TOWARDS A SPLITTER THEOREM FOR INTERNALLY 4-CONNECTED BINARY MATROIDS VIII: SMALL MATROIDS

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Abstract. Our splitter theorem studies pairs of the form (M, N), where M and N are internally 4-connected binary matroids, M has a proper N-minor, and if M' is an internally 4-connected matroid such that M has a proper M'-minor and M' has an N-minor, then $|E(M)| - |E(M')| > 3$. The analysis in the splitter theorem requires the constraint $|E(M)| \geq 16$. In this article, we complement that analysis by describing all such pairs for which $|E(M)| \leq 15$.

1. Introduction

A matroid is *internally* 4-*connected* if it is 3-connected and $\min\{|X|, |Y|\} = 3$ for any 3-separation, (X, Y). For some time, we have been engaged in a project to develop a splitter theorem for internally 4-connected binary matroids [2, 3, 4, 5, 6, 7, 8, 9]. This means that we are concerned with what we refer to here as interesting pairs. If N and M are matroids, we write $N \preceq M$ to mean that M has an N-minor, and $N \prec M$ to mean that M has a proper N-minor. An *interesting pair* is a pair (M, N), where M and N are internally 4-connected binary matroids such that

- $|E(N)| \geq 6$;
- $N \prec M$;
- if M' is an internally 4-connected matroid for which $N \preceq M' \prec M$, then $|E(M)| - |E(M')| > 3$.

Note that the last condition means that $|E(M)| - |E(N)| > 3$. We say that an interesting pair, (M, N), is a *fascinating pair* if M' is isomorphic to N whenever M' is an internally 4-connected matroid satisfying $N \preceq M' \prec M$. Thus an interesting pair is fascinating if there is no intermediate internally 4-connected matroid in the minor order.

It has been known for some time (see, for example, [11]) that there are fascinating pairs with $|E(M)| - |E(N)|$ arbitrarily large; indeed, this is true even if we insist that M and N are graphic matroids, since we can produce a fascinating pair by setting N to be the graphic matroid of a cubic planar ladder, and letting M be the graphic matroid of a quartic planar ladder on the same number of vertices. However, our project has shown that only a

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small number of constructions are needed to build M from N, whenever (M,N) is a fascinating pair.

The analysis in our project requires $E(M)$ to have at least 16 elements. To complement this analysis, this article describes all interesting pairs for which $|E(M)| \leq 15$. Our first theorem will describe the fascinating pairs. Up to duality, there are exactly 31. Before that, we introduce some important matroids and graphs.

For $n \geq 3$, we denote the *cubic Möbius ladder* on $2n$ vertices by CM_{2n}. This graph is obtained from a cycle on $2n$ vertices by joining each vertex to the vertex of distance n. Similarly, for $n \geq 2$, the *quartic Möbius ladder* on $2n+1$ vertices is denoted by QM_{2n+1}, and is obtained from a cycle with $2n+1$ vertices by joining each vertex to the two vertices of distance n. Note that QM_5 is isomorphic to K_5, and CM_6 is isomorphic to $K_{3,3}$.

The Möbius matroids have been discovered in several contexts [13, 14]. For each positive integer $n \geq 3$, let $\mathcal{W}_n$ be the wheel with $n+1$ vertices, and let B be the set of spoke edges. Thus B is a basis of the rank-n binary matroid $M(\mathcal{W}_n)$. Let M_n be the binary matroid obtained from $M(\mathcal{W}_n)$ by adding a single element, γ, so that the fundamental circuit, $C(\gamma, B)$, is $B \cup \gamma$. Kingan and Lemos [13] denote M_n by F_{2n+1}. Observe that M_3 is the Fano matroid, and $M_4 \cong M^*(K_{3,3})$. When n is odd, M_n^* is the *rank-$(n+1)$ triadic Möbius matroid*, denoted by Υ_{n+1}. Hence $\Upsilon_4 \cong F_7^*$. Moreover, Υ_6 is isomorphic to any single-element deletion of T_{12}, the rank-6 binary matroid introduced by Kingan [12]. We also observe that $\Upsilon_{n+1} \backslash \gamma \cong M^*(QM_n)$.

For $n \geq 3$, we construct the graph G^+_{n+2} by starting with an n-vertex cycle, C, and then adding two additional vertices, u and w, and making both of them adjacent to every vertex in C. We then join u and w with an edge, γ. Note that the planar dual of $G^+_{n+2} \backslash \gamma$ is CM_{2n}. Let x and y be adjacent vertices in C. Let Δ_{n+1} be the binary matroid that is obtained from $M(G^+_{n+2})$ by deleting the element xy and adding a new element so that it forms a circuit with the elements wx and uy. This new element also forms a circuit with ux and wy. Then Δ_{n+1} is the *rank-$(n+1)$ triangular Möbius matroid*. We define Δ_3 to be F_7. Note that $\Delta_{n+1} \backslash \gamma \cong M^*(CM_{2n})$. Kingan and Lemos [13] use B_{3n+1} to denote G^+_{n+2}, and S_{3n+1} to denote Δ_{n+1}.

Now we give our description of fascinating pairs. Any graphs or matroids which we have not yet defined will be introduced in Section 3. For now, we note that Q_3 is the cube graph; H_1, H_2, and H_3 are graphs with 13 edges, and, respectively, 6, 7, and 8 vertices; $Q_3^\times$ and Y_9 have 14 edges and, respectively, 8 and 9 vertices; A_1, A_2, A_3, A_4, and A_5 are non-graphic matroids with rank 8 and 14 elements, whereas A_6 has rank 7 and 14 elements; the matroids P and Q have rank 4 and 11 elements; each matroid of the form B_i or C_j has rank 8 and 15 elements; both R and S have rank 5 and 11 elements, while D_1 and E_1 have rank 9 and 15 elements.

Theorem 1.1. *Assume that (M_0, N_0) is a fascinating pair and $|E(M_0)| \leq 15$. Then, for some pair, (M, N) in $\{(M_0, N_0), (M_0^*, N_0^*)\}$, one of the following statements holds.*

(1) *M is one of $M(Q_3)$ or $M(K_5) \cong M(QM_5)$, and N is $M(K_4)$;*
(2) *M is one of Υ_6 or Υ_6^*, and N is $F_7 \cong \Upsilon_4^*$;*
(3) *M is one of $M(H_1)$, $M(H_2)$, $M(H_3)$, or $M(QM_7)$, and N is $M(K_{3,3}) \cong M(CM_6)$;*
(4) *M is one of $M(Q_3^\times)$, $M(Y_9)$, $M(QM_7)$, or $M(CM_{10})$, and N is $M(K_5) \cong M(QM_5)$;*
(5) *M is one of A_1, A_2, A_3, A_4, A_5, A_6, or Υ_8, and N is Δ_4;*
(6) *M is one of B_1, B_2, B_3, B_4, or B_5, and N is P;*
(7) *M is one of C_1, C_2, C_3, or C_4, and N is Q;*
(8) *$(M, N) = (D_1, R)$;*
(9) *$(M, N) = (E_1, S)$; or*
(10) *$(M, N) = (\Upsilon_8, \Upsilon_6)$.*

With Theorem 1.1 in hand, it is easy to find the pairs that are interesting but not fascinating: there are only three (up to duality).

Theorem 1.2. *Assume that (M_0, N_0) is an interesting pair that is not fascinating and that $|E(M_0)| \leq 15$. Then there is a pair, (M, N) in $\{(M_0, N_0), (M_0^*, N_0^*,)\}$, such that (M, N) is either $(M(QM_7), M(K_4))$, $(\Upsilon_8, F_7,)$, or (Υ_8^*, F_7).*

The following table shows the number of interesting pairs (up to duality), where the larger matroid has m elements in its ground set, and the smaller has n elements. Note that none of the pairs we have listed consists of two self-dual matroids, so if we were not taking duality into account, we would just double the numbers in the table.

n \ m	10	11	12	13	14	15
6	1		1		1	
7		2				2
8						
9				3	1	
10					9	2
11						12

Next we note the specialisation of our theorems to graphic matroids. Any graphs not already defined are described in Section 3. Let G be a simple, 3-connected graph. For any partition, (X, Y), of the edge set, let $V(X, Y)$ be the set of vertices incident with edges in both X and Y. We say that G is *internally* 4-*connected* if, whenever $3 \leq |X| \leq |Y|$ we have that $|V(X, Y)| \geq 3$, with equality implying that X is either a triangle or the set of edges incident with a vertex of degree 3. In other words, G is internally 4-connected if and only if $M(G)$ is an internally 4-connected matroid.

Theorem 1.3. *Assume G_1 and G_2 are internally 4-connected graphs such that $|E(G_1)| \leq 15$, and G_1 has a proper G_2-minor. Assume also that if G is an internally 4-connected graph such that G_1 has a proper G-minor, and G has a G_2-minor, then $|E(G_1)| - |E(G)| > 3$. Then one of the following statements holds.*

(1) *G_1 is one of K_5, Q_3, $K_{2,2,2}$, or QM_7, and G_2 is K_4;*
(2) *G_1 is one of H_1, H_2, H_3, or QM_7, and G_2 is $K_{3,3}$;*
(3) *G_1 is one of $Q_3^{\times}$, Y_9, QM_7, or CM_{10}, and G_2 is K_5.*

In many of the pairs in Theorems 1.1 and 1.2, we encounter structures that are familiar from the analysis in the rest of the project. These structures lead to operations that we can use to produce a smaller internally 4-connected matroid from a larger one. Four such operations will be documented in Section 2. In the following results, we explain exactly when it is possible to perform them on our interesting pairs.

Theorem 1.4. *Let the pair (M, N) be as described in one of the statements* (1)–(10) *in* Theorem 1.1. *If (M, N) is not one of $(M(Q_3), M(K_4))$, $(M(K_5), M(K_4))$, (Υ_6, F_7), (Υ_6^*, F_7), $(M(QM_7), M(K_{3,3}))$, or (Υ_8, Δ_4), then N can be obtained from M (or N^* can be obtained from M^*) by one of the following four operations:*

(1) *trimming a ring of bowties;*
(2) *deleting the central cocircuit of a good augmented 4-wheel;*
(3) *a ladder-compression move; or*
(4) *trimming an open rotor chain.*

The next corollary deals with the three interesting pairs identified in Theorem 1.2.

Corollary 1.5. *Let (M, N) be $(M(QM_7), M(K_4))$, (Υ_8, F_7), or (Υ_8^*, F_7). Then there is an internally 4-connected binary matroid, M_0, such that $N \prec M_0 \prec M$, and either M_0 can be obtained from M (or M_0^* can be obtained from M^*) by a ladder-compression move.*

Three of the six exceptional pairs in Theorem 1.4 are covered by specific scenarios from our main theorem [9]. In particular, since $\Delta_3 \cong F_7$, we see that if (M, N) is (Υ_6, F_7) or (Υ_8, Δ_4), then M is a triadic Möbius matroid of rank $2r$, and N is a triangular Möbius matroid of rank r. If (M, N) is $(M(QM_7), M(K_{3,3}))$, then M is the cycle matroid of a quartic Möbius ladder, and N is the cycle matroid of a cubic Möbius ladder, $K_{3,3} \cong CM_6$, and furthermore, $r(N) = r(M) - 1$. Thus the only truly exceptional pairs are $(M(K_5), M(K_4))$, $(M(Q_3), M(K_4))$, and (Υ_6^*, F_7).

We prove Theorems 1.1 and 1.2 with an exhaustive search, using the matroid functionality of the Sage mathematics package (Version 6.10) [17]. All the computations performed in this search were performed on a single desktop computer, and took a total of approximately 55 hours. In Section 4 we will sketch the procedures we used. Full details can be found at `arXiv:1501.`

00327. Some of the objects created during the search, such as a catalogue of 3-connected binary matroids with at most 15 elements, required a non-trivial amount of computation. Those objects, along with the Sage worksheet, `BinarySplitter.sws`, used in the search, are available for download at `http://homepages.ecs.vuw.ac.nz/~mayhew/splittertheorem`. The files are also hosted on SageMathCloud at `https://cloud.sagemath.com/projects/fa8ea5db-9456-4875-a4a6-56f202168fdc/files/`.

2. Winning Moves

In this section, we describe four different structures that appear naturally when we examine internally 4-connected binary matroids. Each structure allows us to perform certain deletions and contractions to obtain an internally 4-connected proper minor. These operations play an essential role in the statement of our splitter theorem. In Section 3, we analyse the pairs in Theorems 1.1 and 1.2, and demonstrate that, in many cases, these structures appear there also.

A 4-element fan is a set $\{x_1, x_2, x_3, x_4\}$, where $\{x_1, x_2, x_3\}$ is a triangle and $\{x_2, x_3, x_4\}$ is a triad. A 3-connected matroid, M, is $(4, 4, S)$-*connected* if, for every 3-separation, (X, Y), of M, one of X and Y is a triangle, a triad, or a 4-element fan.

A *bowtie* consists of a pair of disjoint triangles whose union contains a 4-element cocircuit. Assume $k \geq 2$, and $T_0, T_1, \ldots, T_k$ is a sequence of pairwise disjoint triangles. Let T_i be $\{a_i, b_i, c_i\}$ for $i \in \{0, 1, \ldots, k\}$. Assume $D_i = \{b_i, c_i, a_{i+1}, b_{i+1}\}$ is a cocircuit for $i \in \{0, 1, \ldots, k-1\}$, and, in addition, $D_k = \{b_k, c_k, a_0, b_0\}$ is a cocircuit. Then we say that $T_0, D_0, T_1, D_1, \ldots, T_k, D_k$ is a *ring of bowties.* Although the matroid M we are dealing with need not be graphic, we follow the convention begun in [1] of using a modified graph diagram to keep track of some of the circuits and cocircuits in M. Figure 1 shows such a modified graph diagram. Each of the cycles in such a graph diagram corresponds to a circuit of M while a circled vertex indicates a known cocircuit of M. If M' is obtained from M by deleting the dashed edges, then we say that M' is obtained from M by *trimming a ring of bowties.*

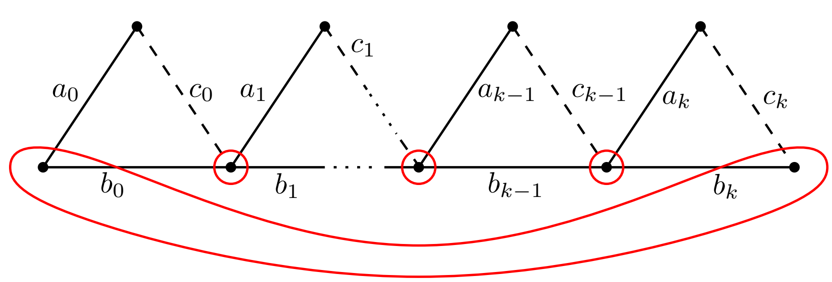

Figure 1. A ring of bowties. All elements are distinct.

An *augmented* 4-*wheel* is represented by the diagram in Figure 2, where the four dashed edges form the *central cocircuit*. If a matroid, M, contains the structure in Figure 2 and $M\backslash e$ is $(4,4,S)$-connected, then we say that the augmented 4-wheel is *good*.

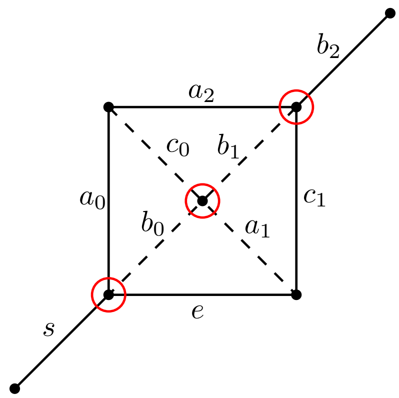

FIGURE 2. An augmented 4-wheel. All elements are distinct.

Our third structure requires a special four-element move. If M contains the structure in Figure 3, then we say that $M\backslash c_1, c_2/d_1, b_2$ is obtained from M by a *ladder-compression move*.

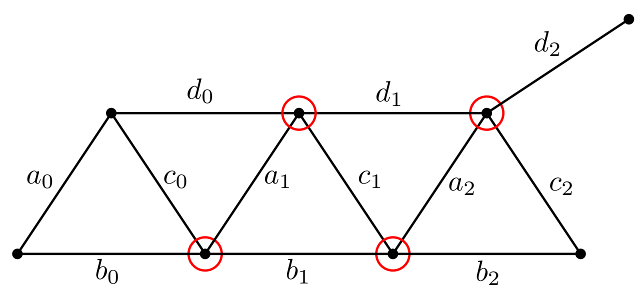

FIGURE 3. A ladder-segment. All elements are distinct.

Finally, we consider the structure in Figure 4. Note that n may be either even or odd. When there are at least three dashed elements, we refer to the structure in Figure 4 as an *open rotor chain* and we refer to the operation of deleting the dashed elements as *trimming an open rotor chain*.

3. THE SPECIAL GRAPHS AND MATROIDS

This section has two purposes. First, we introduce the graphs and matroids from Theorem 1.1 that have not already been defined. In many of the pairs from that theorem, it is possible to apply one of the four operations described in Section 2. Thus the second purpose of this section is

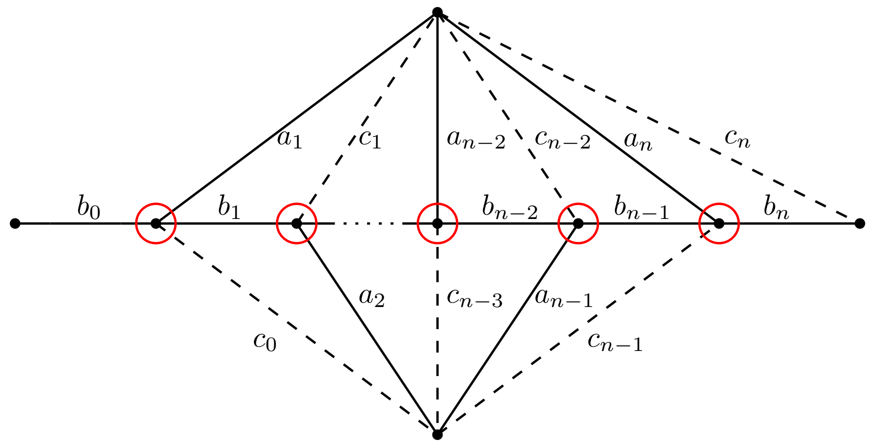

FIGURE 4. An open rotor chain. All elements are distinct.

to document when we are able to perform these operations, and thereby prove Theorem 1.4. For reference, we list the pairs from Theorem 1.1. The bolded pairs are those that appear in Theorem 1.4; that is, the pairs that do not admit one of the operations from Section 2 (or the dual of such an operation).

(1) $\boldsymbol{(M(Q_3), M(K_4))}$, $\boldsymbol{(M(K_5), M(K_4))}$
(2) $\boldsymbol{(\Upsilon_6, F_7)}$, $\boldsymbol{(\Upsilon_6^*, F_7)}$;
(3) $(M(H_1), M(K_{3,3}))$, $(M(H_2), M(K_{3,3}))$, $(M(H_3), M(K_{3,3}))$, $\boldsymbol{(M(QM_7), M(K_{3,3}))}$;
(4) $(M(Q_3^\times), M(K_5))$, $(M(Y_9), M(K_5))$, $(M(QM_7), M(K_5))$, $(M(CM_{10}), M(K_5))$;
(5) (A_i, Δ_4) for $i = 1, \ldots, 6$, $\boldsymbol{(\Upsilon_8, \Delta_4)}$;
(6) (B_i, P) for $i = 1, \ldots, 5$;
(7) (C_i, Q) for $i = 1, \ldots, 4$;
(8) (D_1, R);
(9) (E_1, S);
(10) (Υ_8, Υ_6).

Now we start describing various graphs and matroids, beginning with the graphs K_4, K_5, and Q_3, all of which are illustrated in Figure 5. The graph Q_3 is known as the *cube graph*. Figure 5 also shows the *octahedron graph*, $K_{2,2,2}$, which is the planar dual of Q_3.

In Lemma 2.3 of [10], Geelen and Zhou describe five internally 4-connected graphs having $K_{3,3} \cong CM_6$ as a minor. One of the five is CM_8, which has only 12 edges. Another is isomorphic to QM_7. Let the other three graphs be H_1, H_2, and H_3. These are shown in Figure 6.

Proposition 3.1. *Let (M, N) be one of the pairs $(M(H_1), M(K_{3,3}))$, $(M(H_2), M(K_{3,3}))$, or $(M^*(H_3), M^*(K_{3,3}))$. Then N is obtained from M by*

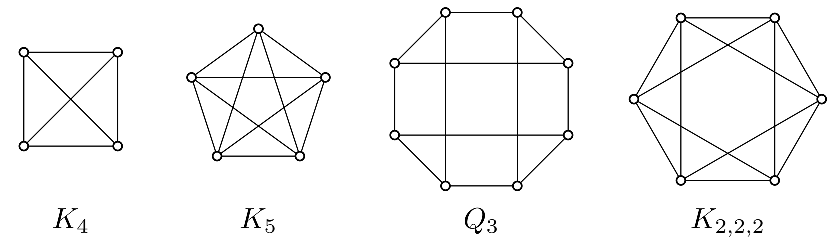

FIGURE 5. Graphs K_4, K_5, Q_3, and $K_{2,2,2}$.

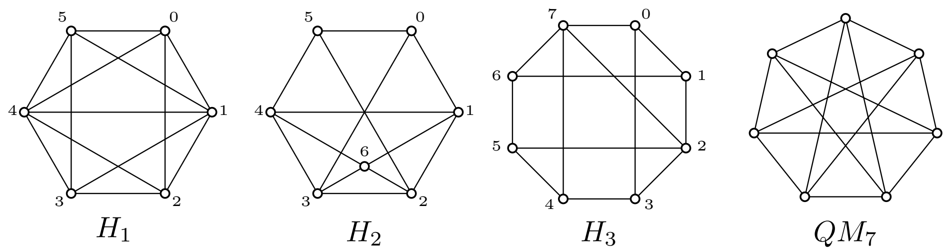

FIGURE 6. Graphs H_1, H_2, H_3, and QM_7.

trimming a bowtie ring, deleting the central cocircuit from a good augmented 4-wheel, or a ladder-compression move.

Proof. Note that $M(H_1)$ has the bowtie ring shown in Figure 7, and trimming this ring yields $M(K_{3,3})$. Also, $M(H_2)$ has a good augmented 4-wheel whose central cocircuit is the set of edges incident with vertex 6. Deleting this cocircuit yields $M(K_{3,3})$. Finally, $M^*(H_3)$ has the ladder segment shown in Figure 3, where edges $(16, 12, 01, 07, 03, 23, 34, 47, 45, 25, 56, 67)$ correspond to $(a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2)$. If we delete c_1 and c_2, and contract d_1 and b_2, then we obtain $M^*(K_{3,3})$. □

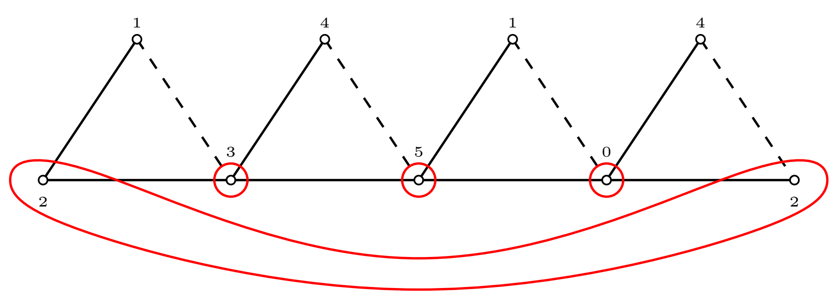

FIGURE 7. Bowtie ring in H_1.

Observe that of all the pairs in statements (1), (2), and (3) are either bolded, or dealt with by Proposition 3.1. Thus we have verified Theorem 1.4 for these pairs.

The graphs $Q_3^\times$ and Y_9 are shown in Figure 8, along with CM_{10}.

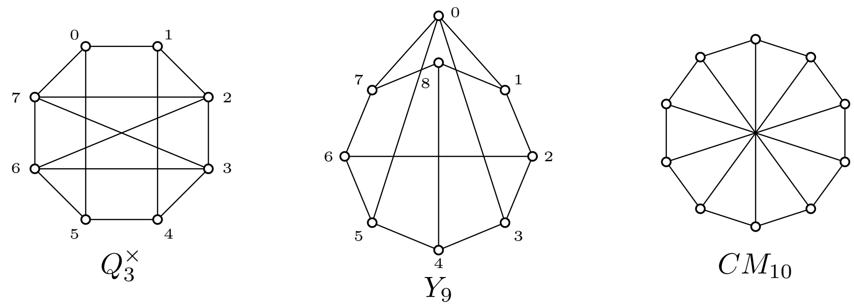

FIGURE 8. Graphs $Q_3^\times$, Y_9, and CM_{10}.

Proposition 3.2. *Let (M, N) be one of the pairs $(M^*(Q_3^\times), M^*(K_5))$, $(M^*(Y_9), M^*(K_5))$, $(M(QM_7), M(K_5))$, or $(M^*(CM_{10}), M^*(K_5))$. Then N is obtained from M by trimming a bowtie ring, deleting the central cocircuit from a good augmented 4-wheel, or a ladder-compression move.*

Proof. Figure 9 shows a labelling of some of the edges in $Q_3^\times$, along with a good augmented 4-wheel in $M^*(Q_3^\times)$. Deleting the central cocircuit of this augmented wheel produces $M^*(K_5)$. Figure 10 shows the labelling of a bowtie ring in $M^*(Y_9)$. Trimming this ring produces $M^*(K_5)$. Similarly, by trimming the bowtie ring shown in Figure 11, we can obtain $M^*(K_5)$ from $M^*(CM_{10})$. Finally, it is clear that $M(QM_{n-2})$ is obtained from $M(QM_n)$ by a ladder-compression move, so in particular this applies to $M(QM_7)$ and $M(QM_5) \cong M(K_5)$. □

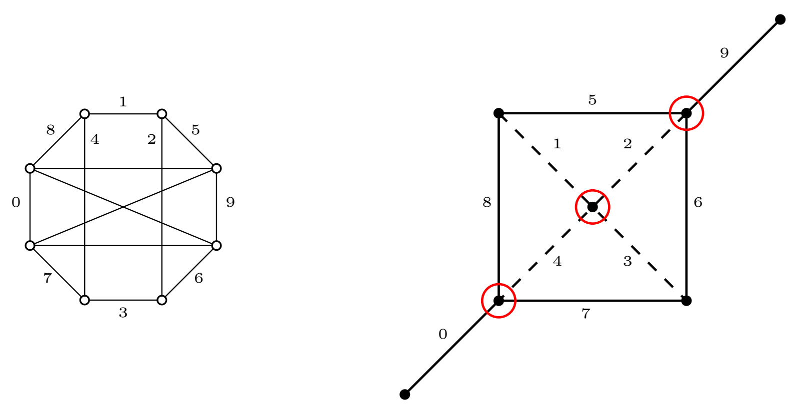

FIGURE 9. $Q_3^\times$ and a good augmented 4-wheel in $M^*(Q_3^\times)$.

Since Proposition 3.2 verifies Theorem 1.4 for the pairs listed in statement (4), we now turn to non-graphic binary matroids. We shall describe each of

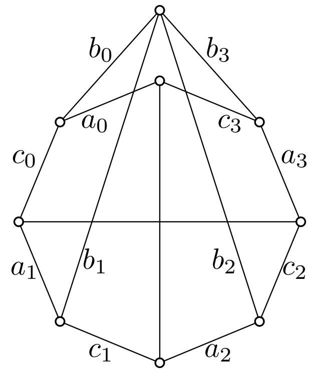

FIGURE 10. A bowtie ring in $M^*(Y_9)$.

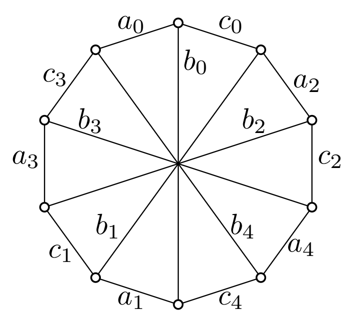

FIGURE 11. A bowtie ring in $M^*(CM_{10})$.

these matroids via reduced binary representations. For example, Figure 12 shows a matrix, A, where $[I_4 \mid A]$ represents Δ_4 over GF(2).

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccccc} 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

FIGURE 12. Representations of Δ_4 and P.

The matroids A_1, A_2, A_3, A_4, and A_5 have as reduced representations the reduced matrices shown in Figure 13. Thus each A_i, for $i = 1, \ldots, 5$, is a rank-8 binary matroid with 14 elements, and each contains a 4-element independent set whose contraction produces a minor isomorphic to Δ_4. The matroid A_6 is represented in Figure 14. We can produce a Δ_4-minor from A_6 by contracting a 3-element independent set and deleting a single element.

$$\underset{A_1}{\left[\begin{array}{cccccc} \multicolumn{6}{c}{A} \\ \hline 1&0&1&0&0&1 \\ 1&0&0&0&1&0 \\ 0&1&1&0&1&0 \\ 0&0&0&1&1&0 \end{array}\right]} \quad \underset{A_2}{\left[\begin{array}{cccccc} \multicolumn{6}{c}{A} \\ \hline 1&0&1&0&0&1 \\ 1&0&0&0&1&0 \\ 0&1&0&1&0&0 \\ 0&1&0&0&1&1 \end{array}\right]} \quad \underset{A_3}{\left[\begin{array}{cccccc} \multicolumn{6}{c}{A} \\ \hline 1&0&1&0&0&0 \\ 1&0&0&0&1&0 \\ 0&1&0&1&0&1 \\ 0&0&0&1&1&0 \end{array}\right]}$$

$$\underset{A_4}{\left[\begin{array}{cccccc} \multicolumn{6}{c}{A} \\ \hline 1&0&1&0&0&0 \\ 1&0&0&0&1&0 \\ 0&1&0&1&0&0 \\ 0&0&0&1&1&0 \end{array}\right]} \quad \underset{A_5}{\left[\begin{array}{cccccc} \multicolumn{6}{c}{A} \\ \hline 1&0&0&0&1&0 \\ 0&1&1&0&1&0 \\ 0&1&0&1&0&1 \\ 0&0&1&1&0&0 \end{array}\right]}$$

FIGURE 13. Representations of A_1, A_2, A_3, A_4, and A_5.

$$\underset{A_6}{\left[\begin{array}{cccccc|c} &&&&&& 1 \\ &&&&&& 1 \\ \multicolumn{6}{c|}{A} & 0 \\ &&&&&& 0 \\ \cline{1-6} 1&1&0&0&1&0&0 \\ 1&0&0&0&1&0&0 \\ 1&0&0&0&0&0&1 \end{array}\right]}$$

FIGURE 14. A representation of A_6.

Proposition 3.3. *Let (M,N) be one of the pairs (A_1^*, Δ_4^*), (A_2^*, Δ_4^*), (A_3^*, Δ_4^*), (A_4^*, Δ_4^*), (A_5^*, Δ_4^*), or (A_6^*, Δ_4^*). Then N is obtained from M by trimming a bowtie ring, trimming an open rotor chain, or deleting the central cocircuit from a good augmented 4-wheel.*

Proof. We will check that Δ_4^* is obtained from each of A_1^*, A_2^*, A_3^*, and A_5^* by trimming a bowtie ring. In Figure 13, assume that the matrices inherit the labels on rows and columns from A, so that the first four rows of any matrix are labelled 0, 1, 2, 3, the columns are labelled 4, 5, 6, 7, 8, 9, and the last four rows are labelled 10, 11, 12, and 13. Now A_1^* contains a bowtie ring, as in Figure 1, where $n = 3$, and the labelling is given as follows

$$(a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3) = (3, 0, 10, 9, 2, 12, 1, 5, 11, 8, 7, 13).$$

Trimming this ring produces Δ_4^*. Similar statements apply to A_2^*, A_3^*, and A_5^*. In those cases, the bowtie rings, $(a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3)$, are

- $(4, 8, 11, 5, 7, 12, 0, 3, 10, 2, 6, 13)$;
- $(4, 6, 10, 3, 2, 12, 1, 5, 11, 7, 8, 13)$; and
- $(1, 0, 12, 2, 9, 11, 7, 6, 13, 8, 4, 10)$

respectively.

The matroid A_4^* contains an open rotor chain, as in Figure 4, where $n = 3$, and we label so that

$$(b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3) = (2, 10, 3, 6, 13, 4, 8, 11, 7, 5, 12).$$

Trimming this rotor chain produces Δ_4^*.

Finally, for A_6, we assume the matrix in Figure 14 inherits the labels from A, and we label the extra column 10, and the extra rows as $11, 12$, and 13. Then A_6^* contains an augmented 4-wheel, as in Figure 2, where we label so that the elements $(e, s, a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2)$ are replaced by $(1, 0, 13, 10, 4, 11, 12, 5, 8, 7)$. Now $A_6^* \backslash 1$ is $(4, 4, S)$-connected, and $A_6^* \backslash 4, 10, 11, 12 \cong \Delta_4^*$, so the proof of the proposition is complete. □

Before we continue, we recall some introductory material. A simple rank-r binary matroid, M, can be considered as a subset, E, of points in the projective geometry $\mathrm{PG}(r-1, 2)$. The *complement* of M is the binary matroid corresponding to the set of points of $\mathrm{PG}(r-1, 2)$ not in E. The complement of M is well-defined by [15, Proposition 10.1.7], meaning that it depends only on M, and not on the choice of E. In particular, if two simple rank-r binary matroids have isomorphic complements, then they are themselves isomorphic. The complement of $M^*(K_{3,3})$ in $\mathrm{PG}(3, 2)$ is $U_{2,3} \oplus U_{2,3}$, and the complement of Δ_4 is $U_{2,2} \oplus U_{2,3}$. The complement of $M(K_5)$ in $\mathrm{PG}(3, 2)$ is $U_{4,5}$. From this, it follows that $M(K_5)$ has a unique simple rank-4 binary extension on 11 elements. We denote this extension by P, so the complement of P is $U_{4,4}$. The matrix B, shown in Figure 12, represents P over $\mathrm{GF}(2)$. Note that $P \backslash 10$ is isomorphic to $M(K_5)$, and that 10 is in triangles with $\{4, 9\}$, $\{5, 8\}$, and $\{6, 7\}$, where each of these pairs corresponds to a matching in K_5. The matroids B_1, B_2, B_3, B_4, and B_5 are represented by the matrices in Figure 15.

Proposition 3.4. *Let (M, N) be one of the pairs (B_1^*, P^*), (B_2^*, P^*), (B_3^*, P^*), (B_4^*, P^*), (B_5^*, P^*). Then N is obtained from M by trimming a bowtie ring.*

Proof. We assume that each matrix, B_i, inherits the labels on B, and that the extra rows are labelled 11, 12, 13, and 14. In B_1^*, there is a bowtie ring,

$$\underset{B_1}{\left[\begin{array}{ccccccc} \multicolumn{7}{c}{B} \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]} \qquad \underset{B_2}{\left[\begin{array}{ccccccc} \multicolumn{7}{c}{B} \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]}$$

$$\underset{B_3}{\left[\begin{array}{ccccccc} \multicolumn{7}{c}{B} \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]} \qquad \underset{B_4}{\left[\begin{array}{ccccccc} \multicolumn{7}{c}{B} \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]}$$

$$\underset{B_5}{\left[\begin{array}{ccccccc} \multicolumn{7}{c}{B} \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]}$$

FIGURE 15. Representations of B_1, B_2, B_3, B_4, and B_5.

as in Figure 1, with $n = 3$, where $(a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3)$ is relabelled as $(1, 3, 12, 0, 6, 11, 5, 9, 13, 7, 8, 14)$. Similarly, for B_2^*, B_3^*, B_4^*, and B_5^*, the relevant relabellings are

- $(1, 8, 12, 10, 5, 13, 2, 0, 11, 6, 3, 14)$;
- $(8, 5, 13, 0, 2, 11, 3, 9, 14, 4, 10, 12)$;
- $(10, 8, 14, 3, 1, 11, 0, 4, 12, 7, 5, 13)$; and
- $(8, 1, 12, 7, 2, 13, 5, 0, 11, 6, 3, 14)$. $\square$

Let Q be the binary matroid represented by the matrix C, below. Note that Q is obtained by extending Δ_4 by the element 10 in such a way that $\{0, 8, 10\}$ is a triangle. The complement of Q in $\mathrm{PG}(3, 2)$ is $U_{1,1} \oplus U_{2,3}$.

$$C = \begin{array}{c|c} & \begin{array}{ccccccc} 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

The matroids C_1, C_2, C_3, and C_4 are represented by the matrices in Figure 16.

$$\underset{C_1}{\begin{bmatrix} & & & C & & & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}} \quad \underset{C_2}{\begin{bmatrix} & & & C & & & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}}$$

$$\underset{C_3}{\begin{bmatrix} & & & C & & & \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{C_4}{\begin{bmatrix} & & & C & & & \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}}$$

FIGURE 16. Representations of C_1, C_2, C_3, and C_4.

Proposition 3.5. *Let (M, N) be one of the pairs (C_1^*, Q^*), (C_2^*, Q^*), (C_3^*, Q^*), (C_4^*, Q^*). Then N is obtained from M by trimming a bowtie ring.*

Proof. We assume that each matrix C_i inherits the row and column labels from C, and the extra rows are labelled 11, 12, 13, and 14. For C_1^*, C_2^*, C_3^*, and C_4^*, we relabel the elements $(a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3)$ in Figure 1 as

- $(1, 6, 12, 7, 9, 13, 2, 0, 11, 8, 10, 14)$;
- $(4, 9, 12, 2, 0, 11, 3, 7, 14, 8, 5, 13)$;
- $(9, 4, 12, 8, 6, 14, 1, 10, 11, 3, 5, 13)$; and
- $(7, 0, 11, 4, 1, 12, 5, 2, 13, 6, 3, 14)$. □

Propositions 3.3 to 3.5 verify Theorem 1.4 for the pairs listed in statements (5), (6), and (7). There are two matrices in Figure 17. The matrix D

represents the binary matroid R. Note that R is obtained from $M(K_5)$ by coextending by the element 10 so that 10 is in a triad with two elements that correspond to a 2-edge matching in K_5. Therefore R is isomorphic to the matroid obtained from P by performing a Δ-Y-operation on the triangle $\{4, 9, 10\}$.

$$D = \begin{array}{cc} & \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 10 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \underset{D_1}{\begin{bmatrix} \multicolumn{6}{c}{D} \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}$$

FIGURE 17. Representations of R and D_1.

Proposition 3.6. *R^* can be obtained from D_1^* by trimming a bowtie ring.*

Proof. Label the extra rows in D_1 that are not in D as 11, 12, 13, and 14. Then $(8, 3, 12, 6, 0, 11, 5, 2, 13, 7, 1, 14)$ is the appropriate bowtie ring. $\square$

The matroid S is represented by the matrix E, and E_1 is represented by the matrix shown in Figure 18. We can obtain S from Δ_4 by coextending by the element 10 so that it is in a triad with 0 and 8. Thus S can also be obtained from Q by a Δ-Y-operation.

$$E = \begin{array}{cc} & \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 10 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \qquad \underset{E_1}{\begin{bmatrix} \multicolumn{6}{c}{E} \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}}$$

FIGURE 18. Representations of S and E_1.

Proposition 3.7. *S^* can be obtained from E_1^* by trimming a bowtie ring.*

Proof. Label the extra rows in E_1 that are not in E as 11, 12, 13, and 14. Then $(1, 5, 11, 4, 9, 12, 7, 6, 14, 3, 2, 13)$ is the appropriate bowtie ring. $\square$

Recall that the Möbius matroids are defined in Section 1.

Proposition 3.8. *When $r \geq 6$ is an even integer, the matroid Υ_r^* can be obtained from Υ_{r+2}^* by a ladder-compression move.*

Proof. Recall that $\Upsilon_{r+2}^* = M_{r+1}$ and $\Upsilon_r^* = M_{r-1}$, where M_k is an extension of the rank-k wheel by the element γ. Assume that the spokes of $M(\mathcal{W}_{r+1})$, in cyclic order, are $x_0, x_1, \ldots, x_r$ and that $\{x_i, y_i, x_{i+1}\}$ is a triangle of $M(\mathcal{W}_{r+1})$ for $i = 0, 1, \ldots, r$. (We interpret subscripts modulo $r+1$.) Then, for $i = 0, 1, \ldots, r$, the set $\{y_i, x_{i+1}, y_{i+1}, \gamma\}$ is a cocircuit of M_{r+1}. We obtain M_{r-1} from M_{r+1} by contracting y_{r-1} and y_r, and deleting x_{r-1} and x_0, and then relabelling x_r as x_0. To see this, observe that M_{r+1} has $\{x_0, \ldots, x_r, \gamma\}$ and $\{x_{r-1}, x_r, y_{r-1}\}$ as circuits, so their symmetric difference, $C = \{x_0, \ldots, x_{r-2}, y_{r-1}, \gamma\}$, is a disjoint union of circuits. Orthogonality with the cocircuits containing γ implies that C is a circuit of M_{r+1}. Next we note that $\{x_{r-1}, x_r, y_{r-2}, y_r\}$ is the symmetric difference of $\{y_{r-2}, x_{r-1}, y_{r-1}, \gamma\}$ and $\{y_{r-1}, x_r, y_r, \gamma\}$, and is therefore a disjoint union of cocircuits. This implies that y_r is not in the closure of C in M_{r+1}. Therefore $C - y_{r-1} = \{x_0, \ldots, x_{r-2}, \gamma\}$ is a spanning circuit of $M_{r+1}/y_{r-1}, y_r \backslash x_{r-1}, x_0$, and it follows easily that this matroid is M_{r-1}, up to relabelling.

Now we need only show that this operation is a ladder-compression move. We note that M_{r+1} contains a ladder segment, as depicted in Figure 3, where the labels a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, b_2, c_2, and d_2 are replaced by x_{r-4}, y_{r-4}, x_{r-3}, y_{r-3}, x_{r-2}, y_{r-2}, x_{r-1}, y_{r-1}, x_r, y_r, x_0, and y_0, respectively. Because $r \geq 6$, these elements are all distinct. $\square$

Proposition 3.8 now implies that Υ_6^* can be obtained from Υ_8^* by a ladder-compression move. Thus we have completed the proof of Theorem 1.4.

Proof of Corollary 1.5. If (M, N) is $(M(QM_7), M(K_4))$, then we can set M_0 to be $M(QM_5) \cong M(K_5)$, and M_0 can be obtained from M by a ladder-compression move. If (M, N) is (Υ_8, F_7) or (Υ_8^*, F_7), then we can set M_0 to be Υ_6 or Υ_6^*, respectively. In either case, by Proposition 3.8, we can use a ladder-compression move to obtain M_0^* from M^* (in the first case), or M_0 from M (in the second). $\square$

4. A proof sketch

In this section we sketch our proofs of Theorems 1.1 and 1.2. All computation was carried out using Sage (Version 6.10). A full account is at `arXiv:1501.00327`. Assume that (M, N) is a fascinating pair that contradicts the statement of Theorem 1.1. We start by restricting the size of N.

4.1.1. $|E(N)| \in \{10, 11\}$.

Certainly $|E(N)| \leq 11$, since $|E(M)| \leq 15$, and (M, N) is a fascinating pair, so $|E(M)| - |E(N)| > 3$. Assume that $|E(N)| < 10$. First consider the case that $|E(N)| = 6$, so that N is isomorphic to $M(K_4)$. If M has a proper minor, M', such that $|E(M)| - |E(M')| \leq 3$, and M' is internally

4-connected, then M' has an $M(K_4)$-minor [16, Corollary 12.2.13], and hence (M,N) is not a fascinating pair. Therefore M has no such minor, so we can apply our chain theorem [1, Theorem 1.3]. Since $|E(M)| \leq 15$, it follows from that theorem that M is the cycle matroid of a planar or Möbius quartic ladder, or the dual of such a matroid. The only planar quartic ladder with fewer than 16 edges is the octahedron, $K_{2,2,2}$, which is the dual graph of Q_3, the cube. The only Möbius quartic ladders with fewer than 16 edges have 14 or 10 edges. The former has the latter as a minor, and the latter is isomorphic to K_5. From this we deduce that, up to duality, (M,N) is $(M(Q_3), M(K_4))$ or $(M(K_5), M(K_4))$, and that therefore (M,N) is not a counterexample after all. Hence $6 < |E(N)| < 10$. Up to duality, the only internally 4-connected binary matroids satisfying this constraint are F_7 and $M(K_{3,3})$ [10, Lemma 2.1].

From this point, we use almost exactly the same arguments as in [4, Lemma 2.3]. Assume N is F_7, so $|E(M)| \geq 11$. We can use [18, Corollary 1.2] to deduce that M is isomorphic to $T_{12}\backslash e \cong \Upsilon_6$ or $T_{12}/e \cong \Upsilon_6^*$, so (M,N) fails to contradict the theorem. Therefore we assume N is $M(K_{3,3})$, and hence $|E(M)| \geq 13$. Now we can use [10, Lemma 2.3]. This lemma defines five graphs, but only four of them have at least 13 edges. Therefore we can deduce that M is isomorphic to one of the graphic matroids $M(H_1)$, $M(H_2)$, $M(H_3)$, or $M(QM_7)$. Again this is a contradiction, as it implies that (M,N) is not a counterexample, so the proof of 4.1.1 is complete.

At this point, it is appropriate to verify that the pairs mentioned in the proof of 4.1.1 are indeed fascinating. Given a pair, (M,N), we consider all flats, F, of M such that $0 \leq r(F) \leq r(M) - r(N)$. If M/F has a proper N-minor, then we examine subsets, D, of $E(M/F)$ such that $|E(N)| < |E(M/F\backslash D)| < |E(M)|$. If $M/F\backslash D$ is internally 4-connected and has an N-minor, then we have found a certificate that (M,N) is not fascinating. If we fail to find any such certificate, then (M,N) is fascinating. In this way, we confirm that all the pairs in statements (1), (2), and (3) of Theorem 1.1 are fascinating.

By duality, we may assume that $r(M) \leq r^*(M)$. As $|E(M)| \leq 15$, the next result is a consequence.

4.1.2. *$r(N) \leq r(M) \leq 7$.*

Next we create a catalogue of all 3-connected binary matroids with ground sets of cardinality between 6 and 15 and rank at most 7. Every 3-connected binary matroid with at least 6 elements contains an $M(K_4)$-minor [16, Corollary 12.2.13]. We populate our catalogue by starting with this matroid, and enlarging the catalogue through single-element extensions and coextensions. When we extend, we ensure we produce no coloops, no loops, and no parallel pairs. Dually, when we coextend, we create no loops, coloops, or series pairs. Thus we only ever create 3-connected matroids [15, Proposition 8.1.10]. Every 3-connected binary matroid can be constructed in this way, with the exception of wheels [16, Theorem 8.8.4], so we initiate by adding the wheels of rank 3, 4, 5, 6, and 7. In this way, we guarantee that our catalogue will contain every 3-connected binary matroid with suitable size and rank.

The generation of the catalogue is initially quick, but it becomes time-consuming as we process larger matroids. In total, populating the catalogue takes about 24 hours. The file, `catalogue.sobj`, which contains the catalogue, is available for download. Table 1 shows the number of 3-connected binary matroids with rank r and size n.

n \ r	3	4	5	6	7
6	1	0	0	0	0
7	1	1	0	0	0
8	0	3	0	0	0
9	0	4	4	0	0
10	0	4	16	4	0
11	0	3	37	37	3
12	0	2	68	230	68
13	0	1	98	983	983
14	0	1	121	3360	10035
15	0	1	140	10012	81218

TABLE 1. 3-connected binary matroids.

We examine each of these 3-connected matroids to find those that are internally 4-connected. In this way, we create a catalogue file, `ifccatalogue.sobj`, containing all internally 4-connected binary matroids with size at most 15 and rank at most 7. Table 2 shows the number of such matroids.

n \ r	3	4	5	6	7
6	1	0	0	0	0
7	1	1	0	0	0
8	0	0	0	0	0
9	0	1	1	0	0
10	0	2	2	2	0
11	0	2	7	7	2
12	0	2	24	46	24
13	0	1	52	272	272
14	0	1	84	1389	3385
15	0	1	116	5816	36962

TABLE 2. Internally 4-connected binary matroids.

Now we know that there are exactly 24 internally 4-connected binary matroids with ground sets of cardinality 10 or 11. We think of these as "target" matroids. We process each of the internally 4-connected matroids in our catalogue with a ground set of cardinality 11, 12, 13, or 14, and record in a lookup table, `targetminors.sobj`, which of the 24 target matroids it has as a proper minor.

Next we search for fascinating pairs of the form (M, N), where $|E(M)| = 15$. In this case, N must be one of the 24 target matroids. For each internally 4-connected matroid, M, with $|E(M)| = 15$, we seek to eliminate target matroids as candidates for N. If a target matroid is not isomorphic to a minor of M, then it is certainly not a candidate for N. Having eliminated any such target matroids, we then process internally 4-connected matroids of size 11, 12, 13, and 14. Let M' be such a matroid. If M has an M' minor, then we use the lookup table to find the target matroids that are isomorphic to minors of M'. Any such target matroid cannot be N, because M' is an intermediate matroid in the minor-order. If we eliminate every target matroid as a candidate for N, then we know that M does not appear in a fascinating pair, and we stop processing it. On the other hand, if we have considered every possible M', and the target matroid N has not been eliminated, then we know that (M, N) is a fascinating pair. Processing the 15-element matroids in this way takes approximately 21 hours and produces a list of 14 pairs.

We repeat this procedure for fascinating pairs, (M, N), where $|E(M)| = 14$. In this case, $|E(N)| = 10$, so we need consider only six of the target matroids. Furthermore, the potential intermediate matroid, M', can be assumed to have size 11, 12, or 13. This procedure take only 17 minutes, and produces 11 pairs. However, two of these pairs are duals of other pairs, so up to duality, the procedure discovers 9 new pairs. Therefore, amongst fascinating pairs, (M, N), with $|E(M)| \leq 15$, there are, up to duality, two containing $M(K_4)$, two containing F_7, and four containing $M(K_{3,3})$. The computer search finds an additional 23 pairs. The proof of Theorem 1.1 is completed by simply checking that the 23 pairs found by the computer are all contained in the statement of the theorem, up to duality.

From Theorem 1.1, it is straightforward to prove Theorem 1.2. If (M, M_0) is interesting but not fascinating, then there is an internally 4-connected matroid, M_1, satisfying $M_0 \prec M_1 \prec M$. Now (M, M_1) is an interesting pair, so we can repeat this argument. Continuing in this way, we see that if (M, M_0) is interesting but not fascinating, then $M_0 \prec N \prec M$ for some internally 4-connected matroid, N, such that (M, N) is a fascinating pair.

This observation gives us our strategy for finding all interesting pairs. Let (M, N) range over all fascinating pairs (up to duality) with $|E(M)| \leq 15$. Consider each matroid, T, from the catalogue of internally 4-connected matroids, such that N has a proper T-minor. We test to see whether any proper minor of M produced by deleting and contracting at most three elements is internally 4-connected with a T-minor. If not, then (M, T) is interesting but not fascinating. Applying this check to all the fascinating pairs in Theorem 1.1 produces the pairs $(M(QM_7), M(K_4))$, (Υ_8^*, F_7), and (Υ_8^*, F_7^*). This completes the proof of Theorem 1.2.

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