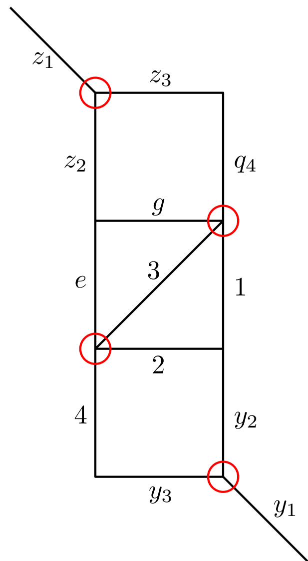

FIGURE 22

We are interested in applying [1, Lemma 8.1] with the element labelled 4 here playing the role of the element labelled 6 in the original lemma. We now know that neither (i) nor (ii) of that lemma holds. The rest of the proof of Lemma 7.2 will involve showing that none of [1, Lemma 8.1] holds.

First suppose that (iv) of that lemma holds. Then M has a circuit $\{3, 4, q_4, y_2\}$ and a triad $\{q_4, y_1, y_2\}$ where $\{y_1, y_2\}$ avoids $\{1, 2, 3, 4, e, g, q_4\}$. As $\{v_2, v_3, 3, 1, e\}$ is a cocircuit of M, by orthogonality, $\{q_4, 4, y_2\}$ meets $\{v_2, v_3, 1, e\}$. Thus $y_2 \in \{v_2, v_3\}$, so M has a 4-fan; a contradiction.

Next suppose that Lemma 8.1(v) of [1] holds. Then M has a circuit $\{e, 4, x_2, x_3\}$ and a triad $\{x_1, x_2, x_3\}$ where $\{x_1, x_2, x_3\}$ avoids $\{1, 2, 3, 4, e, g, q_4\}$ except that, possibly, $x_1 = q_4$. Now $\{v_2, v_3, 3, 1, e\}$ is a cocircuit of M, so $\{v_2, v_3, 3, 1\}$ meets $\{4, x_2, x_3\}$. But $\{x_2, x_3\}$ avoids $\{3, 1\}$ and $\{v_2, v_3, 3, 1\}$ avoids 4. Thus $\{v_2, v_3\}$ meets $\{x_2, x_3\}$, so M has a 4-fan; a contradiction. Hence Lemma 8.1(v) of [1] does not hold.

We deduce now that Lemma 8.1(iii) of [1] holds. By symmetry, Lemma 8.1(iii) of [1] also holds when we interchange the roles of 4 and q_4. It follows that M has a 4-circuit $\{2, 4, y_2, y_3\}$ and a triad $\{y_1, y_2, y_3\}$ where $1, 2, 3, 4, e, g, q_4, y_1, y_2, y_3$ are distinct except that possibly $y_1 = q_4$. In addition, M has a 4-circuit $\{g, q_4, z_2, z_3\}$ and a triad $\{z_1, z_2, z_3\}$ where $1, 2, 3, 4, e, g, q_4, z_1, z_2, z_3$ are distinct except that possibly $z_1 = 4$. Combining all this, we see that M contains the structure shown in Figure 22.

Next we show the following.

7.2.8. *If $h \in \{y_2, y_3\} \cap \{z_2, z_3\}$, then M has no 4-cocircuit that contains $\{2, g, h\}$ and meets $\{1, 3\}$.*

Without loss of generality, let $h = y_2 = z_2$. Suppose that M has a cocircuit C^* that contains $\{2, g, h\}$ and meets $\{1, 3\}$. Then orthogonality implies that $\{y_1, y_3\}$ meets $\{g, q_4, z_3\}$. If $y_3 \in \{g, q_4, z_3\}$, then $4 \in \mathrm{cl}(\{1, 2, 3, e, g, q_4, z_2, z_3\})$, so $\lambda(\{1,2,3,4,e,g,q_4,z_2,z_3\}) \leq 2$; a contradiction. Thus $y_1 \in \{g,q_4,z_3\}$. Then $\lambda(\{1,2,3,4,e,g,q_4,y_1,z_2,z_3\}) \leq 2$; a contradiction. We conclude that 7.2.8 holds.

Next observe that, by 7.2.6, $N \preceq M/3\backslash 2,g$, so $N \preceq M\backslash 2,g$. We shall complete the proof of the lemma by showing that $M\backslash 2,g$ is internally 4-connected. First we observe that

7.2.9. *$M\backslash 2,g$ is 3-connected.*

To see this, note that $M\backslash 2$ is 3-connected as 2 is in a triangle of M. Now g is in the guts of a 4-fan in $M\backslash 2$. Thus we may assume that g is in a triad of $M\backslash 2$ otherwise 7.2.9 holds. Hence M has a 4-cocircuit D^* containing $\{2,g\}$. By orthogonality, D^* must meet $\{1,3\},\{4,y_2,y_3\}$, and $\{q_4,z_2,z_3\}$. Thus there is an element h in $\{y_2,y_3\} \cap \{z_2,z_3\}$ such that D^* contains $\{2,g,h\}$ and meets $\{1,3\}$. This contradiction to 7.2.8 implies that 7.2.9 holds.

7.2.10. *$M\backslash 2,g$ is sequentially 4-connected.*

Suppose (U,V) is a non-sequential 3-separation of $M\backslash 2,g$. We may assume that $\{1,3,q_4\} \subseteq U$. If e or 4 is in U, then $(U \cup 2 \cup g, V)$ is a $(4,3)$-violator of M; a contradiction. Thus $\{e,4\} \subseteq V$. Now $\{y_1,y_2,y_3\} \not\subseteq U$ otherwise $4 \in \mathrm{cl}(U)$ and we can move 4 into U to get a contradiction. Thus we may assume that $\{y_1,y_2,y_3\} \subseteq V$. Then, in $M\backslash 2,g$, we see that 3 is in the coclosure of V, so $(U-3, V \cup 3 \cup 2 \cup g)$ is a $(4,3)$-violator of M; a contradiction. Thus 7.2.10 holds.

We may now assume that $M\backslash 2,g$ has a 4-fan $(\alpha,\beta,\gamma,\delta)$ otherwise the lemma holds. Then M has a cocircuit C^* such that $\{\beta,\gamma,\delta\} \subsetneqq C^* \subseteq \{2,g,\beta,\gamma,\delta\}$. If $2 \in C^*$, then orthogonality implies that $\{1,3\}$ and $\{4,y_2,y_3\}$ meet C^*. Now no element in $\{4,y_2,y_3\}$ is in a triangle of M, so $\delta \in \{4,y_2,y_3\}$ and, without loss of generality, $\gamma \in \{1,3\}$. Orthogonality between $\{\alpha,\beta,\gamma\}$ and $\{1,3,q_4\}$ implies that 4 or q_4 is in a triangle of M; a contradiction to Lemma 2.8. We deduce that $2 \notin C^*$. Since a symmetric argument establishes that $g \notin C^*$, we obtain the contradiction that $C^* = \{\beta,\gamma,\delta\}$. We conclude that Lemma 7.2 holds. □

Lemma 7.3. *If, in Lemma 7.1, part (iii)(b) holds, then either M has an internally 4-connected matroid M' such that $1 \leq |E(M)-E(M')| \leq 2$ and M' has an N-minor, or M has a good bowtie.*

Proof. Assume that part (iii)(b) of Lemma 7.1 holds but that the lemma is false. Then, in addition to the structure shown in Figure 21(b), we know that M has $\{u_1,g,u_3\}$ as a triangle and has $\{2,g,u_3,1,e\}$ as a cocircuit.

7.3.1. *None of $1,2,3,4,e,q_2$, or q_4 is in any triangles apart from those shown in Figure 21(b).*

To see this, first we observe that, by Lemma 2.8, M has no triangle containing 4 and no triangle containing q_4. The cocircuits $\{1,q_4,2,q_2\}$ and $\{2,3,4,e\}$ along with the existing triangles now guarantee that none of $1,2,3,e$, or q_2 is any triangles apart from those shown in the figure. Thus 7.3.1 holds.

Since $M\backslash 1,e$ has $(u_1,g,u_3,2)$ as a 4-fan, either $M\backslash 1,e/2$ or $M\backslash 1,e,u_1$ has an N-minor. First we show that

7.3.2. *$M\backslash 1,e,u_1$ has no N-minor.*

Assume the contrary. Then $M\backslash u_1$ is not internally 4-connected.

7.3.3. *$M\backslash u_1$ is sequentially 4-connected.*

Since u_1 is in a triangle, $M\backslash u_1$ is 3-connected. Let (U, V) be a non-sequential 3-separation of $M\backslash u_1$. Then we may assume that $\{1, g, q_2\} \subseteq U$. It follows that $u_3 \in V$. Suppose that $2 \in U$. Then we may assume that $e \in U$. It follows that we can move u_3 into U; a contradiction. We may now assume that $2 \in V$. Then both e and 3 are in V otherwise we can move 2 into U. Now V spans $\{1, g, q_2\}$ so all these elements can be moved into V to obtain a contradiction. Hence 7.3.3 holds.

Next we establish a useful property of 4-fans in $M\backslash u_1$.

7.3.4. *In every 4-fan of $M\backslash u_1$, the coguts element is in $\{g, u_3\}$ but no internal element is in this set.*

Let $(\alpha, \beta, \gamma, \delta)$ be a 4-fan in $M\backslash u_1$. Then $\{u_1, \beta, \gamma, \delta\}$ is a cocircuit in M and orthogonality with $\{g, u_1, u_3\}$ implies that $\{\beta, \gamma, \delta\}$ meets $\{g, u_3\}$. Suppose that g or u_3 is an internal element of the fan; without loss of generality, let it be γ. Then $\{\alpha, \beta, \gamma\}$ meets $\{1, 2, e, g, u_3\}$. Thus $\{\alpha, \beta, \gamma\}$ meets $\{1, 2, e\}$, and 7.3.1 implies that $\{\alpha, \beta, \gamma\}$ is $\{1, g, q_2\}$ or $\{3, e, g\}$. Hence $\gamma = g$ and orthogonality with the triangles $\{1, g, q_2\}$ or $\{3, e, g\}$ implies that $\{\beta, \delta\}$ meets $\{3, e\}$ and $\{1, q_2\}$. Orthogonality with $\{1, 2, 3\}$ implies that $\{u_1, \beta, \gamma, \delta\}$ is $\{1, 3, g, u_1\}$ or $\{e, g, q_2, u_1\}$. Then $(\{1, 2, 3\}, \{g, u_1, u_3\}, \{1, 3, g, u_1\})$ or $(\{2, e, q_2\}, \{g, u_1, u_3\}, \{e, q_2, g, u_1\})$ is a good bowtie, since $M\backslash 1$ and $M\backslash e$ are $(4, 4, S)$-connected. This contradiction implies that $\{g, u_3\}$ avoids $\{\gamma, \delta\}$. Thus $\delta \in \{g, u_3\}$ and 7.3.4 holds.

Now $M\backslash u_1$ does not have a 5-cofan of the form $(g, \alpha, \beta, \gamma, u_3)$ otherwise we obtain the contradiction that $\{g, u_1, u_3, \alpha, \beta, \gamma\}$ is a $(4, 3)$-violator of M. We conclude that $M\backslash u_1$ is $(4, 4, S)$-connected. As it is not internally 4-connected, it has a 4-fan whose coguts element is in $\{g, u_3\}$. Then Lemma 2.8 implies that M has a good bowtie. This contradiction completes the proof of 7.3.2.

As 7.3.2 holds, it follows that $M\backslash 1, e/2$ has an N-minor. Thus $N \preceq M/2\backslash 3, q_2$, so $N \preceq M\backslash 3\backslash q_2$. To complete the proof of Lemma 7.3, we shall prove the following.

7.3.5. *$M\backslash 3\backslash q_2$ is internally 4-connected.*

As 3 is in no triad in M, it follows that $M\backslash 3$ is 3-connected. Now $M\backslash 3$ has q_2 in the guts of a 4-fan, so either $M\backslash 3\backslash q_2$ is 3-connected, or q_2 is in a triad of $M\backslash 3$. The latter implies, by orthogonality, that q_2 is in a triad of $M\backslash 3$ meeting $\{1, g\}$ and $\{2, e\}$, so $\lambda_M(\{1, 2, 3, e, g, q_2\}) \leq 2$; a contradiction. We conclude that $M\backslash 3\backslash q_2$ is 3-connected.

Suppose that (U, V) is a non-sequential 3-separation of $M\backslash 3\backslash q_2$. Without loss of generality, $\{2, 4, e\} \subseteq U$. As U does not span $\{3, q_2\}$, we must have $\{1, g\} \subseteq V$. Moreover, $q_4 \in V$ otherwise we can move 1 into V. Then $(U - 2, V \cup 2 \cup 3 \cup q_2)$ is a $(4, 3)$-violator of M; a contradiction. Thus $M\backslash 3\backslash q_2$ is sequentially 4-connected.

Let $(\alpha, \beta, \gamma, \delta)$ be a 4-fan in $M\backslash 3\backslash q_2$. Then M has a cocircuit C^* such that $\{\beta, \gamma, \delta\} \subseteq C^* \subseteq \{3, q_2, \beta, \gamma, \delta\}$. Since M has no 4-fan, we know that C^* contains 3 or q_2. As the next step towards proving 7.3.5, we show that

7.3.6. *$g \in \{\beta, \gamma\}$ and $\delta \in \{1, 2, e\}$. Moreover, $\{1, 2, e\}$ avoids $\{\alpha, \beta, \gamma\}$.*

It is an immediate consequence of 7.3.1 that $M\backslash 3\backslash q_2$ has no triangle meeting $\{1, 2, e\}$, so $\{1, 2, e\}$ avoids $\{\alpha, \beta, \gamma\}$. Now suppose that $\{3, q_2\} \subseteq C^*$. Then the triangles in $M|\{1, 2, 3, e, g, q_2\}$ imply that $\{\beta, \gamma, \delta\}$ meets $\{1, 2\}, \{2, e\}, \{1, g\}$, and $\{e, g\}$. Thus $g \in \{\beta, \gamma\}$ and $\delta = 2$, so 7.3.6 holds. We may now assume that $|C^* \cap \{3, q_2\}| = 1$. By orthogonality with the circuits $\{1, 3, e, q_2\}$ and $\{2, 3, g, q_2\}$, it follows that $\{\beta, \gamma, \delta\}$ meets $\{1, e\}$ and $\{2, g\}$. Thus $\delta \in \{1, e\}$ and $g \in \{\beta, \gamma\}$. We conclude that 7.3.6 holds.

Since M has $\{1, 2, e, g, u_3\}$ as a cocircuit and g is in the triangle $\{u_1, g, u_3\}$, orthogonality implies that $u_3 \in \{\alpha, \beta, \gamma\}$, so this triangle is $\{g, u_1, u_3\}$. Thus $\{\beta, \gamma\} = \{g, u_i\}$ for some i in $\{1, 3\}$. Now the cocircuit C^* is contained in $\{g, u_i, 3, q_2, \delta\}$ and $\delta \in \{1, 2, e\}$. Thus $\lambda_M(\{1, 2, 3, e, g, q_2, u_1, u_3\}) \leq 2$; a contradiction. Thus 7.3.5 holds.

Since $M\backslash 3\backslash q_2$ is internally 4-connected and has an N-minor, we have a contradiction that completes the proof of Lemma 7.3. □

On combining Lemmas 7.1, 7.2, and 7.3, we immediately obtain the following theorem, the main result of the section.

Theorem 7.4. *Let M and N be internally 4-connected matroids with $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Assume that M has a triangle $\{e, f, g\}$ such that $M\backslash e$ is $(4, 4, S)$-connected having $(1, 2, 3, 4)$ as a 4-fan. Suppose that $M\backslash e, 1$ has an N-minor and that Hypothesis F holds. Then one of the following holds.*

(i) *M has a good bowtie or a pretty good bowtie; or*
(ii) *M has an internally 4-connected matroid M' such that $1 \leq |E(M) - E(M')| \leq 3$ and M' has an N-minor.*

8. The contraction case with just one 4-fan

In view of the theorem in the last section combined with the results from the three previous sections, we may strengthen Hypothesis F and assume the following for the rest of the proof of Theorem 3.1.

Hypothesis D. *Whenever a triangle of M contains an element t such that $M\backslash t$ is $(4, 4, S)$-connected having an N-minor, $M\backslash t$ has a unique 4-fan (t_1, t_2, t_3, t_4) and $M\backslash t\backslash t_1$ has no N-minor; and, whenever a triad of M contains an element t such that M/t is $(4, 4, S)$-connected having an N-minor, M/t has a unique 4-fan (t_1, t_2, t_3, t_4) and $M/t/t_4$ has no N-minor.*

As before, we are assuming that M has a triangle T containing an element e such that $N \preceq M\backslash e$ and $(1, 2, 3, 4)$ is a 4-fan of $M\backslash e$. Moreover, by Hypothesis D, the matroid $M\backslash e\backslash 1$ has no N-minor. Thus, by Lemma 2.5, $N \preceq M\backslash e/4$ and $M\backslash e/4$ is $(4, 4, S)$-connected. By orthogonality, T meets $\{2, 3, 4\}$ and we shall assume that $T = \{3, e, g\}$.

Lemma 8.1. *The matroid $M/4$ is $(4, 4, S)$-connected unless*

(i) *M has a cocircuit $\{u_1, u_2, u_3\}$ that is disjoint from $\{1, 2, 3, 4, e, g\}$ such that each of $\{2, e, u_2, u_3\}$, $\{4, e, u_1, u_2\}$, and $\{2, 4, u_1, u_3\}$ is a circuit; or*
(ii) *$M/4$ has a quad $\{e, g, t_1, t_2\}$ with the element 3 in the guts; $\{3, e, g\}$ and $\{3, t_1, t_2\}$ are circuits of $M/4$; and $\{e, g, t_1, t_2, 4\}$ and $\{3, 4, t_1, t_2\}$ are circuits of M.*

Proof. Let (U, V) be a $(4, 4, S)$-violator of $M/4$. Then neither U nor V contains $\{2, 3, e\}$. Hence we may assume that one of the following holds.

(a) $\{2, 3\} \subseteq U$ and $e \in V$;
(b) $\{2, e\} \subseteq U$ and $3 \in V$; or
(c) $\{3, e\} \subseteq U$ and $2 \in V$.

8.1.1. *If (a) or (b) holds, then $g \in V$.*

Suppose that $g \in U$. Then $(U \cup \{e,3\} \cup 4, V - \{e,3\})$ is a 3-separation of M. Thus $|V| = 4$ and V is sequential in $M/4$; a contradiction. Hence 8.1.1 holds.

Now consider (a). Then $g \in V$. Clearly $(U, V-e)$ is a 3-separation of $M/4\backslash e$, so $(U \cup 4, V-e)$ is a 3-separation of $M\backslash e$. As $M\backslash e$ is $(4,4,S)$-connected, $|V-e| \leq 4$. If $|V-e| = 4$, then we have a contradiction since $V-e$ is not the unique 4-fan of $M\backslash e$. Hence $|V| = 4$. As $M/4$ has (U,V) as a $(4,4,S)$-violator, we deduce that V is a quad $\{e,g,t_1,t_2\}$ having 3 in its guts. Since $\{3,e,g\}$ is a circuit of $M/4$, we deduce that $\{3,t_1,t_2\}$ is a circuit of $M/4$. As $\{e,g,t_1,t_2,4\}$ is a circuit of M, so is $\{3,4,t_1,t_2\}$ so (ii) of the lemma holds.

Next consider (b). By 8.1.1, $g \in V$. Suppose $1 \in U$. Then $(U \cup 3 \cup 4, V-3)$ is a 3-separation of M, so V is a 4-fan in $M/4$; a contradiction. Thus $1 \in V$. Then $\{2,e\} \subseteq \mathrm{cl}_{M/4}(V)$, so U is a 5-fan $(e,u_2,u_1,u_3,2)$ in $M/4$. Hence $\{u_1,u_2,u_3\}$ is a triad of M and $\{e,4,u_1,u_2\}$ and $\{u_1,u_3,2,4\}$ are circuits of M. Thus $\{e,2,u_2,u_3\}$ is also a circuit of M and (i) holds.

Finally, consider (c). Then $1 \notin U$ otherwise $(U \cup 2 \cup 4, V-2)$ is 3-separating in M, so V is a 4-fan of $M/4$; a contradiction. Suppose $g \in V$. Then $\{3,e\} \subseteq \mathrm{cl}_{M/4}(V)$, so $(U - \{3,e\}, V \cup \{3,e\} \cup 4)$ is 3-separating in M. Thus $M/4$ has U as a 5-fan $(3,u_1,u_2,u_3,e)$ with g in the guts. Therefore $M/4\backslash e$ has a 5-fan; a contradiction. We deduce that $g \notin V$. Thus $g \in U$. Then $(U-3, V \cup 3)$ is a 3-separation of $M/4$. We have reduced to a case symmetric to (a) unless $(U-3, V\cup 3)$ is not a $(4,4,S)$-violator of $M/4$. In the exceptional case, U is a 5-fan in $M/4$ having 3 in the guts. But U also contains the triangle $\{3,e,g\}$. Thus $\{e,g\}$ is contained in a triad of $M/4$ and hence of M; a contradiction. □

Lemma 8.2. *Suppose $M/4$ has a quad $\{e,g,t_1,t_2\}$ and has $\{3,e,g\}$ and $\{3,t_1,t_2\}$ as circuits. Then $M\backslash g$ is internally 4-connected having an N-minor, or M has a good bowtie.*

Proof. As $\{e,g,t_1,t_2\}$ is a quad of $M/4$, by [2, Lemma 2.2], $M/4\backslash e \cong M/4\backslash g$. Thus, as $M/4\backslash e$ has an N-minor, so does $M\backslash g$. Since g is in a triangle of M, it follows that $M\backslash g$ is 3-connected. We show next that

8.2.1. *$M\backslash g$ is sequentially 4-connected.*

Suppose (U,V) is a non-sequential 3-separation of $M\backslash g$. Without loss of generality, we may assume that $\{1,2,3\} \subseteq U$. If $e \in U$, then $(U \cup g, V)$ is a non-sequential 3-separation of M; a contradiction. We may now assume that the cocircuit $\{e,t_1,t_2\}$ of $M\backslash g$ is contained in V. If $4 \in U$, then $(U \cup e \cup g, V-e)$ is a non-sequential 3-separation of M; a contradiction. Thus $4 \in V$ and, since $\{3,4,t_1,t_2\}$ is a circuit, $(U-3, V \cup 3 \cup g)$ is a non-sequential 3-separation of M; a contradiction. We conclude that 8.2.1 holds.

Now suppose that $M\backslash g$ is not internally 4-connected. Then it has a 4-fan $(\alpha,\beta,\gamma,\delta)$. As $\{g,\beta,\gamma,\delta\}$ is a cocircuit of M, it follows that $\{e,3\}$ meets $\{\beta,\gamma,\delta\}$ in a single element. Since $\{e,3,g\}$ is a triangle of M but $g \notin \{\alpha,\beta,\gamma\}$, it follows that $|\{e,3\} \cap \{\alpha,\beta,\gamma\}| \leq 1$.

Suppose that $e = \gamma$. Then, by orthogonality, $\{2,3,4\}$ meets $\{\alpha,\beta\}$. By Lemma 2.8, $4 \notin \{\alpha,\beta\}$ so $2 \in \{\alpha,\beta\}$. Suppose $2 = \beta$. Then, as $\{1,2,3\}$ is a circuit of M, orthogonality implies that $\delta = 1$. Thus $\{2,e,1,g\}$ is a cocircuit of M, so $(3,1,2,g)$ is a 4-fan in $M\backslash e$; a contradiction to Hypothesis D. We may now assume that $2 = \alpha$. Then, by orthogonality and symmetry, we may assume that $\beta = t_1$. Hence $\{t_1, 2, e\} \triangle \{1, 2, 3\} \triangle \{3, e, g\}$ is a circuit, $\{1, g, t_1\}$, of M, so $M\backslash e$ has a second 4-fan, $(1, t_1, g, t_2)$; a contradiction to Hypothesis D. Thus $e \neq \gamma$.

Suppose next that $3 = \gamma$. Then $\{\alpha, \beta\}$ meets $\{2, 4, e\}$. Lemma 2.8 implies that 4 is not in a triangle, and $e \notin \{\alpha, \beta, \gamma\}$, so $2 \in \{\alpha, \beta\}$. Thus $\{\alpha, \beta\} = \{1, 2\}$. Orthogonality between $\{\beta, \gamma, \delta, g\}$ and the circuit $\{3, 4, t_1, t_2\}$ of M implies that $\delta \in \{4, t_1, t_2\}$, so $\lambda_M(\{1, 2, 3, 4, e, g, t_1, t_2\}) \leq 2$; a contradiction. We conclude that $3 \neq \gamma$. Hence $\delta \in \{2, 3\}$. Thus every 4-fan in $M\backslash g$ has e or 3 as its coguts element, but not as an internal element. Hence $M\backslash g$ has no 5-fans. Moreover, if $M\backslash g$ has a 5-cofan F, then $\{e, 3\} \subseteq F$, so $F \cup g$ is a 6-element 3-separating set in M; a contradiction. Therefore $M\backslash g$ is $(4, 4, S)$-connected. Moreover, $M\backslash g$ has a 4-fan with e or 3 in its coguts, and Lemma 2.8 implies that M has a good bowtie. □

By the last two lemmas, we need to consider when $M/4$ is $(4, 4, S)$-connected but not internally 4-connected and when (ii) of Lemma 8.1 holds. We shall break the remainder of the argument up into the following three cases.

(A) $M/4$ is $(4, 4, S)$-connected and every 4-fan of $M/4$ has e as its guts end;
(B) $M/4$ is $(4, 4, S)$-connected and $M/4$ has a 4-fan that does not have e its guts end; and
(C) M has a cocircuit $\{u_1, u_2, u_3\}$ that is disjoint from $\{1, 2, 3, 4, e, g\}$ such that each of $\{2, e, u_1, u_3\}$, $\{4, e, u_1, u_2\}$, and $\{2, 4, u_2, u_3\}$ is a circuit of M.

Note that Hypothesis D cannot be applied directly in (A) or (B) since we do not know that 4 is in a triad of M.

Lemma 8.3. *If* (y_1, y_2, y_3, y_4) *is a* 4*-fan in* $M/4$*, then* $\{y_1, y_2, y_3, 4\}$ *is a circuit of* M *and* $y_1 \in \{2, 3, e\}$*, while* $\{y_2, y_3\}$ *avoids* $\{1, 2, 3, 4, e, g\}$*.*

Proof. Since M has no 4-fans, $\{y_1, y_2, y_3, 4\}$ is a circuit of M. By orthogonality with the cocircuit $\{2, 3, 4, e\}$, it follows that $\{y_1, y_2, y_3\}$ meets $\{2, 3, e\}$. Since $\{y_2, y_3, y_4\}$ is a cocircuit of M and this matroid has no 4-fans, it follows that $\{1, 2, 3, e, g\}$ avoids $\{y_2, y_3\}$. Thus $y_1 \in \{2, 3, e\}$. □

Now we begin the consideration of case A. We recall that $M\backslash e/4$ is $(4, 4, S)$-connected having an N-minor.

Lemma 8.4. *Assume that* $M/4$ *is* $(4, 4, S)$*-connected and that every* 4*-fan of* $M/4$ *has* e *as its guts end. Let* (x_1, x_2, x_3, x_4) *be a* 4*-fan of* $M\backslash e/4$ *and* (e, y_2, y_3, y_4) *be a* 4*-fan of* $M/4$*. Then* $\{x_1, x_2, x_3, 4\}$ *is a circuit of* M *and* $\{x_2, x_3, x_4, e\}$ *is a cocircuit of* M*. Moreover,* $x_4 \in \{y_2, y_3\}$ *and* $g \in \{x_2, x_3\}$ *and* $x_1 = 2$*. In particular, if* $x_4 = y_2$ *and* $g = x_2$*, then* $\{2, g, x_3, 4\}$ *is a circuit;* $\{g, x_3, y_2, e\}$ *and* $\{y_3, y_4, x_4\}$ *are cocircuits; and* $x_3 \notin \{1, 2, 3, 4, e, g, y_2, y_3, y_4\}$*.*

Proof. Recall that $M\backslash e$ has $(1, 2, 3, 4)$ as its unique 4-fan. If $\{x_1, x_2, x_3\}$ is a circuit of M, then (x_1, x_2, x_3, x_4) is a 4-fan of $M\backslash e$ that differs from $(1, 2, 3, 4)$; a contradiction. We deduce that $\{x_1, x_2, x_3, 4\}$ is a circuit of M. By orthogonality with the cocircuit $\{2, 3, 4, e\}$, it follows that $\{2, 3\}$ meets $\{x_1, x_2, x_3\}$.

Now either $\{x_2, x_3, x_4\}$ or $\{x_2, x_3, x_4, e\}$ is a cocircuit of M. In the first case, (x_1, x_2, x_3, x_4) is a 4-fan of $M/4$ that does not have e as its guts end; a contradiction. Thus $\{x_2, x_3, x_4, e\}$ is a cocircuit of M. Now $\{e, 4, y_2, y_3\}$ is a circuit of M. Thus, by orthogonality, $\{y_2, y_3\}$ meets $\{x_2, x_3, x_4\}$. By symmetry, we may assume that $y_2 \in \{x_2, x_3, x_4\}$.

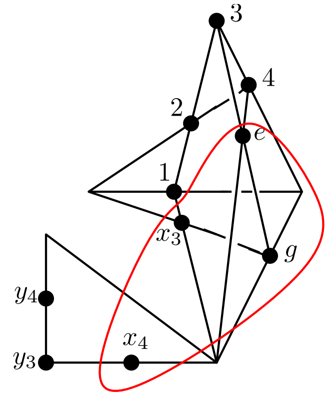

FIGURE 23

Suppose $y_2 \in \{x_2, x_3\}$. Then the circuit $\{x_1, x_2, x_3, 4\}$ must contain another element of the triad $\{y_2, y_3, y_4\}$. Suppose $y_3 \in \{x_1, x_2, x_3, 4\}$. Then $\{x_1, x_2, x_3, 4\}$ contains at least three elements of the 4-circuit $\{e, 4, y_2, y_3\}$. Hence these two 4-circuits are equal. This contradicts the fact that $\{x_1, x_2, x_3\}$ meets $\{2, 3\}$ since $\{y_2, y_3, y_4\}$ is a triad but neither 2 nor 3 is in a triad. Thus $y_3 \notin \{x_1, x_2, x_3, 4\}$ and so $y_4 \in \{x_1, x_2, x_3, 4\}$. Therefore $\{x_1, x_2, x_3, 4\}$ contains $\{y_2, y_4, 4\}$ and either 2 or 3. Then $M/4$ has a 4-fan (y_0, y_2, y_4, y_3) where $y_0 \in \{2, 3\}$; a contradiction. We deduce that $y_2 = x_4$.

As M has $\{x_2, x_3, x_4, e\}$ as a cocircuit and has $\{3, e, g\}$ as a circuit, it follows that $|\{3, g\} \cap \{x_2, x_3, x_4\}| = 1$. As $\{x_4, y_3, y_4\}$ is a triad of M, and each of 3 and g is in a triangle, we deduce that $\{3, g\} \cap \{x_2, x_3\} \neq \emptyset$. Suppose $3 \in \{x_2, x_3\}$. Without loss of generality, $3 = x_2$. As $\{1, 2, 3\}$ is a circuit, $\{1, 2\}$ meets $\{x_3, x_4\}$. But x_4 is in a triad, so $x_3 \in \{1, 2\}$. Letting x_0 be the element of $\{1, 2\} - x_3$, we see that $(x_0, 3, x_3, x_4)$ is a 4-fan of $M \backslash e$ yet $x_4 \neq 4$; a contradiction.

We may now assume that $g \in \{x_2, x_3\}$ so, without loss of generality, $g = x_2$. We also know that 2 or 3 is in $\{x_1, x_2, x_3\}$, so 2 or 3 is in $\{x_1, x_3\}$. As $\{x_2, x_3, x_4, e\}$ is a cocircuit and $\{3, e, g\}$ is a circuit, $3 \neq x_3$. If $2 = x_3$, then $\{g, 2, x_4, e\}$ is a cocircuit meeting the triangle $\{1, 2, 3\}$. Thus $x_4 = 1$, so $\{g, 2, 1, e\}$ is a quad of M; a contradiction. Thus $x_3 \notin \{2, 3\}$. We deduce that $x_1 \in \{2, 3\}$. Hence $\{x_1, x_2, x_3, 4\}$ is $\{3, g, x_3, 4\}$ or $\{2, g, x_3, 4\}$. In the first case, as $\{3, g, x_3, 4\} \triangle \{3, g, e\} = \{e, x_3, 4\}$, we have $\{e, 4\}$ in a triangle; a contradiction. Thus $\{2, g, x_3, 4\}$ is a circuit of M and $(x_1, x_2, x_3, x_4) = (2, g, x_3, y_2)$. Hence $\{g, x_3, y_2, e\}$ is a cocircuit of M.

Clearly $x_3 \notin \{2, 4, e, g, y_2\}$. If $x_3 \in \{1, 3\}$, then the symmetric difference of $\{1, 2, 3\}$ and $\{2, g, x_3, 4\}$ is a triangle containing 4; a contradiction to Lemma 2.8. If $x_3 \in \{y_3, y_4\}$, then the symmetric difference of $\{y_3, y_4, x_4\}$ and $\{g, x_3, x_4, e\}$ is a triad containing $\{e, g\}$; a contradiction. We conclude that $x_3 \notin \{1, 2, 3, 4, e, g, y_2, y_3, y_4\}$ and the lemma follows. □

Figure 23 is a geometric illustration of the situation arising in the next lemma. As usual, the ring around $\{e, g, x_3, x_4\}$ is to indicate that it is a cocircuit of M.

Lemma 8.5. *Assume that $M/4$ is $(4, 4, S)$-connected and that every 4-fan of $M/4$ has e as its guts end. Let $(2, g, x_3, x_4)$ be a 4-fan of $M \backslash e/4$ and (e, x_4, y_3, y_4) be a*

4-fan of $M/4$. Then either M^* has a good bowtie, or M/y_4 is internally 4-connected having an N-minor.

Proof. First we show the following.

8.5.1. $N \preceq M/4/y_4$.

Since $M\backslash e/4$ has an N-minor and has $(2, g, x_3, x_4)$ as a 4-fan, either $N \preceq M\backslash e/4\backslash 2$, or $N \preceq M\backslash e/4/x_4$. Observe that $M/4\backslash e\backslash 2 \cong M\backslash e\backslash 2/3 \cong M/3\backslash e\backslash 1$. Thus if $N \preceq M\backslash e/y_4\backslash 2$, then $N \preceq M\backslash e\backslash 1$, a contradiction to Hypothesis D. We deduce that $N \preceq M\backslash e/4/x_4$. Now $M\backslash e/4/x_4 \cong M/4/x_4\backslash y_3 \cong M/4\backslash y_3/y_4$. Hence $N \cong M/4/y_4$, that is, 8.5.1 holds.

8.5.2. *M/y_4 is sequentially 4-connected.*

Let (U, V) be a non-sequential 3-separation of M/y_4. Then exactly one of y_3 and x_4 is in U. Moreover, we may assume that $\{1, 2, 3\} \subseteq U$. If e or g is in U, then we may assume that both are in U so we can add 4 to U. Then the circuit $\{e, 4, y_3, x_4\}$ means that we may assume that $\{y_3, x_4\} \subseteq U$; a contradiction. We deduce that $\{e, g\} \subseteq V$. This means that $4 \in V$. Moreover, the circuit $\{3, e, g\}$ means that we can add 3 to V. Then we can move 2 and then 1 into V; a contradiction. Thus 8.5.2 holds.

Next we establish the following.

8.5.3. *Every 4-fan in M/y_4 has x_4 or y_3 as its guts element.*

Let $(\alpha, \beta, \gamma, \delta)$ be a 4-fan in M/y_4. Then $\{y_4, \alpha, \beta, \gamma\}$ is a circuit of M and $\{\beta, \gamma, \delta\}$ is a cocircuit, and exactly one of x_4 and y_3 is in $\{\alpha, \beta, \gamma\}$. Orthogonality implies that $\{y_3, x_4\}$ meets $\{\alpha, \beta, \gamma\}$. Suppose first that $\{y_3, x_4\}$ meets $\{\beta, \gamma\}$. By orthogonality between $\{\beta, \gamma, \delta\}$ and the circuit $\{4, e, x_4, y_3\}$, we deduce that $\{\beta, \gamma, \delta\}$ contains exactly two elements in $\{4, e, x_4, y_3\}$. Since $|\{x_4, y_3\} \cap \{\beta, \gamma\}| = 1$, we may assume that $\beta \in \{x_4, y_3\}$. If $4 \notin \{\gamma, \delta\}$, then e is in the triad $\{\beta, \gamma, \delta\}$ and so $M/4$ has a 5-cofan; a contradiction. We deduce that $4 \in \{\gamma, \delta\}$.

Now $\{2, g, x_3\}$ or $\{2, g, x_3, 4\}$ is a circuit of M. By orthogonality with the cocircuit $\{2, 3, 4, e\}$, it follows that $\{2, g, x_3, 4\}$ is a circuit of M. Thus, by orthogonality, $x_3 \in \{\gamma, \delta\}$. Then $\lambda_M(\{1, 2, 3, 4, e, g, x_3, x_4, y_3, y_4\}) \leq 2$; a contradiction. We conclude that $\{y_3, x_4\}$ avoids $\{\beta, \gamma\}$. Hence $\alpha \in \{x_4, y_3\}$ and 8.5.3 holds.

It follows, by 8.5.3, that if F is a 5-fan or a 5-cofan in M/y_4, then both x_4 and y_3 are in F. Hence $F \cup y_4$ is 3-separating in M; a contradiction. We conclude that M/y_4 is $(4, 4, S)$-connected. Then either M/y_4 is internally 4-connected, or $M^*\backslash y_4$ has a 4-fan with its coguts element in $\{x_4, y_3\}$ and so in the triangle $\{x_4, y_3, y_4\}$ of M^*. In the former case, the lemma holds while, in the latter case, Lemma 2.8 implies that M^* has a good bowtie and, again, the lemma holds. □

We have now completed the proof of Theorem 3.1 in case A, that is, when $M/4$ is $(4, 4, S)$-connected and every 4-fan of $M/4$ has e as its guts end. Next we consider case B, that is, $M/4$ is $(4, 4, S)$-connected and $M/4$ has a 4-fan that does not have e as its guts end (see Figure 24(a) and (b)). By orthogonality, every 4-fan of $M/4$ must have a member of $\{2, 3, e\}$ as its guts end.

Lemma 8.6. *Suppose that $M/4$ is $(4, 4, S)$-connected and that $M/4$ has a 4-fan (α, s_1, s_2, s_3) where $\alpha \in \{2, 3\}$. Then $M/4\backslash e/s_3$ has an N-minor. Moreover, M/s_3 is sequentially 4-connected.*

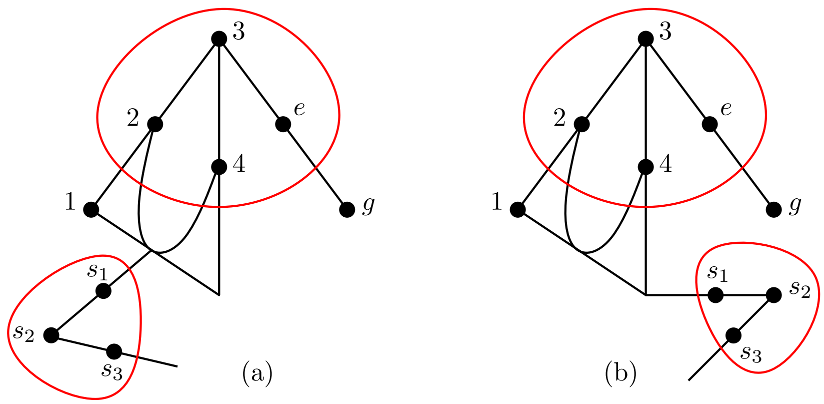

FIGURE 24. (a) $M/4$ has $(2, s_1, s_2, s_3)$ as a 4-fan. (b) $M/4$ has $(3, s_1, s_2, s_3)$ as a 4-fan.

Proof. As $M/4\backslash e$ is $(4, 4, S)$-connected, it has (α, s_1, s_2, s_3) as a 4-fan. Thus we may assume that $N \preceq M/4\backslash e\backslash \alpha$ otherwise the first part of the lemma holds. Now $M\backslash e\backslash \alpha$ has $\{4, \beta\}$ as a cocircuit where $\{\alpha, \beta\} = \{2, 3\}$. Then $N \preceq M\backslash e\backslash \alpha/\beta$. But $M\backslash e/\beta\backslash \alpha \cong M\backslash e/\beta\backslash 1$, so $N \preceq M\backslash e, 1$; a contradiction to Hypothesis D. Hence $N \preceq M/4\backslash e/s_3$.

Since s_3 is in a triad of M, it follows that M/s_3 is 3-connected. Now suppose that M/s_3 has a non-sequential 3-separation (U, V). Then we may assume that $\{1, 2, 3\} \subseteq U$. Now exactly one of s_1 and s_2 is in U so we may assume that $s_1 \in U$ and $s_2 \in V$. As $\{\alpha, 4, s_1, s_2\}$ is a circuit of M, it follows that $4 \in V$. The cocircuit $\{2, 3, 4, e\}$ now implies that $e \in V$ while the circuit $\{3, e, g\}$ implies that $g \in V$. But now we can move 3 into V, and then 2 into V, and finally s_1 into V; a contradiction. We conclude that M/s_3 is sequentially 4-connected. □

Lemma 8.7. *Suppose that $M/4$ is $(4, 4, S)$-connected and that $M/4$ has a 4-fan (α, s_1, s_2, s_3) where $\alpha \in \{2, 3\}$. Then either M^* has a good bowtie, or M/s_3 is internally 4-connected having an N-minor.*

Proof. By Lemma 8.6, M/s_3 is sequentially 4-connected having an N-minor. First we will identify the possible guts elements in a 4-fan in M/s_3.

8.7.1. *If (j, k, l, m) is a 4-fan in M/s_3, then $j \in \{s_1, s_2\}$.*

Clearly $\{s_3, j, k, l\}$ is a circuit of M and $\{k, l, m\}$ is a cocircuit. The cocircuit $\{s_1, s_2, s_3\}$ and the circuit $\{s_3, j, k, l\}$ imply that $|\{s_1, s_2\} \cap \{j, k, l\}| = 1$. Assume that 8.7.1 fails. Then $s_i \in \{k, l\}$, for some i in $\{1, 2\}$. By symmetry, we may assume that $s_1 = k$. The circuit $\{4, s_1, s_2, \alpha\}$ implies that $4 \in \{l, m\}$, as α is in no triad of $M/4$ otherwise $M/4$ is not $(4, 4, S)$-connected. Then 4 is in a triad of M, and $M/4$ is $(4, 4, S)$-connected with s_3 in the coguts of a 4-fan. But, by Lemma 8.6, $M/4/s_3$ has an N-minor. This contradiction to Hypothesis D completes the proof of 8.7.1.

Now if F is a 5-fan or a 5-cofan in M/s_3 containing $\{s_1, s_2\}$, then $F \cup s_3$ is a 6-element 3-separating set in M; a contradiction. It follows by 8.7.1 that M/s_3 is $(4, 4, S)$-connected. If M/s_3 is internally 4-connected, then the lemma holds. Thus we may assume that $M^*\backslash s_3$ has a 4-fan. By 8.7.1, the coguts element of this 4-fan is in $\{s_1, s_2\}$ and so is contained in a triangle in M^*. It follows by Lemma 2.8 that M^* has a good bowtie. We conclude that Lemma 8.7 holds. □

The last lemma completes the proof of Theorem 3.1 in case B. It remains to consider case C, that is, M has a cocircuit $\{u_1, u_2, u_3\}$ that is disjoint from $\{1, 2, 3, 4, e, g\}$ such that each of $\{2, e, u_2, u_3\}$, $\{4, e, u_1, u_2\}$, and $\{2, 4, u_1, u_3\}$ is a circuit. This structure can be illustrated geometrically as in Figure 25. First observe that we have the following.

Lemma 8.8. *If M contains the structure shown in Figure 25, then the only triangle of M containing 2 is $\{1, 2, 3\}$.*

Proof. Suppose that M has a triangle T other than $\{1, 2, 3\}$ containing 2. Then, by orthogonality with the cocircuit $\{2, 3, 4, e\}$, one of $\{2, e\}$ or $\{2, 4\}$ is in T. Then the symmetric difference of T with $\{2, e, u_2, u_3\}$ or $\{2, 4, u_1, u_3\}$ is a triangle containing $\{u_2, u_3\}$ or $\{u_1, u_3\}$, so M has a 4-fan; a contradiction. □

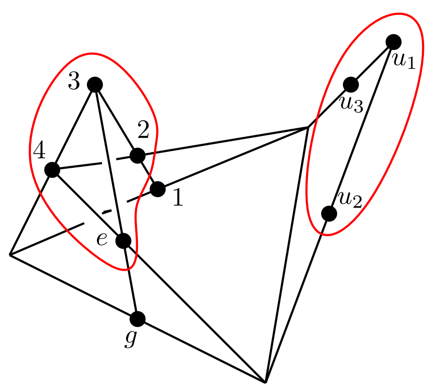

FIGURE 25

Lemma 8.9. *Suppose $|E(M)| \geq 16$ and $|E(N)| \geq 7$. If M contains the structure shown in Figure 25, then M has a proper minor M' such that $|E(M)| - |E(M')| \leq 3$ and M' is internally 4-connected with an N-minor, or M or M^* has a good bowtie or a pretty good bowtie.*

Proof. We shall assume that the lemma fails. By Hypothesis D, $N \not\preceq M\backslash e, 1$ so $N \preceq M\backslash e/4$. Now, in $M/4$, we have a 5-fan $(e, u_2, u_1, u_3, 2)$. Moreover,

8.9.1. $|\{1, 2, 3, 4, e, g, u_1, u_2, u_3\}| = 9$.

Next we show the following.

8.9.2. *$N \preceq M/4/u_2\backslash e$ and $N \preceq M/4/u_3\backslash 2$.*

To see this, we observe that the desired result holds unless $N \preceq M/4\backslash e\backslash 2$. In the exceptional case, since $\{4, 3\}$ is a cocircuit of $M\backslash e\backslash 2$, we deduce that $N \preceq M\backslash e, 2/3$. Hence $N \preceq M\backslash e, 1$; a contradiction. Thus 8.9.2 holds.

Next we show the following.

8.9.3. *M/u_2 and M/u_3 are sequentially 4-connected.*

The asymmetry that exists between u_2 and u_3 arises because $M\backslash e$ is $(4,4,S)$-connected while we do not know that $M\backslash 2$ is $(4,4,S)$-connected. We will prove that M/u_2 is sequentially 4-connected without using this differentiating information. It will then immediately follow that M/u_3 is sequentially 4-connected. Let (U,V) be a non-sequential 3-separation of M/u_2. Then, without loss of generality, we may assume that $\{u_1,4,e\} \subseteq U$ and $\{u_3,1,g\} \subseteq V$. Suppose $3 \in U$. Then we may also assume $2 \in U$, so we can move u_3 into U and then add u_2 to U to get a non-sequential 3-separation of M; a contradiction. Thus $3 \in V$. Then we can assume e and 2 are in V. This means we can move 4 and then u_1 into V and then add u_2 to V to get a non-sequential 3-separation of M; a contradiction. We conclude that M/u_2 is sequentially 4-connected and so 8.9.3 holds.

8.9.4. *Suppose (j,k,l,m) is a 4-fan of M/u_h for some $h \in \{2,3\}$. Then one of the following holds.*

(i) *(u_h,j) is (u_2,e) and $\{k,l\} = \{u_1,4\}$; or*
(ii) *(u_h,j) is $(u_3,2)$ and $\{k,l\} = \{u_1,4\}$; or*
(iii) *$u_1 \in \{k,l\}$ and $m = 4$; or*
(iv) *$j \in \{u_1,u_i\}$ where $\{h,i\} = \{2,3\}$.*

Moreover, if M/u_h is $(4,4,S)$-connected, then (i) or (ii) holds.

Suppose that $j \notin \{u_1,u_i\}$ where $\{h,i\} = \{2,3\}$. Clearly $\{k,l,m\}$ is a triad of M and $\{u_h,j,k,l\}$ is a circuit of M. By orthogonality with the triad $\{u_1,u_2,u_3\}$, the circuit $\{u_h,j,k,l\}$ contains u_1 or u_i. Without loss of generality, $k \in \{u_1,u_i\}$. Suppose $k = u_i$. Then, by orthogonality between the triad $\{k,l,m\}$ and the circuit $\{u_2,u_3,2,e\}$ in M, we deduce that 2 or e is in a triad of M; a contradiction. Thus $k = u_1$.

By orthogonality between $\{k,l,m\}$ and the circuits $\{u_1,u_2,4,e\}$ and $\{u_1,u_3,4,2\}$, either $4 \in \{l,m\}$, or $\{l,m\} = \{2,e\}$. As 2 is not in a triad, $4 \in \{l,m\}$. If $4 = l$, then $j = e$ when $u_h = u_2$, and $j = 2$ when $u_h = u_3$. Thus either (u_h,j,l) is $(u_2,e,4)$ or $(u_3,2,4)$; or $m = 4$. We conclude that one of (i)–(iv) holds.

Now assume that M/u_h is $(4,4,S)$-connected. Suppose (iii) holds. Then $m = 4$. As M/u_h has a triangle containing 4, we obtain the contradiction that M/u_h has a 5-fan. Hence (iii) does not hold. If (iv) holds, then M^* has $(\{k,l,m\},\{u_1,u_2,u_3\},\{u_h,j,k,l\})$ as a good bowtie, a contradiction. We conclude that 8.9.4 holds.

8.9.5. *Neither M/u_2 nor M/u_3 is $(4,4,S)$-connected.*

Assume M/u_h is $(4,4,S)$-connected for some h in $\{2,3\}$. Since the lemma holds if M/u_h is internally 4-connected with an N-minor, M/u_h has a 4-fan (j,k,l,m). By 8.9.4, $\{k,l\} = \{u_1,4\}$, and (u_h,j) is either (u_2,e) or $(u_3,2)$.

Suppose that $(u_h,j,k,l) = (u_2,e,u_1,4)$. Since 8.9.2 implies that $N \preceq M/4/u_2\backslash e$, and $M/4/u_2\backslash e \cong M/4/u_2\backslash u_1 \cong M/u_2\backslash u_1/m$, we deduce that $N \preceq M/u_2/m$. As M/u_2 is $(4,4,S)$-connected, this is a contradiction to Hypothesis D.

Next assume that $(u_h,j,k,l) = (u_3,2,u_1,4)$. Again 8.9.2 implies that $N \preceq M/4/u_3\backslash 2$, and $M/4/u_3\backslash 2 \cong M/4/u_3\backslash u_1 \cong M/u_3\backslash u_1/m$. Thus $N \preceq M/u_3/m$; a contradiction to Hypothesis D. We conclude that 8.9.5 holds.

Next we observe the following.

8.9.6. *If F is a 5-fan or a 5-cofan in M/u_h for some h in $\{2,3\}$, then $|F \cap \{u_1,u_2,u_3\}| \leq 1$.*

To see this, observe that if the conclusion does not hold, then $F \cup u_h$ is 3-separating in M; a contradiction.

8.9.7. *For each h in $\{2,3\}$, the matroid M/u_h has a 5-fan $(\xi, 4, u_1, w_4, w_5)$ for some elements w_4 and w_5, where ξ is e or 2 depending on whether u_h is u_2 or u_3.*

Let i be the element of $\{2,3\} - \{h\}$. By 8.9.3 and 8.9.5, M/u_h has a 5-fan or a 5-cofan. Suppose that M/u_h has a 5-cofan $(z_1, z_2, z_3, z_4, z_5)$. Then M/u_h has (z_2, z_3, z_4, z_5) and (z_4, z_3, z_2, z_1) as 4-fans. We shall apply 8.9.4 to these two 4-fans. Suppose that (i) or (ii) holds for (z_2, z_3, z_4, z_5). Then $\{z_3, z_4\} = \{u_1, 4\}$. Thus (iii) or (iv) must hold for (z_4, z_3, z_2, z_1). But (iii) yields the contradiction that $z_1 = 4$. Thus (iv) holds, so $z_4 \in \{u_1, u_i\}$. Hence, by 8.9.6, $(z_3, z_4) = (4, u_1)$. Thus M/u_h has $(z_1, \xi, 4, u_1, z_5)$ as a 5-cofan. But M has $\{2, e, u_2, u_3\}$ as a circuit. Thus, by orthogonality with $\{z_1, \xi, 4\}$, we deduce that $z_1 \in \{u_2, u_3\}$ and we contradict 8.9.6.

Using symmetry, we now know that (iii) or (iv) holds for (z_2, z_3, z_4, z_5) and for (z_4, z_3, z_2, z_1). But (iii) cannot hold for both fans, nor can (iv). Hence we may assume that (iii) holds for the first and (iv) holds for the second. Thus $u_1 \in \{z_3, z_4\}$ and $z_5 = 4$; and $z_4 \in \{u_1, u_i\}$. Hence, by 8.9.6, $z_4 = u_1$ and $z_5 = 4$, so $(z_1, z_2, z_3, z_4, z_5) = (z_1, z_2, z_3, u_1, 4)$. As M/u_h has $\{u_1, 4, \xi\}$ as a circuit, it follows that $(z_2, z_3, u_1, 4, \xi)$ is a 5-fan of M/u_h, so 8.9.7 holds if M/u_h has a 5-cofan.

Next assume that $(w_1, w_2, w_3, w_4, w_5)$ is a 5-fan of M/u_h. Then (w_1, w_2, w_3, w_4) and (w_5, w_4, w_3, w_2) are 4-fans. Assume that (i) or (ii) holds for the first of these. Then $w_1 = \xi$ and $\{w_2, w_3\} = \{u_1, 4\}$. Thus (iii) or (iv) holds for the second 4-fan. By 8.9.6, $w_5 \notin \{u_1, u_i\}$. Thus $w_2 = 4$ and $u_1 \in \{w_4, w_3\}$. Hence $(w_1, w_2, w_3) = (\xi, 4, u_1)$ and again 8.9.7 holds.

It remains to consider the case when (iii) and (iv) hold for both (w_1, w_2, w_3, w_4) and (w_5, w_4, w_3, w_2). Using 8.9.6, we see that (iii) cannot hold for both 4-fans; nor can (iv). Thus we may assume that $u_1 \in \{w_2, w_3\}$ and $w_4 = 4$; and $w_5 \in \{u_1, u_i\}$. This again contradicts 8.9.6. We conclude that 8.9.7 holds.

8.9.8. *M/u_2 has a 5-fan $(e, 4, u_1, b, a)$ and M/u_3 has a 5-fan $(2, 4, u_1, d, c)$ for some elements $a, b, c,$ and d. Moreover, $d = b$.*

By 8.9.7, M/u_2 has a 5-fan $(e, 4, u_1, b, a)$ and M/u_3 has a 5-fan $(2, 4, u_1, d, c)$. As M has $\{u_1, 4, b\}$ and $\{u_1, 4, d\}$ as triads, we must have $b = d$. Hence 8.9.8 holds.

By 8.9.8, we obtain the structure illustrated in Figure 26, where $\{u_1, 4, b\}$ is a triad. Note that we are not asserting that $\{a, b, c\}$ is a triad.

8.9.9. $|\{1, 2, 3, 4, e, g, u_1, u_2, u_3, a, b, c\}| = 12$.

Let $Z = \{1, 2, 3, 4, e, g, u_1, u_2, u_3\}$. By 8.9.1, $|Z| = 9$. If $\{a, b, c\}$ meets Z, then $r(Z \cup \{a, b, c\}) \leq 5$, so $\lambda(Z \cup \{a, b, c\}) \leq 2$ as the last set contains the cocircuits $\{2, 3, 4, e\}$, $\{u_1, u_2, u_3\}$, and $\{4, b, u_1\}$. This contradiction establishes 8.9.9.

8.9.10. *$N \preceq M/u_2/b\backslash a$ and $N \preceq M/u_3/b\backslash c$.*

We know that $(e, 4, u_1, b, a)$ and $(2, 4, u_1, b, c)$ are 5-fans of M/u_2 and M/u_3, respectively. By 8.9.2, $N \preceq M/4/u_2\backslash e$. But $M/4/u_2\backslash e \cong M/u_2/b\backslash a$ so the first part of 8.9.10 holds. The second part follows by symmetry.

Now we have symmetry between c and a provided we do not use the fact that $M\backslash e$ is $(4, 4, S)$-connected. This symmetry will be exploited in the next part of the argument. If $M\backslash c$ or $M\backslash a$ is internally 4-connected, then the lemma holds, so we assume not.

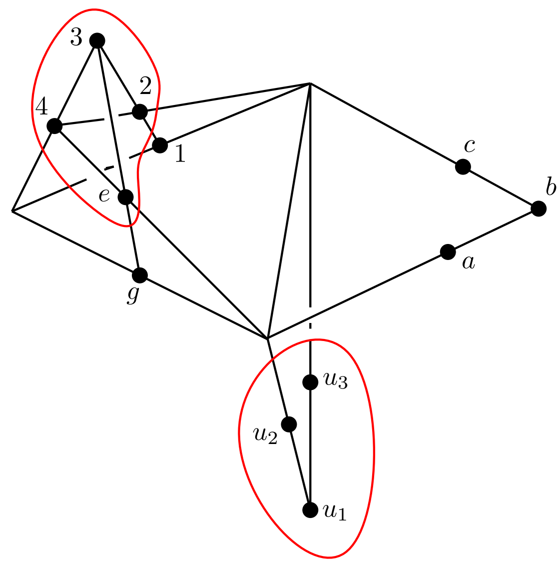

FIGURE 26. M also has $\{u_1, 4, b\}$ as a triad.

8.9.11. *$M\backslash c$ and $M\backslash a$ are sequentially 4-connected.*

Suppose (U, V) is a non-sequential 3-separation of $M\backslash c$. Then we may assume that $\{u_1, u_2, u_3\} \subseteq U$. If a or b is in U, then the various 4-circuits we have mean that we can add c to U to get a non-sequential 3-separation of M; a contradiction. Hence we must have that $\{a, b\} \subseteq V$. If $4 \in U$, then the cocircuit $\{4, b, u_1\}$ means that we can move b into U; a contradiction. Thus $4 \in V$. This means we can move u_1, then u_2, and finally u_3 into V. This gives us a contradiction as we can now add c to V. Hence $M\backslash c$ is sequentially 4-connected and, by symmetry, so is $M\backslash a$.

8.9.12. *If (z_1, z_2, z_3, z_4) is a 4-fan in $M\backslash c$, then $z_4 = b$ and $a \in \{z_2, z_3\}$. Symmetrically, if (y_1, y_2, y_3, y_4) is a 4-fan in $M\backslash a$, then $y_4 = b$ and $c \in \{y_2, y_3\}$.*

Let (z_1, z_2, z_3, z_4) be a 4-fan on $M\backslash c$. Then M has $\{z_2, z_3, z_4, c\}$ as a cocircuit and has $\{c, a, u_2, u_3\}$, $\{c, b, 2, 4\}$, and $\{c, b, u_1, u_3\}$ as circuits. Thus, by orthogonality, $\{z_2, z_3, z_4\}$ meets $\{a, u_2, u_3\}$, $\{b, 2, 4\}$, and $\{b, u_1, u_3\}$. But, since each element of $\{z_1, z_2, z_3\}$ is in a triangle while each element of $\{u_1, u_2, u_3, b, 4\}$ is in a triad, these two sets are disjoint. Thus

(i) $z_4 \in \{a, u_2, u_3\}$ or $a \in \{z_2, z_3\}$;
(ii) $z_4 \in \{b, 2, 4\}$ or $2 \in \{z_2, z_3\}$; and
(iii) $z_4 \in \{b, u_1, u_3\}$.

We look first at (iii) and suppose that $z_4 = u_1$. Then, by (i) and (ii), $\{z_2, z_3\} = \{a, 2\}$. Thus M has a triangle containing $\{a, 2\}$; a contradiction to Lemma 8.8. Next suppose that $z_4 = u_3$. Then, by (ii) and symmetry, we may assume that $2 = z_3$. By Lemma 8.8 again, it follows that $\{z_1, z_2, z_3\} = \{1, 2, 3\}$. Then, by orthogonality between the circuit $\{c, a, 1, g\}$ and the cocircuit $\{c, z_2, z_3, z_4\}$, we deduce that $z_2 = 1$ and $z_1 = 3$. Letting $Z = \{1, 2, 3, 4, e, g, u_1, u_2, u_3, a, b, c\}$, we see that $r(Z) \leq 6$ while $r^*(Z) \leq |Z| - 4$ since Z contains the cocircuits $\{2, 3, 4, e\}$, $\{u_1, u_2, u_3\}$, $\{4, b, u_1\}$, and $\{1, 2, c, u_3\}$. Hence $\lambda(Z) \leq 2$. This is a contradiction as $|E(M)| \geq 16$.

We may now assume that $z_4 = b$. Then, by (i), $a \in \{z_2, z_3\}$. Hence the first part of 8.9.12 holds, and the second part follows by symmetry.

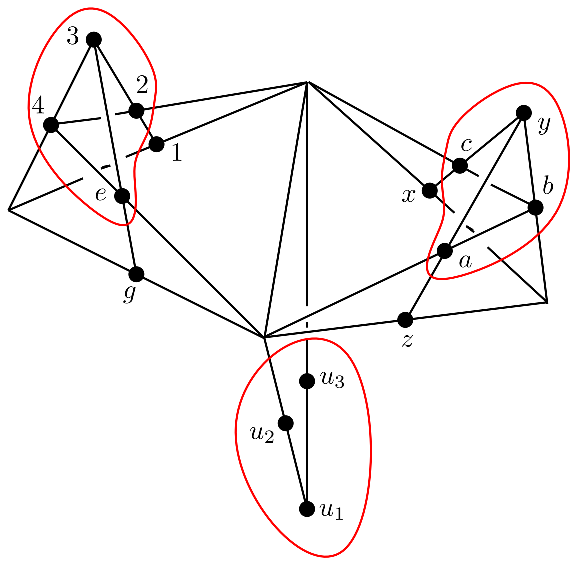

FIGURE 27. M also has $\{u_1, 4, b\}$ as a triad.

By 8.9.12, $M\backslash c$ and $M\backslash a$ have 4-fans (z, a, y, b) and (x, c, w, b) for some elements z, y, x, and w. Then $\{a, b, c, y\}$ and $\{a, b, c, w\}$ are cocircuits of M, so $w = y$. Thus M contains the structure illustrated in Figure 27, where $\{u_1, 4, b\}$ is a triad. Moreover,

8.9.13. $|\{1, 2, 3, 4, e, g, u_1, u_2, u_3, a, b, c, x, y, z\}| = 15.$

By 8.9.9, $|Z'| = 12$ where $Z' = \{1, 2, 3, 4, e, g, u_1, u_2, u_3, a, b, c\}$. If $\{x, y, z\}$ meets Z', then $r(Z' \cup \{x, y, z\}) \leq 6$, so $\lambda(Z') \leq 2$; a contradiction. Hence 8.9.13 holds.

We continue the proof of the lemma by showing that

8.9.14. *$M\backslash e/4, u_2$ is internally 4-connected, or M has a triad $\{p, q, s\}$ such that $\{p, q, u_2, u_i\}$ is a circuit of M for some i in $\{1, 3\}$.*

Assume that this does not hold. Since $M\backslash e/4$ is $(4, 4, S)$-connected having $(2, u_3, u_1, u_2)$ as a 4-fan, we see that $M\backslash e/4, u_2$ is 3-connected. Observe that this matroid has $\{u_1, u_3, 2\}$ as a circuit. Next we show the following.

8.9.15. *$M\backslash e/4, u_2$ has a $(4, 3)$-violator (U, V) such that $\{u_1, u_3\} \subseteq U$.*

Assume that this fails. Certainly $M\backslash e/4, u_2$ has a $(4, 3)$-violator (U, V). Then we may assume that $\{u_i, 2\} \subseteq U$ and $u_h \in V$ where $\{i, h\} = \{1, 3\}$. Then $u_h \in \text{cl}_{M\backslash e/4,u_2}(U)$, so $(U \cup u_h, V - u_h)$ is a 3-separation of $M\backslash e/4, u_2$. Since, by assumption, it cannot be a $(4, 3)$-violator, we deduce that V is a 4-fan (u_h, p, q, s) of $M\backslash e/4, u_2$. Then $\{p, q, s\}$ or $\{e, p, q, s\}$ is a cocircuit of M. Suppose $\{e, p, q, s\}$ is a cocircuit. Then orthogonality with the circuits $\{e, 3, g\}, \{e, 4, u_1, u_2\}$, $\{e, 2, u_2, u_3\}$, and $\{a, c, e, 2\}$ implies that $\{p, q, s\} \subseteq \{3, g, u_1, u_3, a, c\}$, so $\lambda_M(\{1, 2, 3, 4, e, g, a, b, c, u_1, u_2, u_3\}) \leq 2$; a contradiction.

We deduce that $\{p,q,s\}$ is a triad of M. Now $\{u_2,u_h,p,q\},\{4,u_h,p,q\}$, or $\{4,u_2,u_h,p,q\}$ is a circuit C of M. If $4 \in C$, then orthogonality with the cocircuit $\{2,3,4,e\}$ implies that $2,3$, or e is in the triad $\{p,q,s\}$; a contradiction. Thus $\{p,q,s\}$ is a triad of M and $\{p,q,u_2,u_h\}$ is a circuit of M, so 8.9.14 holds; a contradiction. We conclude that 8.9.15 holds.

Since $\{u_1,u_3\} \subseteq U$, we see that $(U \cup u_2, V)$ is a $(4,3)$-violator of $M\backslash e/4$. But the last matroid is $(4,4,S)$-connected, so either $U \cup u_2$ or V is a 4-fan of $M\backslash e/4$. As $|U|,|V| \geq 4$, it follows that V is a 4-fan (t,p,q,s) of $M\backslash e/4$, and $V = \mathrm{fcl}_{M\backslash e/4}(V)$. As (t,p,q,s) is not the unique 4-fan of $M\backslash e$, we deduce that $\{4,t,p,q\}$ is a circuit of M. Observe that, since $\{u_1,4,b\}$ and $\{a,c,y,b\}$ are cocircuits of M, so is their symmetric difference, $\{a,c,y,4,u_1\}$.

Since M has $\{b,u_1,4\}$, $\{2,3,4,e\}$, and $\{a,c,y,4,u_1\}$ as cocircuits and $\{4,t,p,q\}$ as a circuit, and $\{t,p,q\} \subseteq V$, it follows by orthogonality that $b \in V$ and that each of $\{2,3\}$ and $\{a,c,y\}$ meets V. Thus at least two elements in $b,2,3,a,y$, or c are contained in $\{p,q,s\}$. Each of $2,3,a,y$, and c is in a triangle of M so none is in a triad. Hence $\{p,q,s\}$ is not a triad of M, so $\{e,p,q,s\}$ is a cocircuit of M.

Suppose $b \in \{p,q,s\}$. Then the triangle $\{b,c,2\}$ of $M\backslash e/4$ meets this triad in at least two elements, so c or 2 is in $\{p,q,s\}$. Also, the circuit $\{u_1,u_2,a,b\}$ meets the triad $\{p,q,s\}$ of $M\backslash e/4$ in two elements. But $\{u_1,u_2\}$ avoids $\{t,p,q,s\}$. Thus $a \in \{p,q,s\}$. Hence $\{e,b,a,c\}$ or $\{e,b,a,2\}$ is a cocircuit of M that meets the circuit $\{u_1,u_2,e,4\}$ in exactly one element; a contradiction. We conclude that $b \notin \{p,q,s\}$.

We may now assume that $b = t$. Thus $\{4,b,p,q\}$ is a circuit of M. Orthogonality between this circuit and the cocircuits $\{2,3,4,e\}$ and $\{a,b,c,y\}$ implies that $\{p,q\}$ meets both $\{2,3\}$ and $\{a,c,y\}$. Suppose first that $3 \in \{p,q\}$. Then, as neither $\{4,b,3,a\}$ nor $\{4,b,3,c\}$ is a circuit of M, it follows that $\{4,b,3,y\}$ is a circuit of M, that is, $\{p,q\} = \{3,y\}$. Then $\{e,3,y,s\}$ is a cocircuit of M. Thus, by orthogonality with the circuits $\{3,2,1\}$, $\{y,c,x\}$ and $\{y,a,z\}$, we obtain a contradiction. We deduce that $3 \notin \{p,q\}$. Thus $2 \in \{p,q\}$, so the circuit $\{4,b,p,q\}$ has three elements in common with the circuit $\{4,b,2,c\}$ so must equal this circuit. Hence $\{p,q\} = \{2,c\}$, so $\{e,2,c,s\}$ is a cocircuit of M. As M is binary and $\{a,c,e,2\}$ is a circuit, we see that $s = a$, so M has $\{a,c,e,2\}$ as a quad; a contradiction. This completes the proof of 8.9.14.

By 8.9.2, $N \preceq M\backslash e/4/u_2$. Thus, by 8.9.14, M contains the structure illustrated in Figure 28, or M contains the same structure with the line $\{p,q\}$ moved so that it goes through the point where the lines $\{4,e\}$ and $\{a,b\}$ meet.

8.9.16. $|\{1,2,3,4,e,g,u_1,u_2,u_3,a,b,c,x,y,z,p,q,s\}| = 18$.

By 8.9.13, $|Z'| = 15$ where $Z' = \{1,2,3,4,e,g,u_1,u_2,u_3,a,b,c,x,y,z\}$. Suppose $\{p,q,s\}$ meets Z'. Since $\{p,q,s\}$ is a triad, it cannot contain any element that is in a triangle. Thus $\{p,q,s\} \cap Z' \subseteq \{4,b,u_1,u_2,u_3\}$. The circuit $\{p,q,u_2,u_i\}$ implies that $\{p,q\}$ avoids $\{u_1,u_2,u_3\}$. Suppose $\{p,q,s\}$ meets $\{4,b\}$. Then, by orthogonality with the circuit $\{4,2,b,c\}$, it follows that $\{p,q,s\}$ contains $\{4,b\}$. Hence the triads $\{p,q,s\}$ and $\{4,b,u_1\}$ coincide. Now M has a 4-circuit C that contains $\{p,q,u_2\}$ and exactly one of u_1 and u_3. Thus $u_3 \in C$. It follows that $\{4,b,u_2,u_3\}$ is a circuit of M. As this set is also a cocircuit, we have a quad in M; a contradiction. We deduce that $\{p,q,s\}$ avoids $\{4,b\}$. The only remaining possibility is that $s \in \{u_1,u_2,u_3\}$. But now one of the circuits $\{4,2,u_3,u_1\}$ or $\{2,e,u_2,u_3\}$ gives a contradiction to orthogonality. We conclude that 8.9.16 holds.

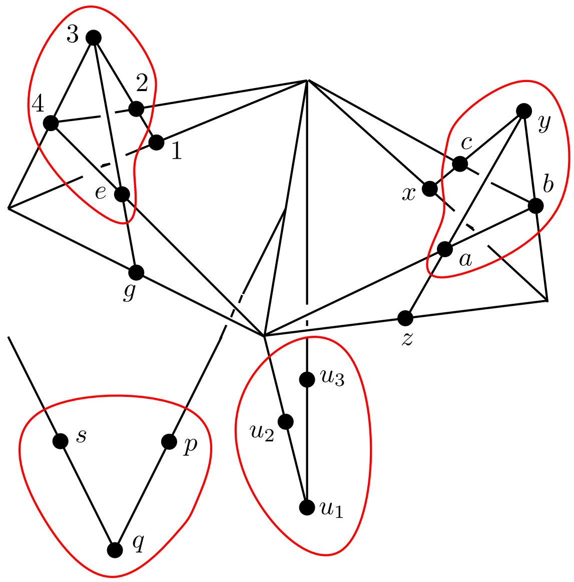

FIGURE 28

Now $M\backslash e/4, u_2$ has an N-minor and has (u_i, p, q, s) as a fan. Thus N is a minor of $M\backslash e/4, u_2\backslash u_i$ or $M\backslash e/4, u_2, s$. Suppose $N \preceq M\backslash e/4, u_2\backslash u_i$ Then, as $\{u_2, u_h\}$ is a cocircuit in $M\backslash u_i$ where $\{i, h\} = \{1, 3\}$, we deduce that $N \preceq M\backslash e/4, u_h$ As $\{2, u_i\}$ is a circuit in the last matroid, $N \preceq M\backslash e, 2$. But $\{3, 4\}$ is a cocircuit of the last matroid, so $N \preceq M\backslash e, 2/3$. Hence $N \preceq M/3\backslash e\backslash 1$, so $N \preceq M\backslash e, 1$; a contradiction. We conclude that $N \preceq M\backslash e/4, u_2, s$.

We finish the proof of this lemma by showing that M^* has a good bowtie. First we show that

8.9.17. *M/s is $(4, 4, S)$-connected.*

Suppose instead that (U, V) is a $(4, 4, S)$-violator of M/s. Without loss of generality, we may assume that $p \in U$ and $q \in V$. If U contains $\{u_2, u_i\}$, then $(U \cup q \cup s, V - q)$ is a $(4, 3)$-violator of M; a contradiction. By the symmetry of $\{p, q\}$, we may assume without loss of generality that $u_2 \in U$ and $u_i \in V$. Then $(U \cup u_i \cup q \cup s, V - q - u_i)$ or $(U - p - u_2, V \cup u_2 \cup p \cup s)$ is a 3-separation of M depending on whether u_h is in U or V. Hence either V is a 5-cofan (u_i, q, t_3, t_4, t_5) of M/s avoiding $\{u_2, u_h\}$, or U is a 5-cofan (u_2, p, t_3, t_4, t_5) in M/s avoiding $\{u_1, u_3\}$. In the first case, as M has $\{u_1, u_3, b, c\}$ and $\{u_1, u_3, 2, 4\}$ as circuits, it follows by orthogonality with the triad $\{u_i, q, t_3\}$ that $\{q, t_3\}$ meets both $\{b, c\}$ and $\{2, 4\}$. Thus, by 8.9.16, $t_3 \in \{b, c\} \cap \{2, 4\}$; a contradiction. In the second case, as M has $\{u_1, u_2, e, 4\}$ and $\{u_1, u_2, a, b\}$ as circuits, it follows by orthogonality that $t_3 \in \{e, 4\} \cap \{a, b\}$; a contradiction. We conclude that 8.9.17 holds.

If M/s is internally 4-connected, then the lemma holds, so we may assume that M/s contains a 4-fan (s_1, s_2, s_3, s_4). Then $\{s, s_1, s_2, s_3\}$ is a circuit of M, which, by orthogonality, must contain p or q. Without loss of generality, $q \in \{s_1, s_2, s_3\}$. Then p avoids $\{s_1, s_2, s_3, s_4\}$. Suppose $q \in \{s_2, s_3, s_4\}$. If $i = 3$, then, by orthogonality, $\{s_2, s_3, s_4\}$ contains at least two elements in each of the circuits

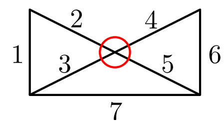

FIGURE 29

$\{2,e,p,q\}, \{a,c,p,q\}$, and $\{u_2,u_3,p,q\}$. Since $p \notin \{s_1,s_2,s_3,s_4\}$, this is a contradiction. If $i=1$, then, by orthogonality, $\{s_2,s_3,s_4\}$ contains at least two elements in each of the circuits $\{4,e,p,q\}, \{a,b,p,q\}$, and $\{u_1,u_2,p,q\}$. Again we obtain a contradiction. We deduce that $q=s_1$, and $(\{p,q,s\},\{s_2,s_3,s_4\},\{q,s,s_2,s_3\})$ is a good bowtie in M^*. This completes the proof of Lemma 8.9 □

9. The implications of a pretty good bowtie

Pretty good bowties appear in Theorem 3.1 but not in Theorem 1.3. In this section, we show why.

Lemma 9.1. *Let (x_1,x_2,x_3,x_4) be a 4-fan in a $(4,4,S)$-connected binary matroid M_0 that has at least ten elements. Then M_0 has no triangle containing x_4.*

Proof. If M_0 has triangle T containing x_4, then, by orthogonality, $|T \cap \{x_2,x_3,x_4\}|=2$. Then $\{x_1,x_2,x_3,x_4\} \cup T$ is a 5-element 3-separating set in M_0; a contradiction as $|E(M_0)| \geq 10$. □

Lemma 9.2. *Let M be an internally 4-connected binary matroid with $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Suppose M has a pretty good bowtie labelled as in Figure 29, where each of $M\backslash 2$ and $M\backslash 7$ has an N-minor, $M\backslash 2$ is $(4,5,S,+)$-connected, and $M\backslash 7$ is $(4,4,S)$-connected. Then*

(i) *M has an internally 4-connected minor M' having an N-minor such that $|E(M)| - E(M')| = 1$; or*
(ii) *M has a good bowtie; or*
(iii) *M has a good augmented 4-wheel labelled as in Figure 30.*

Proof. As $M\backslash 2$ is $(4,5,S,+)$-connected, we see that

9.2.1. *$\{3,4,5,6,7\}$ is fully closed in $M\backslash 2$.*

Assume that none of (i)–(iii) holds. Then $M\backslash 7$ is not internally 4-connected. Thus $M\backslash 7$ has a 4-fan (x_1,x_2,x_3,x_4). By Lemma 9.1, $x_4 \notin \{1,2,3,4,5,6,7\}$. As $\{x_2,x_3,x_4,7\}$ is a cocircuit of M, it follows by orthogonality with the triangle $\{3,5,7\}$ that exactly one of 3 and 5 is in $\{x_2,x_3\}$. Therefore, by symmetry, we may assume that $x_3 \in \{3,5\}$.

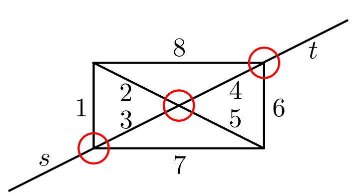

FIGURE 30

9.2.2. *If* $x_3 = 3$*, then* $(x_1, x_2, x_3) = (2, 1, 3)$.

To see this, observe that, by Lemma 2.8, $\{x_1, x_2, x_3\} = \{1, 2, 3\}$. If $x_2 = 2$, then M has $\{2, 3, x_4, 7\}$ as a cocircuit. But $x_4 \notin \{1, 2, 3, 4, 5, 6, 7\}$. Thus $x_4 \in \text{cl}^*_{M\backslash 2}(\{3, 4, 5, 6, 7\})$, a contradiction to 9.2.1. Hence 9.2.2 holds.

9.2.3. $x_3 \neq 5$.

Suppose $x_3 = 5$. Then, by Lemma 2.8, $\{x_1, x_2, x_3\} = \{4, 5, 6\}$. Thus $\{4, 5, 7, x_4\}$ or $\{5, 6, 7, x_4\}$ is a cocircuit of M, which contradicts 9.2.1. Hence 9.2.3 holds.

On combining 9.2.2 and 9.2.3, we see that M has a cocircuit $\{1, 3, 7, s\}$. Now $N \preceq M\backslash 2$. The last matroid has $(7, 3, 5, 4, 6)$ as a 5-fan. Hence $N \preceq M\backslash 2\backslash 6$, so $N \preceq M\backslash 6$. As (i) of the lemma does not hold, $M\backslash 6$ is not internally 4-connected.

Next we show the following.

9.2.4. $M\backslash 6$ *is* $(4, 4, S)$*-connected.*

Suppose $M\backslash 6$ is not $(4, 4, S)$-connected. Then, by Lemma 2.6, $\{4, 5, 6\}$ is the central triangle of a quasi rotor whose other triangles are $\{1, 2, 3\}$, $\{x, y, 8\}$, and $\{8, 9, 0\}$ and whose cocircuits are $\{2, 3, 4, 5\}$ and $\{y, 6, 8, 9\}$, for some x in $\{2, 3\}$ and some y in $\{4, 5\}$. If $x = 3$, then $\{3, 4, 5, 6, 8, 9\}$ is 3-separating in $M\backslash 2$; a contradiction. Hence $x = 2$. The triangle $\{3, 5, 7\}$ implies that $y \neq 5$. Hence $y = 4$ and M contains the configuration shown in Figure 30 where $t = 9$; a contradiction. We conclude that 9.2.4 holds.

Now $M\backslash 1$ has $(5, 7, 3, s)$ as a 4-fan, so it is not internally 4-connected. Thus (iv) of Lemma 2.6 does not hold. Therefore (ii) or (iii) of that lemma holds. But (ii) gives that M has a good bowtie. Hence (iii) holds. Then M has a triangle $\{x, y, z\}$ and a cocircuit $\{y, z, 6, u\}$ where $x \in \{2, 3\}$ and $y \in \{4, 5\}$. Now $(x, y) \neq (3, 5)$ otherwise $z = 7$ and $\{3, 4, 5, 6, 7, u\}$ is 3-separating in $M\backslash 2$; a contradiction. Similarly, $(x, y) \neq (3, 4)$ otherwise $\{3, 4, 5, 6, 7, z\}$ is 3-separating in $M\backslash 2$; a contradiction. We deduce that $x = 2$. If $y = 4$, then the configuration in Figure 30 occurs as a restriction of M; a contradiction. We may now assume that $y = 5$. Then, as $\{5, z, 6, u\}$ is a cocircuit of M and $\{3, 5, 7\}$ is a triangle, orthogonality implies that $\{3, 7\}$ meets $\{z, u\}$. But, by (iii) of Lemma 2.6, $|\{1, 2, 3, 4, 5, 6, z, u\}| = 8$. Thus $7 \in \{z, u\}$. Clearly $z \neq 7$, so $u = 7$. Then $\{3, 4, 5, 6, 7, z\}$ is 3-separating in $M\backslash 2$; a contradiction. We conclude that Lemma 9.2 holds. □

10. The proof the main result

This section completes the final details of the proof of the main result. We begin with the following.

Proof of Theorem 3.1. In Section 3, we outlined the steps in the proof of this theorem. These steps were completed in Sections 4–8, so the theorem holds. □

Proof of Theorem 1.3. By combining Theorem 3.1 with Lemma 9.2, we immediately obtain that Theorem 1.3 holds when $|E(N)| \geq 7$. But the hypothesis of the last theorem allows $|E(N)|$ to be 6. In that case, $N \cong M(K_4)$. Since every 3-connected binary matroid with at least six elements has an $M(K_4)$-minor, when $N \cong M(K_4)$, provided we maintain internal 4-connectivity, we also preserve an N-minor. It is straightforward to apply the main theorem of [1] to verify that Theorem 1.3 holds when $|E(N)| = 6$. □

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