The Internally 4-Connected Binary Matroids With No $M(K_{3,3})$-Minor.

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Abstract

We give a characterization of the internally 4-connected binary matroids that have no minor isomorphic to $M(K_{3,3})$. Any such matroid is either cographic, or is isomorphic to a particular single-element extension of the bond matroid of a cubic or quartic Möbius ladder, or is isomorphic to one of eighteen sporadic matroids.

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CHAPTER 1

Introduction

The Fano plane, the cycle matroids of the Kuratowski graphs, and their duals, are of fundamental importance in the study of binary matroids. The famous excluded-minor characterizations of Tutte [**Tut58**] show that the classes of regular, graphic, and cographic matroids are all obtained by taking the binary matroids with no minors in some subset of the family

$$\{F_7,\, F_7^*,\, M(K_{3,3}),\, M(K_5),\, M^*(K_{3,3}),\, M^*(K_5)\}.$$

It is natural to consider other classes of binary matroids produced by excluding some subset of this family. A number of authors have investigated such classes of binary matroids. We examine the binary matroids with no minor isomorphic to $M(K_{3,3})$, and completely characterize the internally 4-connected members of this class.

Theorem 1.1. *An internally* 4*-connected binary matroid has no* $M(K_{3,3})$*-minor if and only if it is either:*

(i) cographic;
(ii) isomorphic to a triangular or triadic Möbius matroid; or,
(iii) isomorphic to one of 18 *sporadic matroids of rank at most* 11 *listed in Appendix* B.

The triangular and triadic Möbius matroids form two infinite families of binary matroids. Each Möbius matroid is a single-element extension of a cographic matroid; in particular, a single-element extension of the bond matroid of a cubic or quartic Möbius ladder. For every integer $r \geq 3$ there is a unique triangular Möbius matroid of rank r, denoted by Δ_r, and for every even integer $r \geq 4$ there is a unique triadic Möbius matroid of rank r, denoted by Υ_r. We describe these matroids in detail in Chapter 3.

Seymour's decomposition theorem for regular matroids is one of the most fundamental results in matroid theory, and was the first structural decomposition theorem proved for a class of matroids. The following characterization of the internally 4-connected regular matroids is an immediate consequence of the decomposition theorem.

Theorem 1.2 (Seymour [**Sey80**]). *An internally* 4*-connected regular matroid is either graphic, cographic, or isomorphic to a particular sporadic matroid* (R_{10})*.*

Our theorem, and its proof, bears some similarity to Seymour's theorem. Just as Seymour's theorem leads to a polynomial-time algorithm for deciding whether a binary matroid (represented by a matrix over GF(2)) has a minor isomorphic to F_7 or F_7^*, our characterization leads to a polynomial-time algorithm (to be discussed in a subsequent article) for deciding whether a represented binary matroid has an $M(K_{3,3})$-minor.

Other authors who have studied families of binary matroids with no minors in some subset of $\{F_7, F_7^*, M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$ include Walton and Welsh [**WW80**], who examine the characteristic polynomials of matroids in such classes, and Kung [**Kun86**], who has considered the maximum size obtained by a simple rank-r matroid in one of these classes. Qin and Zhou [**QZ04**] have characterized the internally 4-connected binary matroids with no minor in $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$. Moreover Zhou [**Zho08**] has also characterized the internally 4-connected binary matroids that have no $M(K_{3,3})$-minor, but which do have an $M(K_5)$-minor. Finally we note that the classic result of Hall [**Hal43**] on graphs with no $K_{3,3}$-minor leads to a characterization of the internally 4-connected binary matroids with no minor in $\{F_7, F_7^*, M^*(K_{3,3}), M^*(K_5), M(K_{3,3})\}$, and Wagner's [**Wag37**] theorem on the graphs with no K_5-minor leads to a characterization of the internally 4-connected binary matroids with no minor in $\{F_7, F_7^*, M^*(K_{3,3}), M^*(K_5), M(K_5)\}$.

The proof of Theorem 1.1 is unusual amongst results in matroid theory, in that we rely upon a computer to check certain facts. All these checks have been carried out using the software package Macek, developed by Petr Hliněný. In addition we have written software, which does not depend upon Macek, to provide us with an independent check of the same facts.

The Macek package is available to download, along with supporting documentation. The current website is `http://www.fi.muni.cz/~hlineny/MACEK`. Thus the interested reader is able to download Macek and confirm that it verifies our assertions. Points in the proof where a computer check is required are marked by a marginal symbol ✠ and a number. The numbers provide a reference for the website of the second author (current url: `http://www.maths.uwa.edu.au/~gordon`), which contains a more detailed description of the steps taken to verify each assertion, and the intermediate data produced during that computer check.

We emphasize that none of the computer tests relies upon an exhaustive search of all the matroids of a particular size or rank. The only tasks a program need perform to verify our checks are: determine whether two binary matroids are isomorphic; check whether a binary matroid has a particular minor; and, generate all the single-element extensions and coextensions of a binary matroid. Whenever the proof requires a computer check, the text includes a complete description (independent of any particular piece of software) of the tasks that we ask the computer to perform. Hence a reader who is able to construct software with the capabilities listed above could provide another verification of our assertions.

The article is organized as follows: In Chapter 2 we develop the basic definitions and results we will need to prove Theorem 1.1. In Chapter 3 we introduce the Möbius matroids and consider their properties in detail.

Chapter 4 is concerned with showing that a minimal counterexample to Theorem 1.1 can be assumed to be vertically 4-connected. This chapter depends heavily upon the Δ-Y operation and its dual; we make extensive use of results proved by Oxley, Semple, and Vertigan [**OSV00**]. The central idea of Chapter 4 is that if a binary matroid M with no $M(K_{3,3})$-minor is non-cographic and internally 4-connected but not vertically 4-connected, then by repeatedly performing Y-Δ operations, we can produce a vertically 4-connected non-cographic binary matroid M' with no $M(K_{3,3})$-minor. In Lemmas 4.5 and 4.9 we show that if M' obeys Theorem 1.1, then M also satisfies the theorem. From this it follows that a minimal counterexample to Theorem 1.1 can be assumed to be vertically 4-connected.

The regular matroid R_{12} was introduced by Seymour in the proof of his decomposition theorem. He shows that R_{12} contains a 3-separation, and that this 3-separation persists in any regular matroid that contains R_{12} as a minor. In Chapter 5 we introduce a binary matroid Δ_4^+ that plays a similar role in our proof. The matroid Δ_4^+ is a single-element coextension of Δ_4, the rank-4 triangular Möbius matroid. It contains a four-element circuit-cocircuit, which necessarily induces a 3-separation. We show that this 3-separation persists in any binary matroid without an $M(K_{3,3})$-minor that has Δ_4^+ as a minor. Hence no internally 4-connected binary matroid without an $M(K_{3,3})$-minor can have Δ_4^+ as a minor. In Corollary 5.3 we show that if M is a 3-connected binary matroid without an $M(K_{3,3})$-minor such that M has both a Δ_4-minor and a four-element circuit-cocircuit, then M has Δ_4^+ as a minor.

Suppose that M is a minimal counterexample to Theorem 1.1. It follows easily from a result of Zhou [**Zho04**] that M must have a minor isomorphic to Δ_4. Hence we deduce that if M' is a 3-connected minor of M, and M' has a Δ_4-minor, then M' has no four-element circuit-cocircuit. This is one of the conditions required to apply the connectivity lemma that we prove in Chapter 6.

The hypotheses of the connectivity result in Chapter 6 are that M and N are simple vertically 4-connected binary matroids such that $|E(N)| \geq 10$ and M has a proper N-minor. Moreover, whenever M' is a 3-connected minor of M with an N-minor, then M' has no four-element circuit-cocircuit. Under these conditions, Theorem 6.1 asserts that M has a internally 4-connected proper minor M_0 such that M_0 has an N-minor and $|E(M)| - |E(M_0)| \leq 4$.

The case-checking required to complete our proof would be impossible if we had no more information than that provided by Theorem 6.1. Lemma 6.7 provides a much more fine-grained analysis. It shows that there are nine very specific ways in which M_0 can be obtained from M.

In Chapter 7 we complete the proof of Theorem 1.1. Our strategy is to assume that M is a vertically 4-connected minimal counterexample to the theorem. We then apply Lemma 6.7, and deduce the presence of a non-cographic proper minor M_0 of M that must necessarily obey Theorem 1.1. The rest of the proof consists of a case-check to show that the counterexample M cannot be produced by reversing the nine procedures detailed by Lemma 6.7 and applying them to the Möbius matroids and the 18 sporadic matroids. Thus a counterexample to Theorem 1.1 cannot exist.

The first of the three appendices contains a description of the case-checking required to complete the proof that a 3-connected binary matroid with both a Δ_4-minor and a four-element circuit-cocircuit has a Δ_4^+-minor. The second appendix describes the sporadic matroids, and the third contains information on the three-element circuits of the sporadic matroids.

In a subsequent article [**MRW**] we consider various applications of Theorem 1.1. In particular, we consider the classes of binary matroids produced by excluding any subset of $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$ that contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. We characterize the internally 4-connected members of these classes, and show that each such class has a polynomial-time recognition algorithm (where the input consists of a matrix with entries from GF(2)). We also consider extremal results and the critical exponent for these classes.

CHAPTER 2

Preliminaries

In this chapter we define basic ideas and develop the fundamental tools we will need to prove our main result. Terminology and notation will generally follow that of Oxley [**Oxl92**]. A *triangle* is a three-element circuit and a *triad* is a three-element cocircuit. We denote the simple matroid canonically associated with a matroid M by $\mathrm{si}(M)$, and we similarly denote the canonically associated cosimple matroid by $\mathrm{co}(M)$. Suppose that $\{M_1, \ldots, M_t\}$ is a collection of binary matroids. We denote the class of binary matroids with no minor isomorphic to one of the matroids in $\{M_1, \ldots, M_t\}$ by $\mathcal{EX}(M_1, \ldots, M_t)$.

2.1. Connectivity

Suppose M is a matroid on the ground set E. The function λ_M, known as the *connectivity function* of M, takes subsets of E to non-negative integers. If X is a subset of E, then $\lambda_M(X)$ (or $\lambda(X)$ where there is no ambiguity) is defined to be $r_M(X) + r_M(E - X) - r(M)$. Note that λ is a symmetric function: that is, $\lambda(X) = \lambda(E - X)$. It is well known (and easy to confirm) that the function λ_M is submodular, which is to say that $\lambda_M(X) + \lambda_M(Y) \geq \lambda_M(X \cup Y) + \lambda_M(X \cap Y)$ for all subsets X and Y of $E(M)$. Moreover, if N is a minor of M using the subset X, then $\lambda_N(X) \leq \lambda_M(X)$.

A *k-separation* is a partition (X, Y) of E such that $\min\{|X|, |Y|\} \geq k$ and $\lambda(X) = \lambda(Y) < k$. The subset $X \subseteq E$ is a *k-separator* if $(X, E - X)$ is a k-separation. (Note that this definition of k-separators differs from that used in [**OSW04**].) A k-separation (X, Y) is *exact* if $\lambda(X) = \lambda(Y) = k - 1$. A *vertical k-separation* is a k-separation (X, Y) such that $\min\{r(X), r(Y)\} \geq k$, and a subset $X \subseteq E$ is a *vertical k-separator* if $(X, E - X)$ is a vertical k-separation.

A matroid is *n-connected* if it has no k-separations where $k < n$. It is *vertically n-connected* if it has no vertical k-separations where $k < n$. It is *(n, k)-connected* if it is $(n - 1)$-connected and whenever (X, Y) is an $(n - 1)$-separation, then either $|X| \leq k$ or $|Y| \leq k$.

In addition, we shall say that a matroid M is *almost vertically 4-connected* if it is vertically 3-connected, and whenever (X, Y) is a vertical 3-separation of M, then there exists a triad T such that either $T \subseteq X$ and $X \subseteq \mathrm{cl}_M(T)$, or $T \subseteq Y$ and $Y \subseteq \mathrm{cl}_M(T)$.

We shall use the notion of (n, k)-connectivity only in the case $n = 4$. A $(4, 3)$-connected matroid is *internally 4-connected*.

The next results collect some easily-confirmed properties of the different types of connectivity.

Proposition 2.1. *(i) A simple binary vertically 4-connected matroid is internally 4-connected.*
(ii) A vertically 4-connected matroid with rank at least four has no triads.
(iii) Both vertically 4-connected and internally 4-connected matroids are also almost vertically 4-connected.
(iv) An almost vertically 4-connected matroid with no triads is vertically 4-connected.

Proposition 2.2. *Suppose that (X_1, X_2) is a k-separation of the matroid M, and that N is a minor of M. If $|E(N) \cap X_i| \geq k$ for $i = 1, 2$, then $(E(N) \cap X_1, E(N) \cap X_2)$ is a k-separation of N.*

Proposition 2.3. *Suppose that (X, Y) is a vertical k-separation of the matroid M and that $e \in Y$. If $e \in \mathrm{cl}_M(X)$, then $(X \cup e, Y - e)$ is a vertical k'-separation of M, where $k' \in \{k, k-1\}$.*

Proposition 2.4. *Let e be a non-coloop element of the matroid M. Suppose that (X, Y) is a k-separation of $M\backslash e$. Then $(X \cup e, Y)$ is a k-separation of M if and only if $e \in \mathrm{cl}_M(X)$ and $(X, Y \cup e)$ is a k-separation of M if and only if $e \in \mathrm{cl}_M(Y)$.*

The following result is due to Bixby [**Bix82**].

Proposition 2.5. *Let e be an element of the 3-connected matroid M. Then either $\mathrm{si}(M/e)$ or $\mathrm{co}(M\backslash e)$ is 3-connected.*

Suppose that $(e_1, \ldots, e_t)$ is an ordered sequence of at least three elements from the matroid M. Then $(e_1, \ldots, e_t)$ is a *fan* if $\{e_i, e_{i+1}, e_{i+2}\}$ is a triangle of M whenever $i \in \{1, \ldots, t-2\}$ is odd and a triad whenever i is even. Dually, $(e_1, \ldots, e_t)$ is a *cofan* if $\{e_i, e_{i+1}, e_{i+2}\}$ is a triad of M whenever $i \in \{1, \ldots, t-2\}$ is odd and a triangle whenever i is even. Note that if $(e_1, \ldots, e_t)$ is a fan and t is even, then $(e_t, \ldots, e_1)$ is a cofan. We shall say that the unordered set X is a fan if there is some ordering $(e_1, \ldots, e_t)$ of the elements of X such that $(e_1, \ldots, e_t)$ is either a fan or a cofan. The *length* of a fan X is the cardinality of X. It is straightforward to check that if $\{e_1, \ldots, e_t\}$ is a fan of M, then $\lambda_M(\{e_1, \ldots, e_t\}) \leq 2$. The next result is easy to confirm.

Proposition 2.6. *Suppose that (X, Y) is a 3-separation of the 3-connected binary matroid M. If $|X| \leq 5$ and $r_M(X) \geq 3$ then one of the following holds:*
(i) X is a triad;
(ii) X is a fan with length four or five; or,
(iii) There is a four-element circuit-cocircuit $C^* \subseteq X$ such that $X \subseteq \mathrm{cl}_M(C^*)$ or $X \subseteq \mathrm{cl}^*_M(C^*)$.

The next result follows directly from a theorem of Oxley [**Oxl87**, Theorem 3.6].

LEMMA 2.7. *Let T be a triangle of a 3-connected binary matroid M. If the rank and corank of M are at least three then M has an $M(K_4)$-minor in which T is a triangle.*

2.2. Fundamental graphs

Fundamental graphs provide a convenient way to visualize a representation of a binary matroid with respect to a particular basis. Suppose that B is a basis of the binary matroid M. The *fundamental graph* of M with respect to B, denoted by $G_B(M)$, has $E(M)$ as its vertex set. Every edge of $G_B(M)$ joins an element in B to an element in $E(M) - B$. If $x \in B$ and $y \in E(M) - B$, then x and y are joined by an edge if and only if x is in the unique circuit contained in $B \cup y$. Equivalently, $G_B(M)$ is the bipartite graph represented by the bipartite adjacency matrix A, where M is represented over GF(2) by the matrix $[I_r|A]$ (assuming that the columns of I_r are labeled with the elements of B while the columns of A are labeled by the elements of $E(M) - B$). Thus a labeled fundamental graph of a binary matroid completely determines that matroid. By convention, the elements of B are represented by solid vertices in drawings of $G_B(M)$, while the elements of $E(M) - B$ are represented by hollow vertices.

Suppose that M is a binary matroid on the ground set E and B is a basis of M. Suppose also that $X \subseteq B$ and $Y \subseteq E - B$. It is easy to see that the fundamental graph $G_{B-X}(M/X\backslash Y)$ is equal to the subgraph of $G_B(M)$ induced by $E - (X \cup Y)$.

If $x \in B$ and $y \in E - B$ and x and y are adjacent in $G_B(M)$, then the fundamental graph $G_{(B-x)\cup y}(M)$ is obtained by *pivoting* on the edge xy. In particular, if

$$N_G(u) = \{v \in V(G) - u \mid v \text{ is adjacent to } u\}$$

is the set of neighbors of u in the graph G, then $G_{(B-x)\cup y}(M)$ is obtained from $G_B(M)$ by replacing every edge between $N_{G_B(M)}(x)$ and $N_{G_B(M)}(y)$ with a non-edge, every non-edge between $N_{G_B(M)}(x)$ and $N_{G_B(M)}(y)$ with an edge, and then switching the labels on x and y.

2.3. Blocking sequences

Suppose that B is a basis of the matroid M and let X be a subset of $E(M)$. Define $M[X, B]$ to be the minor of M on the set X produced by contracting $B - X$ and deleting $E(M) - (B \cup X)$. Let X and Y be disjoint subsets of $E(M)$. It is easy to verify that $\lambda_{M[X\cup Y, B]}(X)$ is equal to the parameter $\lambda_B(X, Y)$, defined by Geelen, Gerards, and Kapoor [**GGK00**].

Suppose (X, Y) is a k-separation of $M[X \cup Y, B]$. A *blocking sequence* of (X, Y) is a sequence $e_1, \ldots, e_t$ of elements from $E(M) - (X \cup Y)$ such that:

(i) $(X, Y \cup e_1)$ is not a k-separation of $M[X \cup Y \cup e_1, B]$;
(ii) $(X \cup e_i, Y \cup e_{i+1})$ is not a k-separation of $M[X \cup Y \cup \{e_i, e_{i+1}\}, B]$ for $1 \leq i \leq t-1$;
(iii) $(X \cup e_t, Y)$ is not a k-separation of $M[X \cup Y \cup e_t, B]$; and,
(iv) no proper subsequence of $e_1, \ldots, e_t$ satisfies conditions (i)–(iii).

Blocking sequences were developed by Truemper [**Tru86**], and by Bouchet, Cunningham, and Geelen [**BCG98**]. They were later employed by Geelen, Gerards, and Kapoor in their characterization of the excluded-minors for GF(4)-representability [**GGK00**].

Let (X, Y) be a k-separation of $M[X \cup Y, B]$. We say that (X, Y) *induces* a k-separation of M if there is a k-separation (X', Y') of M such that $X \subseteq X'$ and $Y \subseteq Y'$.

Lemma 2.8. [**GGK00**, Theorem 4.14] *Suppose that B is a basis of the matroid M and that (X, Y) is an exact k-separation of $M[X \cup Y, B]$. Then (X, Y) fails to induce a k-separation of M if and only if there is a blocking sequence of (X, Y).*

Halfan and Zhou use Proposition 4.15 (i) and (iii) of [**GGK00**] to prove the following result ([**Hal02**, Proposition 4.5 (3)] and [**Zho04**, Lemma 3.5 (3)] respectively).

Proposition 2.9. *Let B be a basis of a matroid M. Let (X, Y) be an exact k-separation of $M[X \cup Y, B]$ and let $e_1, \ldots, e_t$ be a blocking sequence of (X, Y). Suppose that $|X| > k$ and that e_1 is either in parallel or in series to $x \in X$ in $M[X \cup Y \cup e_1, B]$ where $x \notin \mathrm{cl}_{M[X \cup Y, B]}(Y)$ and $x \notin \mathrm{cl}^*_{M[X \cup Y, B]}(Y)$. If $\{e_1, x\} \subseteq B$ or if $\{e_1, x\} \cap B = \varnothing$ then let $B' = B$, and otherwise let B' be the symmetric difference of B and $\{x, e_1\}$. Then $M[(X - x) \cup Y \cup e_1, B'] \cong M[X \cup Y, B]$ and $e_2, \ldots, e_t$ is a blocking sequence of the k-separation $((X - x) \cup e_1, Y)$ in $M[(X - x) \cup Y \cup e_1, B']$.*

Proposition 2.10. *Suppose that N is a 3-connected matroid such that $|E(N)| \geq 8$ and N contains a four-element circuit-cocircuit X. If M is an internally 4-connected matroid with an N-minor, then there exists a 3-connected single-element extension or coextension N' of N, such that X is not a circuit-cocircuit of N' and N' is a minor of M.*

Proof. Let B be a basis of M and let X and Y be disjoint subsets of $E(M)$ such that $M[X \cup Y, B] \cong N$, where X is a four-element circuit-cocircuit of $M[X \cup Y, B]$. Thus (X, Y) is an exact 3-separation of $M[X \cup Y, B]$. Since $|X|, |Y| \geq 4$ and M is internally 4-connected it cannot be the case that (X, Y) induces a 3-separation of M. Lemma 2.8 implies that there is a blocking sequence $e_1, \ldots, e_t$ of (X, Y). Let us suppose that B, X, and Y have been chosen so that t is as small as possible.

Since $(X, Y \cup e_1)$ is not a 3-separation of $M[X \cup Y \cup e_1, B]$ it follows that X is not a circuit-cocircuit in $M[X \cup Y \cup e_1, B]$. Thus if $M[X \cup Y \cup e_1, B]$ is 3-connected there is nothing left to prove. Therefore we will assume that

$M[X \cup Y \cup e_1, B]$ is not 3-connected. Hence e_1 is in series or parallel to some element in $M[X \cup Y \cup e_1, B]$, and in fact e_1 is in parallel or series to an element $x \in X$, since X is not a circuit-cocircuit of $M[X \cup Y \cup e_1, B]$. But Proposition 2.9 now implies that our assumption on the minimality of t is contradicted. This completes the proof. □

2.4. Splitters

Suppose that $\mathcal{M}$ is a minor-closed class of matroids. A *splitter* of $\mathcal{M}$ is a matroid $M \in \mathcal{M}$ such that any 3-connected member of $\mathcal{M}$ having an M-minor is isomorphic to M. We present here two different forms of Seymour's Splitter Theorem.

Theorem 2.11. [**Sey80**, (7.3)] *Suppose $\mathcal{M}$ is a minor-closed class of matroids. Let M be a 3-connected member of $\mathcal{M}$ such that $|E(M)| \geq 4$ and M is neither a wheel nor a whirl. If no 3-connected single-element extension or coextension of M belongs to $\mathcal{M}$, then M is a splitter for $\mathcal{M}$.*

Theorem 2.12. [**Oxl92**, Corollary 11.2.1] *Let N be a 3-connected matroid with $|E(N)| \geq 4$. If N is not a wheel or a whirl, and M is a 3-connected matroid with a proper N-minor, then M has a 3-connected single-element deletion or contraction with an N-minor.*

Recall that for a binary matroid M we use $\mathcal{EX}(M)$ to denote the set of binary matroids with no M-minor.

Proposition 2.13. *The only 3-connected matroid in $\mathcal{EX}(M(K_{3,3}))$ that is regular but non-cographic is $M(K_5)$.*

Proof. Walton and Welsh [**WW80**] note (and it is easy to confirm) that $M(K_5)$ is a splitter for $\mathcal{EX}(F_7, F_7^*, M(K_{3,3}))$. This implies the result, since the set of cographic matroids is exactly $\mathcal{EX}(F_7, F_7^*, M(K_{3,3}), M(K_5))$ and the set of regular matroids is $\mathcal{EX}(F_7, F_7^*)$. □

Kingan defined the matroid T_{12} in [**Kin97**]. All single-element deletions of T_{12} are isomorphic, and so are all single-element contractions, so the matroids $T_{12}\backslash e$ and T_{12}/e are well-defined. The matroids N_{10} and $\widetilde{K}_5$ are defined by Zhou, who proved the following result.

Proposition 2.14. [**Zho04**, Corollary 1.2] *If M is an internally 4-connected binary non-regular matroid that is not isomorphic to F_7 or F_7^*, then M contains one of the following as a minor: N_{10}, $\widetilde{K}_5$, $\widetilde{K}_5^*$, $T_{12}\backslash e$, or T_{12}/e.*

The triangular Möbius matroids are defined in Chapter 3. The rank-4 triangular Möbius matroid is denoted by Δ_4, and is isomorphic to $\widetilde{K}_5$.

Corollary 2.15. *If $M \in \mathcal{EX}(M(K_{3,3}))$ is an internally 4-connected non-cographic matroid with no Δ_4-minor, then M is isomorphic to one of the following: F_7, F_7^*, $M(K_5)$, $T_{12}\backslash e$, T_{12}/e, or T_{12}.*

Proof. Suppose that the result is not true. Let M be an internally 4-connected non-cographic member of $\mathcal{EX}(M(K_{3,3}), \Delta_4)$ that is isomorphic to none of the six matroids listed in the statement. Proposition 2.13 tells us that M is non-regular, so we can apply Proposition 2.14. We have noted that $\widetilde{K}_5$ is isomorphic to Δ_4, and both N_{10} and $\widetilde{K}_5^*$ have $M(K_{3,3})$-minors [**Zho04**]. Thus M has $T_{12}\backslash e$ or T_{12}/e as a proper minor. There is only one 3-connected coextension or extension of $T_{12}\backslash e$ in $\mathcal{EX}(M(K_{3,3}), \Delta_4)$, and that is T_{12}. Similarly T_{12} is the only 3-connected coextension or extension
✠1 of T_{12}/e in $\mathcal{EX}(M(K_{3,3}), \Delta_4)$. It follows from Theorem 2.12 that M has a T_{12}-minor. But T_{12} is a splitter for $\mathcal{EX}(M(K_{3,3}), \Delta_4)$, so M is isomorphic
✠2 to T_{12}, a contradiction. □

Geelen and Zhou have proved a splitter-type theorem for internally 4-connected binary matroids.

Theorem 2.16. [**GZ06**, Theorem 5.1] *Suppose that M and N are internally 4-connected binary matroids where $|E(N)| \geq 7$ and M has a proper minor isomorphic to N. Then there exists an element $e \in E(M)$ such that one of $M\backslash e$ or M/e is $(4, 5)$-connected with a minor isomorphic to N.*

2.5. Generalized parallel connections

In this section we discuss the generalized parallel connection of Brylawski [**Bry75**].

A flat F of the matroid M is a *modular flat* if $r(F) + r(F') = r(F \cap F') + r(F \cup F')$ for every flat F' of M. Suppose that M_1 and M_2 are two matroids and $E(M_1) \cap E(M_2) = T$, where $M_1|T = M_2|T$. If $\mathrm{cl}_{M_1}(T)$ is a modular flat of M_1 and every non-loop element in $\mathrm{cl}_{M_1}(T) - T$ is parallel to an element in T, then we can define the *generalized parallel connection* of M_1 and M_2, denoted by $P_T(M_1, M_2)$. The ground set of $P_T(M_1, M_2)$ is $E(M_1) \cup E(M_2)$ and the flats of $P_T(M_1, M_2)$ are those sets F such that $F \cap E(M_i)$ is a flat of M_i for $i = 1, 2$.

Proposition 2.17. [**Oxl92**, Proposition 12.4.14] *Suppose the generalized parallel connection, $P_T(M_1, M_2)$, of M_1 and M_2 is defined.*

(i) If $e \in E(M_1) - T$ then $P_T(M_1, M_2)\backslash e = P_T(M_1\backslash e, M_2)$.
(ii) If $e \in E(M_2) - T$ then $P_T(M_1, M_2)\backslash e = P_T(M_1, M_2\backslash e)$.
(iii) If $e \in E(M_1) - \mathrm{cl}_{M_1}(T)$ then $P_T(M_1, M_2)/e = P_T(M_1/e, M_2)$.
(iv) If $e \in E(M_2) - \mathrm{cl}_{M_2}(T)$ then $P_T(M_1, M_2)/e = P_T(M_1, M_2/e)$

Suppose that $E(M_1) \cap E(M_2)$ contains a single element p. Then $\mathrm{cl}_{M_1}(\{p\})$ is a modular flat of M_1, so $P_T(M_1, M_2)$ is defined, where $T = \{p\}$. If p is neither a coloop nor a loop in M_1 or M_2 then the *2-sum* of M_1 and M_2 along the *basepoint* p, denoted by $M_1 \oplus_2 M_2$, is defined to be $P_T(M_1, M_2)\backslash T$. The circuits of $M_1 \oplus_2 M_2$ are exactly those circuits of M_1 or M_2 that do not contain p, and sets of the form $(C_1 - p) \cup (C_2 - p)$, where C_i is a circuit of M_i such that $p \in C_i$ for $i = 1, 2$.

The 2-sum operation has the following properties.

PROPOSITION 2.18. [**Oxl92**, Proposition 7.1.20] *Suppose that M_1 and M_2 are matroids such that $E(M_1) \cap E(M_2) = \{p\}$ and p is not a loop or a coloop of M_1 or M_2. Then $(M_1 \oplus_2 M_2)^* = M_1^* \oplus_2 M_2^*$.*

PROPOSITION 2.19. [**Sey80**, (2.6)] *If (X, Y) is an exact 2-separation of a matroid M then there exist matroids M_1 and M_2 on the ground sets $X \cup p$ and $Y \cup p$ respectively, where p is in neither X nor Y, such that M is equal to $M_1 \oplus_2 M_2$. Conversely, if M is the 2-sum of M_1 and M_2 along the basepoint p, where $|E(M_1)|, |E(M_2)| \geq 3$, then $(E(M_1) - p, E(M_2) - p)$ is an exact 2-separation of M, and M_1 and M_2 are isomorphic to minors of M.*

The next result is well known, but seems not to appear in the literature.

PROPOSITION 2.20. *Let N be a 3-connected matroid. Suppose that $M = M_1 \oplus_2 M_2$. If M has an N-minor then either M_1 or M_2 has an N-minor.*

PROOF. Assume that M is the 2-sum of M_1 and M_2 along the basepoint p, and that M has an N-minor, but neither M_1 nor M_2 has an N-minor. Moreover, assume that the proposition holds for all matroids smaller than M.

It is easy to see that the result holds if M is equal to N, so assume that there is an element $e \in E(M)$ such that either $M\backslash e$ or M/e has an N-minor. By relabeling if necessary we will assume that $e \in E(M_1) - p$.

First suppose that $M\backslash e$ has an N-minor. If p is not a coloop in $M_1\backslash e$ then Proposition 2.17 implies that $M\backslash e = (M_1\backslash e) \oplus_2 M_2$. The minimality of M now implies that either $M_1\backslash e$ or M_2 has an N-minor. In either case we are done, so we assume that p is a coloop in $M_1\backslash e$. This means that $\{e, p\}$ is a series pair in M_1. From the description of circuits of $M_1 \oplus_2 M_2$ it follows that $(E(M_1) - \{e, p\}, E(M_2) - p)$ is a 1-separation of $M\backslash e$. Proposition 2.2 implies that $E(N)$ is disjoint with either $E(M_1) - \{e, p\}$ or $E(M_2) - p$. The result follows easily.

Next we assume that M/e has an N-minor. Proposition 2.18 implies that M^* and N^* provide a minimal counterexample to the proposition. Furthermore $M^*\backslash e$ has an N^*-minor. We apply the arguments of the last paragraph to show that the proposition holds. $\square$

LEMMA 2.21. *Let M be a 3-connected binary matroid on the ground set E. Suppose that X is a subset of E such that $\lambda_M(X) \leq 2$ and $r_M(X) \geq 3$. Let $Y = E - X$. Assume that e is an element in $X \cap \mathrm{cl}_M(Y)$. Then either $|E| \leq 7$, or there exists an independent set $I \subseteq X$ and an element $f \in X$, such that $\{e, f\}$ is a parallel pair in M/I and $r_M(I \cup Y) = r_M(I) + r_M(Y)$.*

PROOF. Let M be a counterexample, and assume that M has been chosen so that $|E|$ is as small as possible. Let $Y_0 = \mathrm{cl}_M(Y)$, and let $X_0 = X - Y_0$.

Since M is a counterexample, $|E| > 7$, so every circuit and cocircuit of M contains at least three elements. As $e \in \mathrm{cl}_M(Y)$ it follows that there is a circuit contained in $Y \cup e$ which contains e. Therefore $|Y| \geq 2$.

Suppose that $r_M(X_0) < r_M(X)$. Then $\lambda_M(X_0) < 2$. However $r_M(X) \geq 3$ implies that $r_M(Y_0) \leq r(M) - 1$, so X_0 contains a cocircuit of M. Thus $|X_0| \geq 2$. Furthermore $Y \subseteq Y_0$ so $|Y_0| \geq 2$. Hence (X_0, Y_0) is a 2-separation of M, a contradiction. Therefore $r_M(X_0) = r_M(X)$.

We start by assuming that $r_M(X_0) = 3$. As $|X_0|, |Y_0| \geq 2$ it follows that $\lambda_M(X_0) = 2$. Thus X_0 is a cocircuit of M. Since $r_M(X_0) = r_M(X)$ it follows that $e \in \mathrm{cl}_M(X_0)$, so there is a circuit $C \subseteq X_0 \cup e$ that contains e. Since M is binary and X_0 is a rank-3 cocircuit it follows that $|X_0| \leq 4$. Thus $|C| \leq 5$. But it cannot be the case that $|C| = 5$, for then $|X_0| = 4$ and X_0 is a circuit properly contained in C. Thus $|C| \leq 4$. Since C is a circuit and X_0 is a cocircuit it follows that $|C \cap X_0|$ is even, so we deduce that C is a triangle.

Suppose that $C = \{e, f, g\}$. If we let $I = \{g\}$ then we see that e and f are parallel in M/I and $r_M(I \cup Y) = r(M) = r_M(I) + r_M(Y)$. Thus the result holds for M, a contradiction. Therefore $r_M(X_0) > 3$.

Choose an element $x \in X_0$. Suppose that $\mathrm{si}(M/x)$ is 3-connected. If (A, B) is a 2-separation of M/x then either A or B is a parallel pair. Let $Y' = \mathrm{cl}_{M/x}(Y)$, and let $X' = X - (Y' \cup x)$. Note that $r_M(X) \geq 4$ implies $r_M(Y) \leq r(M) - 2$. Thus $r_{M/x}(Y) \leq r(M/x) - 1$, so Y does not span M/x, and hence X' contains a cocircuit of M/x. As every cocircuit of M/x is also a cocircuit of M it follows that $|X'| \geq 3$.

Note that $\lambda_{M/x}(X - x) \leq 2$, so $\lambda_{M/x}(X') \leq 2$. Assume that $r_{M/x}(X') < r_{M/x}(X - x)$. Then $\lambda_{M/x}(X') < 2$. This means that X' is a parallel pair in M/x, a contradiction as $|X'| \geq 3$. This shows that X' spans $X - x$ in M/x.

Let P be a set containing exactly one element from each parallel pair of M/x, so that $M/x \backslash P \cong \mathrm{si}(M/x)$. Since $x \notin \mathrm{cl}_M(Y)$ it follows that any triangle of M that contains x must contain at least one element of $X - x$. Therefore we will choose P so that $P \subseteq X - x$. If a triangle of M contains both e and x, then the third element of this triangle must be in X, for $e \in \mathrm{cl}_M(Y)$, and $x \notin \mathrm{cl}_M(Y)$. Therefore we can also assume that $e \notin P$.

Let $M_0 = M/x \backslash P$. Since $P \subseteq X - x$ it follows that $e \in \mathrm{cl}_{M_0}(Y)$. Furthermore $\lambda_{M_0}(X - (P \cup x)) \leq 2$. Since $r_M(X) \geq 4$ it follows that $r_{M/x}(X - x) \geq 3$. Recall that $Y' = \mathrm{cl}_{M/x}(Y)$ and that $X' = X - (Y' \cup x)$. If $r_{M_0}(X - (P \cup x)) < r_{M/x}(X - x)$ then there must be a parallel pair in M/x that is not in $\mathrm{cl}_{M/x}(X')$, containing one element from X and one element from Y. But X' spans $X - x$ in M/x, so this cannot happen. Thus $r_{M_0}(X - (P \cup x)) \geq 3$.

By our assumption on $|E|$ there is an independent set $I_0 \subseteq X - (P \cup x)$ and an element $f \in X - (P \cup x)$ such that $\{e, f\}$ is a parallel pair in M_0/I_0 and $r_{M_0}(I_0 \cup Y) = r_{M_0}(I_0) + r_{M_0}(Y)$.

Let $I = I_0 \cup x$. Then I is an independent set in M, and $\{e, f\}$ is a parallel pair in M/I. Moreover $r_M(I) = r_{M_0}(I_0) + 1$ and $r_M(I \cup Y) = r_{M_0}(I_0 \cup Y) + 1$. Also $r_M(Y) = r_{M_0}(Y)$ since $x \notin \mathrm{cl}_M(Y)$. Thus the result holds for M, a contradiction.

Now we must assume that $\mathrm{si}(M/x)$ is not 3-connected. Proposition 2.5 implies that $\mathrm{co}(M\backslash x)$ is 3-connected. It follows that if (A, B) is a 2-separation of $M\backslash x$ then either A or B is a series pair in $M\backslash x$.

Let T be a triad of M which contains x. We will show that $T \subseteq X_0$. Assume otherwise, so T has a non-empty intersection with Y_0. We first suppose that T meets Y_0 in two elements. Then $x \in \mathrm{cl}^*_M(Y_0)$, and hence $(X_0 - x, Y_0)$ is a 2-separation of $M\backslash x$. Thus $|X_0 - x| = 2$. But this is a contradiction as $r_M(X_0) \geq 4$.

Suppose that T contains e. Then $x \in \mathrm{cl}^*_M(Y \cup e)$. Thus $(X - \{e, x\}, Y \cup e)$ is a 2-separation of $M\backslash x$, so $|X - \{e, x\}| = 2$. This implies that $|X| = 4$. As $e \in Y_0$ this means that $|X_0| \leq 3$, a contradiction.

Next we suppose that T meets Y_0 in exactly one element y. The previous paragraph shows that $y \neq e$. Now $y \in \mathrm{cl}^*_M(X)$, so $y \notin \mathrm{cl}_M(Y - y)$. Therefore $\lambda_M(X \cup y) \leq 2$. Moreover $e \in \mathrm{cl}_M(Y - y)$, for otherwise there is a circuit $C \subseteq Y \cup e$ which contains e and which meets the triad T in precisely one element, y. It cannot be the case that y is in $\mathrm{cl}_M(X)$, for then $(X \cup y, Y - y)$ would be a 2-separation of M. (Note that $|Y - y| \geq 2$ since $(Y - y) \cup e$ contains a circuit of M.)

Let us suppose that $\mathrm{si}(M/y)$ is 3-connected. Assume that there is a triangle T' of M containing y. It cannot be the case that $T' \subseteq Y_0$, for then T' and T meet in a single element. Nor can it be the case that $T' - y \subseteq X_0$, for that would imply that $y \in \mathrm{cl}_M(X_0)$, and we have already concluded that $y \notin \mathrm{cl}_M(X)$. Thus T' contains exactly two elements from Y_0, and one element from X_0. This means that the single element in $T' \cap X_0$ is in $\mathrm{cl}_M(Y)$, a contradiction. We have shown that y is contained in no triangles of M, so M/y is 3-connected.

Our assumption on $|E|$ means that there is an independent set $I \subseteq X$ of M/y and an element $f \in X$ such that e and f are parallel in $M/y/I$, and $r_{M/y}(I \cup (Y - y)) = r_{M/y}(I) + r_{M/y}(Y - y)$.

There is a circuit $C \subseteq I \cup \{e, f, y\}$ that contains both e and f. It cannot be the case that $y \in C$, for $y \notin \mathrm{cl}_M(X)$. Thus e and f are parallel in M/I. Moreover $r_M(I \cup Y) = r_{M/y}(I \cup (Y - y)) + 1$ and $r_M(Y) = r_{M/y}(Y - y) + 1$. Also $r_M(I) = r_{M/y}(I)$, for $y \notin \mathrm{cl}_M(X)$. Therefore the lemma holds for M, a contradiction.

Therefore it cannot be the case that $\mathrm{si}(M/y)$ is 3-connected, and hence $\mathrm{co}(M\backslash y)$ is 3-connected. However $(X, Y - y)$ is a 2-separation of $M\backslash y$, so $|Y - y| \leq 2$. Since $(Y - y) \cup e$ contains a circuit it must be the case that $|Y - y| = 2$. Therefore $|Y| = 3$. If Y is not independent then it is a triangle, and in this case Y meets the cocircuit T in a single element, a contradiction. Thus Y is independent, and $r_M(X) = r(M) - 1$. Thus Y is a triad.

Since $e \in \mathrm{cl}_M(Y - y)$ it follows that $(Y - y) \cup e$ is a triangle of $M\backslash y$. But to obtain $\mathrm{co}(M\backslash y)$ from $M\backslash y$ we must contract a single element from the series pair $Y - y$, so $\mathrm{co}(M\backslash y)$ contains a parallel pair. Since $\mathrm{co}(M\backslash y)$ is 3-connected this means that $\mathrm{co}(M\backslash y)$ is isomorphic to some restriction of $U_{1,3}$. It is easy to see that this implies $|E| \leq 7$, contrary to our earlier conclusion. Thus we have proved that any triad of M that contains x must be contained in X_0.

Let S be a set containing a single element from each series pair of $M\backslash x$, so that $M\backslash x/S \cong \mathrm{co}(M\backslash x)$. By the previous arguments it follows that $S \subseteq X_0 - x$. We can assume that $e \notin S$. Let $M_0 = M\backslash x/S$.

Note that $\lambda_{M_0}(X-(S\cup x)) \leq 2$ and $e \in \mathrm{cl}_{M_0}(Y)$. Suppose that $r_{M_0}(X-(S \cup x)) \geq 3$. Then by our assumption on the cardinality of E it follows that there is an independent set $I_0 \subseteq X-(S\cup x)$ of M_0 and an element $f \in X-(S\cup x)$ such that e and f are parallel in M_0/I and $r_{M_0}(I_0 \cup Y) = r_{M_0}(I_0)+r_{M_0}(Y)$. We will assume that I_0 is minimal with respect to these properties, so $I_0 \cup \{e, f\}$ is a circuit of M_0.

Let $\mathcal{T}$ be the set of series pairs of $M\backslash x$ which meet $I_0 \cup \{e, f\}$. Each of the series pairs in $\mathcal{T}$ contains exactly one element in S. Let $S_0 \subseteq S$ be the set containing these elements. Let $I = I_0 \cup S_0$. Since $I_0 \cup \{e, f\}$ is a circuit of M_0 it is easy to see that $I \cup \{e, f\}$ must be a circuit in $M\backslash x$, and hence in M. Thus e and f are parallel in M/I.

Suppose that it is not the case that $r_M(I \cup Y) = r_M(I)+r_M(Y)$. Then there is a circuit C of $M\backslash x$ contained in $I \cup Y$ which meets both I and Y. If C' is any circuit and x' is any element contained in a series pair, then $C'-x'$ is a circuit in the matroid produced by contracting x'. It follows that $C-S$ is a circuit of M_0. Our assumption on I_0 and Y means that $C-S$ cannot meet both I_0 and Y. Moreover I_0 is independent in M_0. Therefore $C-S \subseteq Y$. Suppose that $T = \{s, t\}$ is a series pair in $\mathcal{T}$ and that $s \in S \cap C$. Since the circuit C has a non-empty intersection with T it must contain t. Therefore $t \in C-S$, implying that $t \in Y$. But the definition of $\mathcal{T}$ means that $t \in I_0 \cup \{e, f\}$, and this set has an empty intersection with Y. Therefore $r_M(I \cup Y) = r_M(I)+r_M(Y)$ and the lemma holds for M, a contradiction.

Now we have to assume that $r_{M_0}(X-(S\cup x)) < 3$. Suppose that $r_M(X_0-x) < r_M(X_0)$. Thus (X_0-x, Y_0) is a 2-separation of $M\backslash x$, meaning that X_0-x is a series pair. This is a contradiction as $r_M(X_0) \geq 4$. Thus $r_M(X_0-x) = r_M(X_0)$. We observe that $e \in \mathrm{cl}_M(X_0)$ and $\mathrm{cl}_M(X_0-x) = \mathrm{cl}_M(X_0)$. Since $e \notin X_0-x$ it follows that $(X_0-x)\cup e$ contains a circuit of $M\backslash x$. Therefore $X-(S\cup x)$ contains a circuit in M_0.

If M_0 contains a circuit of size at most two, then M_0 is a restriction of $U_{1,3}$, and $|E| \leq 7$, so we are done. Therefore every circuit of M_0 contains at least three elements. Since $r_{M_0}(X-(S\cup x)) < 3$ it follows that $X-(S\cup x)$ is a triangle in M_0.

Suppose that e is contained in a series pair in $M\backslash x$. Then e is contained in a triad of M which also contains x. But this is a contradiction as we have already shown that any such triad must be contained in X_0 and $e \in \mathrm{cl}_M(Y)$. Therefore e is contained in no series pairs in $M\backslash x$.

As $r_M(X_0-x) \geq 4$ and $r_{M_0}(X-(S\cup x)) = 2$ there are at least two series pairs in $M\backslash x$. As $X-(S\cup x)$ contains exactly three elements, and each series pair of $M\backslash x$ contributes one element to $X-(S\cup x)$ it follows that there are no more than three series pairs in $M\backslash x$. However e is in $X-(S\cup x)$, and e is in no series pair in $M\backslash x$. Therefore $M\backslash x$ contains precisely two series pairs. Let these series pairs be $\{s_1, t_1\}$ and $\{s_2, t_2\}$. Assume that $S=\{s_1, s_2\}$.

Now $\{e, t_1, t_2\}$ is a circuit in M_0. Therefore $\{e, s_1, s_2, t_1, t_2\}$ is a circuit in $M\backslash x$. We have already shown that $r_M(X_0-x)=r_M(X_0)$. Therefore there is a circuit $C\subseteq X_0\cup x$ which contains x. Both $\{s_1, t_1, x\}$ and $\{s_2, t_2, x\}$ are triads in M, so C must meet these sets in exactly two elements each. By taking the symmetric difference of C and $\{e, s_1, s_2, t_1, t_2\}$ we see that there is a circuit C' of M such that $|C'|=4$ and $e, x\in C'$.

Let z_1 and z_2 be the elements in $C'\cap\{s_1, t_1\}$ and $C'\cap\{s_2, t_2\}$ respectively. Let $I=\{z_1, z_2\}$. Then e and x are parallel in M/I. Moreover, since $r_M(X)=4$ it follows that $r_M(Y)=r(M)-2$. Clearly $r_M(I)=2$. Suppose that $r_M(I\cup y)\neq r(M)$. Because $z_1, z_2\notin \mathrm{cl}_M(Y)$ this means that $r(I\cup Y)=r(M)-1$, and there is some circuit contained in $I\cup Y$ which contains both z_1 and z_2. But such a circuit meets the triad $\{x, s_1, t_1\}$ in precisely one element, a contradiction. Thus the lemma holds for M. This completes the proof. □

Proposition 2.22. *Suppose that M and N are 3-connected binary matroids such that $|E(M)|>7$. Let e be an element of $E(M)$ such that $M\backslash e$ has a 2-separation (X_1, X_2) where $r_M(X_1), r_M(X_2)\geq 3$. If $M\backslash e$ has an N-minor then so does M/e.*

Proof. Since M is 3-connected it follows that (X_1, X_2) is an exact 2-separation of $M\backslash e$. By Proposition 2.19 there are matroids M_1 and M_2 with the property that $E(M_i)=X_i\cup p$ for $i=1, 2$, where p is in neither X_1 nor X_2, and $M\backslash e=M_1\oplus_2 M_2$. By Proposition 2.20 we can assume that M_2 has an N-minor.

Let $T=\{p\}$ and let P be a set of points in $\mathrm{PG}(r-1, 2)$, where $r=r(M\backslash e)$, such that the restriction of $\mathrm{PG}(r-1, 2)$ to P is isomorphic to $P_T(M_1, M_2)$. We will identify points of $P_T(M_1, M_2)$ with the corresponding points in P and we will blur the distinction between P, and the restriction of $\mathrm{PG}(r-1, 2)$ to P. Thus $P\backslash p=M\backslash e$. It is well known, and easy to verify, that $P|(X_i\cup p)\cong M_i$ for $i=1, 2$. We identify e with the unique point in $\mathrm{PG}(r-1, 2)$ such that the restriction of $\mathrm{PG}(r-1, 2)$ to $(P-p)\cup e$ is isomorphic to M. Let M' be the restriction of $\mathrm{PG}(r-1, 2)$ to $P\cup e$.

The only possible 2-separation of M' is a parallel pair containing p, so either M' or $M'\backslash p$ is 3-connected. Furthermore $(X_1\cup p, X_2\cup e)$ is a 3-separation of M' and $r_M(X_1\cup p)\geq 3$. It cannot be the case that p is a coloop in M_2, so $p\in \mathrm{cl}_{M'}(X_2\cup e)$. Thus we can apply Lemma 2.21 and conclude that there is an independent set $I\subseteq X_1\cup p$ and a point $p'\in X_1\cup p$ such that p and p' are parallel in M'/I and $r_{M'}(I\cup X_2\cup e)=r_{M'}(I)+r_{M'}(X_2\cup e)$. Let us assume that I is minimal with respect to these properties.

Let $D = E(M_1) - (I \cup p \cup p')$. Then $M_1/I\backslash D$ consists of the parallel pair $\{p, p'\}$. Proposition 2.17 implies that $P/I\backslash D\backslash p$ is isomorphic to M_2. Thus $P/I\backslash D\backslash p = M'/I\backslash D\backslash p\backslash e$ has an N-minor.

Suppose that e is not a coloop in $M'\backslash D\backslash p$. Then there is a circuit $C \subseteq I \cup X_2 \cup \{e, p'\}$ that contains e. It cannot be the case that $C \subseteq X_2 \cup e$, for that would imply that $e \in \mathrm{cl}_M(X_2)$ and that $(X_1, X_2 \cup e)$ is a 2-separation of M. The same argument shows that C is not contained in $X_1 \cup e$. Moreover, p' is contained in C, for otherwise the circuit C contradicts the fact that $(I, X_2 \cup e)$ is a 2-separation of $M'|(X_2 \cup I \cup e)$.

Our assumption on the minimality of I means that $I \cup \{p, p'\}$ is a circuit of M'. The symmetric difference of C and $I \cup \{p, p'\}$ is a union of circuits in M'. Let C' be a circuit in this symmetric difference that contains e. Since $p' \in C$ it follows that $p' \notin C'$. Thus C' either demonstrates that $e \in \mathrm{cl}_M(X_1)$, or $e \in \mathrm{cl}_M(X_2)$, or C' meets two different connected components of $M'|(I \cup X_2 \cup e)$. In each of these cases we have a contradiction.

Thus e is a coloop in $M'\backslash D\backslash p$, and therefore a coloop in $M'/I\backslash D\backslash p$. Thus $M'/I\backslash D\backslash p/e = M'/I\backslash D\backslash p\backslash e$, so $M'/I\backslash D\backslash p/e$, and hence M/e, has an N-minor. $\square$

2.6. The Δ-Y operation

Suppose that M_1 and M_2 are matroids such that $E(M_1) \cap E(M_2) = T$. If $M_1 \cong M(K_4)$ and T is a triangle of both M_1 and M_2 then $P_T(M_1, M_2)$ is defined. In this case $P_T(M_1, M_2)\backslash T$ is said to be the matroid produced from M_2 by doing a Δ-Y *operation* on T. The Δ-Y operation for matroids has been studied by Akkari and Oxley [**AO93**] and generalized by Oxley, Semple, and Vertigan [**OSV00**]. This chapter is concerned with results from this last paper and their consequences.

Let T be a triangle of a matroid M. The authors of [**OSV00**] require that T is coindependent for the Δ-Y operation to be defined, and we follow this convention. The matroid produced from M by a Δ-Y operation on T shall be denoted by $\Delta_T(M)$. Suppose that $T = \{a_1, a_2, a_3\}$. Let $T' = \{a_1', a_2', a_3'\}$. We shall assume that $\Delta_T(M) = P_T(M(K_4), M)\backslash T$ where the ground set of $M(K_4)$ is $T \cup T'$ and that $\{a_i, a_i'\}$ is contained in no triangle of $M(K_4)$ for $i = 1, 2, 3$. It will be convenient to relabel a_i' with a_i in $\Delta_T(M)$, so that M and $\Delta_T(M)$ have the same ground set.

Suppose that T is a coindependent triangle in the binary matroid M. Let B be a basis of M that does not contain T, and let $e \in T$ be an element not in B. Then $B \cup e$ is a basis of $\Delta_T(M)$. The fundamental graph $G_{B\cup e}(\Delta_T(M))$ is easily derived from the graph $G_B(M)$, as we now show. The following claims are easy consequences of the definition of the Δ-Y operation and [**Bry75**, Theorem 6.12]. If $T - e$ is contained in B, then we obtain $G_{B\cup e}(\Delta_T(M))$ from $G_B(M)$ by deleting e, adding a new vertex adjacent to those vertices that are adjacent to exactly one element in $T - e$, and labeling this new vertex e.

Suppose that $T - e = \{f, g\}$ and that $f \in B$, while $g \notin B$. Then the sets of neighbors of e and g are exactly the same, with the exception that exactly one of e and g is adjacent to f in $G_B(M)$. Let us suppose that e is not adjacent to f. Then $G_{B\cup e}(\Delta_T(M))$ is obtained from $G_B(M)$ by deleting e and adding a new vertex adjacent to all the neighbors of f apart from g. If g is not adjacent to f then we obtain $G_{B\cup e}(\Delta_T(M))$ by deleting e and adding a new vertex adjacent to all the neighbors of f as well as g. In either case we label the new vertex e.

Finally we note that if $T \cap B = \varnothing$ then $G_{B\cup e}(\Delta_T(M))$ is obtained from $G_B(M)$ by simply deleting e, adding a new vertex adjacent to the two vertices in $T - e$ and labeling this new vertex e.

If T is an independent triad of a matroid M, then T is a coindependent triangle in M^*, so $\Delta_T(M^*)$ is defined. Let $(\Delta_T(M^*))^*$ be denoted by $\nabla_T(M)$. Then $\nabla_T(M)$ is said to be obtained by performing a Y-Δ operation on M. The Δ-Y and Y-Δ operations preserve the property of being representable over a field [**OSV00**, Lemma 3.5].

If T is a coindependent triangle in M, then T is an independent triad of $\Delta_T(M)$, so $\nabla_T(\Delta_T(M))$ is defined. Similarly, if T is a coindependent triad of M, then T is an independent triangle of $\nabla_T(M)$, so $\Delta_T(\nabla_T(M))$ is defined.

Proposition 2.23. [**OSV00**, Lemma 2.11 and Corollary 2.12] *If T is a coindependent triangle of M then $\nabla_T(\Delta_T(M)) = M$. If T is an independent triad then $\Delta_T(\nabla_T(M)) = M$.*

Proposition 2.24. [**OSV00**, Lemma 2.6] *If T is a coindependent triangle of M then $r(\Delta_T(M)) = r(M) + 1$. If T is an independent triad of M then $r(\nabla_T(M)) = r(M) - 1$.*

Proposition 2.25. [**OSV00**, Lemma 2.18 and Corollary 2.19] *If T and T' are disjoint coindependent triangles of a matroid M then $\Delta_T(\Delta_{T'}(M)) = \Delta_{T'}(\Delta_T(M))$. If T and T' are disjoint independent triads then $\nabla_T(\nabla_{T'}(M)) = \nabla_{T'}(\nabla_T(M))$.*

The next result follows easily from Proposition 2.17.

Proposition 2.26. *Suppose that T is a coindependent triangle of the matroid M, and that N is a minor of M such that T is a coindependent triangle of N. Then $\Delta_T(N)$ is a minor of $\Delta_T(M)$.*

Proposition 2.27. [**OSV00**, Lemmas 2.8 and 2.13 and Corollary 2.14] *If T is a coindependent triangle of the matroid M then $\Delta_T(M)\backslash T = M\backslash T$ and $\Delta_T(M)/T = M/T$. Moreover, if $e \in T$ then $\Delta_T(M)/e \cong M\backslash e$. Similarly, if T is an independent triad then $\nabla_T(M)\backslash T = M\backslash T$ and $\nabla_T(M)/T = M/T$. If $e \in T$ then $\nabla_T(M)\backslash e \cong M/e$.*

The function that switches the two members of $T - e$ and fixes every member of $E(M) - T$ is an isomorphism between $\Delta_T(M)/e$ and $M\backslash e$, and between $\nabla_T(M)\backslash e$ and M/e.

COROLLARY 2.28. *Suppose that T is a coindependent triangle in the binary matroid M. Let M' be a single-element coextension of M by the element e, such that e is in a triad of M' with two elements of T. Then $\Delta_T(M)$ is isomorphic to a minor of M'.*

PROOF. Suppose that T' is the triad of M' that contains e and two elements of T. Let e' be the single element in $T - T'$. Since T is coindependent in M there is a basis B of M that avoids T. Then $B \cup e'$ is a basis of $\Delta_T(M)$ and $B \cup e$ is a basis of M'. As noted earlier, we obtain the fundamental graph $G_{B\cup e'}(\Delta_T(M))$ from $G_B(M)$ by deleting e', adding a new vertex adjacent to the two elements of $T - e'$, and labeling this new vertex e'.

In contrast, the fundamental graph $G_{B\cup e}(M')$ is obtained from $G_B(M)$ by adding a vertex labeled with e so that it is adjacent to the two members of $T - e'$. Thus it is obvious that $\Delta_T(M)$ is isomorphic to $M' \backslash e'$, under the isomorphism that labels e with e' and fixes the label on every other element. □

PROPOSITION 2.29. *Suppose that T_1 and T_2 are disjoint coindependent triangles of the matroid M. If $T_1 \cup T_2$ contains a cocircuit C^* of size four then $\Delta_{T_2}(\Delta_{T_1}(M))$ contains a series pair.*

PROOF. For $i = 1, 2$ let e_i be an element in $C^* \cap T_i$. Now $C^* - \{e_1, e_2\}$ is a series pair in $M \backslash e_1 \backslash e_2$. It is not difficult to see, using Propositions 2.17 and 2.27, that $M \backslash e_1 \backslash e_2 \cong \Delta_{T_2}(\Delta_{T_1}(M))/e_1/e_2$. The result follows. □

LEMMA 2.30. *Suppose that $T_1 = \{a_1, b_1, c_1\}$ and $T_2 = \{a_2, b_2, c_2\}$ are disjoint coindependent triangles in the binary matroid M and that a_1 and a_2 are not parallel in M. Suppose that the binary matroid M_1 is produced from M by extending with the elements x, y, and z so that $T_3 = \{x, y, z\}$ is a triangle of M_1 and $\{a_1, y\}$ and $\{a_2, x\}$ are parallel pairs. Suppose also that the binary matroid M_2 is obtained from $\Delta_{T_2}(\Delta_{T_1}(M))$ by extending with the elements x and y so that $\{b_1, c_1, x\}$ and $\{b_2, c_2, y\}$ are triangles and then coextending with the element z so that $\{x, y, z\}$ is a triad. Then*

$$M_2 = \Delta_{T_3}(\Delta_{T_2}(\Delta_{T_1}(M_1))).$$

PROOF. Note that since a_1 and a_2 are not parallel in M it is possible to construct M_1 is the manner described.

Since a_1 and y are parallel in M_1 it follows that they are also parallel in $M_1 \backslash b_1$. But Proposition 2.27 says that $M_1 \backslash b_1$ is isomorphic to $\Delta_{T_1}(M_1)/b_1$ under the isomorphism that swaps a_1 and c_1. Thus there is a circuit of $\Delta_{T_1}(M_1)$ contained in $\{b_1, c_1, y\}$ that contains c_1 and y. As T_1 is a triad in $\Delta_{T_1}(M_1)$ it follows that $\{b_1, c_1, y\}$ must be a triangle in $\Delta_{T_1}(M_1)$.

Similarly, a_2 and x are parallel in $\Delta_{T_1}(M_1)$, and hence in $\Delta_{T_1}(M_1) \backslash b_2$. But $\Delta_{T_1}(M_1) \backslash b_2$ is isomorphic to $\Delta_{T_2}(\Delta_{T_1}(M_1))/b_2$ under the isomorphism that swaps a_2 and c_2. Using the same argument we see that $\{b_2, c_2, x\}$ is a triangle of $\Delta_{T_2}(\Delta_{T_1}(M_1))$.

Thus both $\{b_1, c_1, y\}$ and $\{b_2, c_2, x\}$ are triangles in $\Delta_{T_2}(\Delta_{T_1}(M_1))$, and hence in $\Delta_{T_2}(\Delta_{T_1}(M_1))\backslash z$. As

$$\Delta_{T_2}(\Delta_{T_1}(M_1))\backslash x\backslash y\backslash z = \Delta_{T_2}(\Delta_{T_1}(M))$$

it now follows that $\Delta_{T_2}(\Delta_{T_1}(M_1))\backslash z$ is isomorphic to M_2/z under the isomorphism that swaps x and y. However $\Delta_{T_2}(\Delta_{T_1}(M_1))\backslash z$ is also isomorphic to $\Delta_{T_3}(\Delta_{T_2}(\Delta_{T_1}(M_1)))/z$ under the isomorphism that swaps x and y. As $\{x, y, z\}$ is a triad in both $\Delta_{T_3}(\Delta_{T_2}(\Delta_{T_1}(M_1))$ and M_2 we are done. □

Recall that a matroid is almost vertically 4-connected if it is vertically 3-connected and whenever (X, Y) is a vertical 3-separation then either X contains a triad that spans X, or Y contains a triad that spans Y. We shall say that a *spanning triad* of X is a triad contained in X that spans X. A spanning triad of Y is defined analogously.

LEMMA 2.31. *Suppose that T is an independent triad of an almost vertically 4-connected matroid M where $r(M) \geq 3$. Then $\nabla_T(M)$ is almost vertically 4-connected.*

PROOF. We will say that a separation (X, Y) is *bad* if it is a vertical 1- or 2-separation, or if it is a vertical 3-separation such that neither X nor Y contains a spanning triad. Let us assume that the result is false, so that $\nabla_T(M)$ has a bad separation.

CLAIM 2.31.1. There is a bad separation (X, Y) of $\nabla_T(M)$ such that $T \subseteq X$.

PROOF. Assume that the claim is false. Let (X, Y) be a bad separation. Assume that (X, Y) is a vertical k-separation for some $k \leq 3$. We can assume by relabeling if necessary that X contains two elements of T. Let e be the element in $T \cap Y$. Since T is an independent triad of M it follows that T is a triangle of $\nabla_T(M)$, so $e \in \mathrm{cl}_{\nabla_T(M)}(X)$. Thus Proposition 2.3 implies that $(X \cup e, Y - e)$ is a vertical k'-separation of $\nabla_T(M)$ for some $k' \leq k$.

By assumption $(X \cup e, Y - e)$ is not a bad separation so $k = k' = 3$ and either $X \cup e$ or $Y - e$ contains a spanning triad. Let us suppose that $X \cup e$ contains a spanning triad T'. Since X does not contain a spanning triad, T' contains e. It cannot be the case that $e \in \mathrm{cl}_{\nabla_T(M)}(Y - e)$, for that would imply the existence of a circuit that meets T' in one element. But now $r_{\nabla_T(M)}(Y - e) < r_{\nabla_T(M)}(Y)$, and thus $k' < k$, contrary to hypothesis.

We conclude that $Y - e$ contains a spanning triad. Because Y does not contain a spanning triad it follows that $r_{\nabla_T(M)}(Y - e) < r_{\nabla_T(M)}(Y)$. But this again leads to a contradiction, so the claim holds. □

Let (X, Y) be a bad separation of $\nabla_T(M)$ so that (X, Y) is a vertical k-separation for some $k \leq 3$. By Claim 2.31.1 we will assume that $T \subseteq X$. From Proposition 2.27 we see that $\nabla_T(M)|Y$ is equal to $M|Y$ so $r_{\nabla_T(M)}(Y) = r_M(Y)$. It follows easily from Proposition 2.24 that (X, Y) is also a vertical k-separation of M. Thus $k = 3$ and either X or Y contains a spanning triad in M.

Suppose that Y contains a spanning triad in M. Let e be an element of T. Then Y contains a spanning triad in M/e. But Proposition 2.27 tells us that M/e is isomorphic to $\nabla_T(M)\backslash e$. As $T \subseteq X$ and T is a triangle in $\nabla_T(M)$ it follows that Y contains a spanning triad in $\nabla_T(M)$. This is a contradiction as (X, Y) is a bad separation of $\nabla_T(M)$. Thus X contains a spanning triad in M and therefore X has rank three in M. It now follows easily from Proposition 2.24 that X has rank two in $\nabla_T(M)$, contradicting the fact that (X, Y) is a vertical 3-separation of $\nabla_T(M)$. $\square$

We fix some notation before proving the next result. Let $G = (V, E)$ be a graph. If $V' \subseteq V$ then $G[V']$ is the subgraph induced by V', and $E(V')$ is the set of edges in $G[V']$. If $E' \subseteq E$ then $V(E')$ is the set of vertices in the subgraph induced by E'. The *cyclomatic number* of E', denoted by $\xi(E')$, is the cardinality of the complement of a spanning forest in the subgraph induced by E'. That is, $\xi(E') = |E'| - r_{M(G)}(E')$.

Suppose that G is connected. It is well known, and easy to show, that if $E' \subseteq E$ and the subgraphs induced by both E' and $E - E'$ are connected then

$$\lambda_{M^*(G)}(E') = \lambda_{M(G)}(E') = |V(E') \cap V(E - E')| - 1 \text{ and} \tag{2.1}$$

$$r_{M^*(G)}(E') = \xi(E') + |V(E') \cap V(E - E')| - 1. \tag{2.2}$$

An *edge cut-set* or a *cut-vertex* of a connected graph is a set of edges or a vertex respectively whose deletion disconnects the graph. A *block* is a maximal connected subgraph that has no cut-vertices. Suppose that G is a connected graph. The *block cut-vertex graph* of G, denoted by $\mathrm{bc}(G)$, has the blocks and the cut-vertices of G as its vertex set. Every edge of $\mathrm{bc}(G)$ joins a block to a cut-vertex, and a block B_0 is adjacent to a cut-vertex c in $\mathrm{bc}(G)$ if and only if c is a vertex of B_0. It is well known that $\mathrm{bc}(G)$ is a tree. A connected graph is *n-connected* if it contains no set of fewer than n vertices such that deleting that set disconnects the graph. A *cycle minor* of a graph is a minor isomorphic to a cycle.

PROPOSITION 2.32. *If v_1, v_2, and v_3 are distinct vertices in G, a 2-connected graph, then there is a cycle minor of G in which v_1, v_2, and v_3 are distinct vertices.*

PROOF. By Menger's Theorem there is a cycle C that contains v_1 and v_2. If C also contains v_3 then we are done, so we assume otherwise. Again by Menger's Theorem there are minimal length paths P_1 and P_2 that join v_3 to C such that P_1 and P_2 meet only in the vertex v_3. If either P_1 or P_2 meets C in a vertex other than v_1 or v_2, then we can contract that path to obtain the desired cycle. Thus we assume that, after relabeling, P_1 joins v_3 to v_1, and P_2 joins v_3 to v_2. The result now follows easily. $\square$

LEMMA 2.33. *Suppose that M is an almost vertically 4-connected non-cographic matroid in $\mathcal{EX}(M(K_{3,3}))$ and that T is an independent triad of M. Then $\nabla_T(M)$ is non-cographic.*

PROOF. Let us assume that the lemma is false, and that there is a graph G such that $\nabla_T(M)$ is isomorphic to $M^*(G)$. We can assume that G has no isolated vertices, and since $M^*(G)$ is almost vertically 4-connected by Lemma 2.31 it follows that G is connected. We can also assume that M, and therefore $\nabla_T(M)$, has no loops, so G has no vertices of degree one. Because $M^*(G)$ is almost vertically 4-connected we deduce that G has no parallel edges.

Since T is a triangle in $\nabla_T(M)$ it is a minimal edge cut-set in G, so there exists a partition (A, B) of the vertex set of G such that both $G[A]$ and $G[B]$ are connected, and the set of edges joining vertices in A to vertices in B is exactly T.

First let us assume that only one vertex in A is incident with edges in T. Then A contains only one vertex, for otherwise $M^*(G)$ has a 1-separation. Since G has no parallel edges there are three distinct vertices in B, say b_1, b_2, and b_3, that are incident with edges in T. Let G' be the graph produced by deleting T, and adding edges between each pair of vertices in $\{b_1, b_2, b_3\}$. It is well known that $M^*(G') \cong \Delta_T(\nabla_T(M)) = M$, so M is cographic, contrary to hypothesis.

Next we assume that only two vertices in A, let us call them a_1 and a_2, are incident with edges in T. It follows from Equation (2.1) that $\lambda_{M^*(G)}(E(A)) = 1$.

Let $T = \{e_1, e_2, e_3\}$. By relabeling we can assume that a_1 is incident with e_1 and e_2, and that a_2 is incident with e_3. Suppose that e_1 and e_2 are incident with the vertices b_1 and b_2 in B. Since $G[B]$ is connected there is a path joining b_1 to b_2 in $G[B]$, and hence there is a cycle in $E(B) \cup T$. It follows that the cyclomatic number of $E(B) \cup T$ is at least one, so Equation (2.2) implies that the rank of $E(B) \cup T$ in $M^*(G)$ is at least two.

Since $M^*(G)$ is almost vertically 4-connected, we deduce that $r_{M^*(G)}(E(A)) < 2$. By Equation (2.2) we see that $\xi(E(A)) < 1$. Thus $G[A]$ is a tree, and since G has no degree-one vertices $G[A]$ is a path joining a_1 to a_2. Let e_3' be the edge in $E(A)$ that is incident with a_1 and let $T' = (T - e_3) \cup e_3'$. The edges e_3 and e_3' are in series in G, and hence are in parallel in $M^*(G)$. Therefore T' is a triangle of $M^*(G) \cong \nabla_T(M)$ and $\Delta_{T'}(\nabla_T(M))$ is isomorphic to $\Delta_T(\nabla_T(M)) = M$. But because e_1, e_2, and e_3' are incident with a common vertex in G, we can show as before that $\Delta_{T'}(\nabla_T(M))$ is cographic, a contradiction.

Now we assume that the three edges in T are incident with three distinct vertices in A, and by symmetry, with three distinct vertices in B. Let the three vertices in A incident with edges in T be a_1, a_2, and a_3.

Suppose that $G[A]$ is a tree. Since G has no degree-one vertices there are at most three degree-one vertices in $G[A]$. Suppose that there are two, so that $G[A]$ is a path. By relabeling assume that a_1 has degree two in $G[A]$. Assume also that a_1 is incident with e_1, and the two edges e'_2 and e'_3 in $E(A)$. Then $T' = \{e_1, e'_2, e'_3\}$ is a triangle in $M^*(G)$, and e'_2 and e'_3 are parallel to either e_2 or e_3. It follows that $\Delta_{T'}(\nabla_T(M)) \cong M$, and we again have a contradiction, since the edges e_1, e'_2, and e'_3 are incident with a common vertex.

Since $G[A]$ is not a path it contains a vertex v of degree three. Suppose that v is incident with the three edges e'_1, e'_2 and e'_3. By letting $T' = \{e'_1, e'_2, e'_3\}$ we obtain a similar contradiction.

We have shown that $G[A]$ is not a tree, so it has cyclomatic number at least one, and hence $E(A)$ has rank at least three in $M^*(G)$. By using the same arguments we can also show that $r_{M^*(G)}(E(B)) \geq 3$.

Claim 2.33.1. There is a cycle minor of $G[A]$ in which a_1, a_2, and a_3 are distinct vertices.

Proof. If $G[A]$ is 2-connected then the result follows from Proposition 2.32, so we assume that $G[A]$ contains at least two blocks. We have shown that $G[A]$ is not a tree, so let B_0 be a block of $G[A]$ that contains a cycle.

If B is a block of $G[A]$ with degree one in $\mathrm{bc}(G[A])$ then B contains one of the vertices a_1, a_2, or a_3, and this vertex cannot be a cut-vertex in $G[A]$, for otherwise G contains a cut-vertex, and this would imply that $M^*(G)$ is not connected. Thus there are at most three degree-one vertices in $\mathrm{bc}(G[A])$. Suppose that there are exactly three.

First assume that B_0 has degree less than three in $\mathrm{bc}(G[A])$. Let X be the edge set of B_0. If B_0 has degree one in $\mathrm{bc}(G[A])$ then it contains exactly one of the vertices a_1, a_2, and a_3. If B_0 has degree two in $\mathrm{bc}(G[A])$ then it contains none of these vertices. In either case it follows that $V(X)$ and $V(E(G)-X)$ meet in exactly two vertices. Moreover, the cyclomatic number of X is at least one (since B_0 contains a cycle). We have already concluded that $E(B)$ has rank at least three in $M^*(G)$, so now Equations (2.1) and (2.2) imply that $M^*(G)$ has a vertical 2-separation, a contradiction.

Therefore B_0 has degree at least three in $\mathrm{bc}(G[A])$, and since $\mathrm{bc}(G[A])$ has exactly three degree-one vertices it follows that B_0 has degree exactly three. Let c_1, c_2, and c_3 be the cut-vertices of $G[A]$ that are contained in B_0. There are three vertex-disjoint paths in $G[A]$ that join the vertices in $\{c_1, c_2, c_3\}$ to the vertices in $\{a_1, a_2, a_3\}$. Now B_0 has a cycle minor in which c_1, c_2, and c_3 are disjoint vertices by Proposition 2.32, so we can find a cycle minor of $G[A]$ in which a_1, a_2, and a_3 are disjoint vertices.

Next we will assume that $\mathrm{bc}(G[A])$ has exactly two degree-one vertices, B_1 and B_2, so that $\mathrm{bc}(G[A])$ is a path. By relabeling we will assume that a_i is a vertex of B_i for $i = 1, 2$.

Let X be the edge set of B_0. Suppose that either a_3 is not contained in the block B_0, or that a_3 is a cut-vertex of $G[A]$. Then $V(X)$ and $V(E(G)-X)$ have exactly two common vertices. Moreover, the cyclomatic number of X is at least one, so we can obtain a contradiction as before. Therefore a_3 is contained in B_0, and is not a cut-vertex of $G[A]$. In fact, none of the vertices a_1, a_2, or a_3 can be a cut-vertex of $G[A]$.

Suppose that B_0 is equal to B_1. Let c be the unique cut-vertex of $G[A]$ that is contained in B_0. Then a_1, a_3, and c are distinct vertices in B_0, so B_0 has a cycle minor in which a_1, a_3 and c are distinct vertices by Proposition 2.32. Furthermore, there is a path in $G[A]$ joining c to a_2, so the result follows. Therefore let us assume that B_0 is not equal to B_1. (By symmetry we assume that it is not equal to B_2.) There are two cut-vertices c_1 and c_2 of $G[A]$ that are in B_0. Then B_0 has a cycle minor in which a_3, c_1, and c_2 are distinct vertices. Furthermore, there are two vertex-disjoint paths of $G[A]$ joining the vertices in $\{c_1, c_2\}$ to the vertices in $\{a_1, a_2\}$. Again we can find the desired cycle minor of $G[A]$, so the result follows. $\square$

We can also apply the arguments of Claim 2.33.1 to show $G[B]$ has a cycle minor in which the three vertices in B that meet the edges of T are distinct vertices. Thus G has a minor G' isomorphic to the graph shown in Figure 2.1, where T is the set of edges joining the two triangles.

Proposition 2.26 tells us that $\Delta_T(M^*(G'))$ is a minor of $\Delta_T(M^*(G)) \cong \Delta_T(\nabla_T(M)) = M$. But $\Delta_T(M^*(G'))$ is isomorphic to $M(K_{3,3})$, so we have a contradiction that completes the proof of the lemma. $\square$

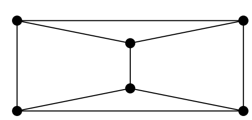

Figure 2.1: G', a minor of G.

Proposition 2.34. *Suppose that T is an independent triad in the matroid M. If M has no $M(K_{3,3})$-minor then neither does $\nabla_T(M)$.*

Proof. Suppose that $\nabla_T(M)$ does have an $M(K_{3,3})$-minor. Since T is a triangle in $\nabla_T(M)$ and $M(K_{3,3})$ has no triangles there is some element $e \in T$ such that $\nabla_T(M)\backslash e$ has an $M(K_{3,3})$-minor. But Proposition 2.27 tells us that $\nabla_T(M)\backslash e \cong M/e$. Thus M has an $M(K_{3,3})$-minor, a contradiction. $\square$

Proposition 2.35. *Suppose that M is an almost vertically 4-connected non-cographic matroid in $\mathcal{EX}(M(K_{3,3}))$ and that T is a triad of M. Every triad of $\nabla_T(M)$ is a triad of M.*

Proof. Lemma 2.33 implies that $\nabla_T(M)$ is non-cographic. Thus if $r(\nabla_T(M)) \leq 3$ it follows that $\mathrm{si}(\nabla_T(M)) \cong F_7$. In this case $\nabla_T(M)$ has no triads and the result follows. Therefore we will assume that $r(\nabla_T(M)) > 3$.

Suppose that T' is a triad of $\nabla_T(M)$ but not of M. It cannot be the case that $T' \cap T = \varnothing$, for then T' would be a triad of $\nabla_T(M)/T$, which is equal to M/T by Proposition 2.27. This would imply that T' is a triad of M. Thus

T' has a non-empty intersection with T. Since M, and hence $\nabla_T(M)$, is binary, and T is a triangle of $\nabla_T(M)$, it follows that T' meets T in exactly two elements. Let X equal $T \cup T'$ and let $Y = E(\nabla_T(M)) - X$. Now (X, Y) is a vertical 3-separation of $\nabla_T(M)$ and it is easy to see that it is also a vertical 3-separation of $\Delta_T(\nabla_T(M)) = M$. It follows easily from Proposition 2.24 that X has rank four in M. Since M is almost vertically 4-connected we conclude that Y contains a spanning triad in M. But now it is easy to check that $\nabla_T(M)$ is cographic (in fact, up to the addition of parallel points, $\nabla_T(M)$ is a minor of the cographic matroid corresponding to the graph shown in Figure 2.1), and this is contrary to our earlier conclusion. □

Let M be a binary matroid, and let $\mathcal{T}$ be a multiset of coindependent triangles of M. For each element $e \in E(M)$ let t_e denote the number of triangles in $\mathcal{T}$ that contain e. Suppose that M_0 is obtained from M by the following procedure: For each element $e \in E(M)$, if $t_e > 1$ then add $t_e - 1$ parallel elements to e. Then $\mathrm{si}(M_0) \cong M$, so each triangle of M_0 naturally corresponds to a triangle of M. We can find a set $\mathcal{T}_0$ of pairwise disjoint coindependent triangles of M_0 such that $|\mathcal{T}_0| = |\mathcal{T}|$ and each triangle in $\mathcal{T}_0$ corresponds to a triangle in $\mathcal{T}$. Now we let $\Delta(M; \mathcal{T})$ be the matroid obtained from M_0 by performing Δ-Y operations in turn on each of the triangles in $\mathcal{T}_0$. It is clear that $\Delta(M; \mathcal{T})$ is well-defined up to isomorphism.

We shall denote $\Delta(M; \mathcal{T})^*$ by $\nabla(M^*; \mathcal{T})$. It is easy to see that $\nabla(M^*; \mathcal{T})$ can be obtained from M_0^* by performing Y-Δ operations on the triads in $\mathcal{T}_0$, where M_0^* is obtained from M^* by adding $t_e - 1$ series elements to each element e of M^* when $t_e > 1$, and $\mathcal{T}_0$ is a set of pairwise disjoint independent triads of M_0^* corresponding to the set $\mathcal{T}$.

Lemma 2.36. *Suppose that M is a binary matroid and that $\mathcal{T}$ is a multiset of coindependent triangles of M. Let T be a triangle in $\mathcal{T}$. If there exists a triangle $T' \neq T$ in $\mathcal{T}$ such that $T \cap T' \neq \varnothing$, then $\Delta(M; \mathcal{T})$ has a minor isomorphic to $\Delta(M; \mathcal{T} - T)$.*

Proof. Let e be an element contained in both T and T'. Then there is an element $e' \in E(M_0) - E(M)$ such that e' is parallel to e in M_0. Let T_0 and T_0' be disjoint triangles of M_0 that correspond to T and T' respectively. We can assume that $e' \in T_0$. It follows from Proposition 2.23 that $\Delta(M; \mathcal{T} - T)$ can be obtained from $\nabla_{T_0}(\Delta(M; \mathcal{T}))$ by deleting those elements of T_0 that are parallel in M_0 to some element of M that is contained in more than one triangle of $\mathcal{T}$. In particular, $\Delta(M; \mathcal{T} - T)$ is a minor of $\nabla_{T_0}(\Delta(M; \mathcal{T})) \backslash e'$. But Proposition 2.27 tells us that $\nabla_{T_0}(\Delta(M; \mathcal{T})) \backslash e'$ is isomorphic to $\Delta(M; \mathcal{T})/e'$. The result follows. □

Several of the sporadic matroids described in Appendix B are best understood as matroids of the form $\nabla(M; \mathcal{T})$. In particular, the matroids $M_{7,15}$, $M_{9,18}$, and $M_{11,21}$ are isomorphic to $\nabla(F_7^*; \mathcal{T})$, where $\mathcal{T}$ is a set of five, six, or seven triangles respectively in the Fano plane F_7. Moreover, if $\mathcal{T}_4^a$ is a set of four triangles in F_7 such that three triangles contain a common point, then $\nabla(F_7^*; \mathcal{T}_4^a) \cong M_{5,12}^a$. If $\mathcal{T}_4^b$ is a set of four triangles in F_7 such that no three triangles contain a common point then $\nabla(F_7^*; \mathcal{T}_4^b) \cong M_{6,13}$.

Let $M = F_7$, and suppose that $\mathcal{T}$ is a non-empty set of triangles in M. Let M_0 be the matroid obtained from M by adding parallel points, as described earlier, and suppose that $\mathcal{T}_0 = \{T_1, \ldots, T_n\}$ is a set of disjoint triangles in M_0 that corresponds to $\mathcal{T}$. For $1 \leq i \leq n$ let M_i be the matroid obtained by performing Δ-Y operations on the triangles $T_1, \ldots, T_i$. Clearly M_0 has no minor isomorphic to $M^*(K_{3,3})$. Therefore, if M_i does have an $M^*(K_{3,3})$-minor for some $i \in \{1, \ldots, n\}$, there exists an integer i such that M_i has an $M^*(K_{3,3})$-minor but M_{i-1} does not.

However, $M^*(K_{3,3})$ contains no triads, and T_i is a triad of M_i. Therefore there is an element $e \in T_i$ such that M_i/e has an $M^*(K_{3,3})$-minor. Now Proposition 2.27 implies that $M_{i-1}\backslash e$ has an $M^*(K_{3,3})$-minor, contrary to our assumption. Therefore M_i does not have an $M^*(K_{3,3})$-minor for any integer $i \in \{1, \ldots, n\}$.

In particular, M_{n-1} does not have an $M^*(K_{3,3})$-minor. Proposition 2.23 tells us that M_{n-1} is isomorphic to $\nabla_{T_n}(\Delta(M; \mathcal{T}))$. Since the triangles in $\mathcal{T}_0$ are disjoint Proposition 2.25 asserts that the order in which we apply the Δ-Y operation to them is immaterial. The above argument now shows that $\nabla_T(\Delta(M; \mathcal{T}))$ has no $M^*(K_{3,3})$-minor for any triangle $T \in \mathcal{T}_0$.

Dualising, we see that neither $\nabla(M^*; \mathcal{T})$ nor $\Delta_T(\nabla(M^*; \mathcal{T}))$ has an $M(K_{3,3})$-minor, for any triangle $T \in \mathcal{T}_0$.

In general, if T is a triangle of a matroid $M \in \mathcal{EX}(M(K_{3,3}))$ and $\Delta_T(M)$ has no $M(K_{3,3})$-minor, then we shall say that T is an *allowable triangle*. The previous arguments assert the following fact.

PROPOSITION 2.37. *Suppose that $\mathcal{T}$ is a non-empty set of triangles in F_7. Then each triangle in $\mathcal{T}$ corresponds to an allowable triangle in $\nabla(F_7^*; \mathcal{T})$.*

CHAPTER 3

Möbius matroids

In this chapter we define in detail the two infinite classes featured in Theorem 1.1. In Section 3.3 we consider their non-cographic minors.

The *cubic Möbius ladder* CM_{2n} is obtained from the even cycle on the vertices $v_0, \ldots, v_{2n-1}$ by joining each vertex v_i to the antipodal vertex v_{i+n} (indices are read modulo $2n$). The *quartic Möbius ladder* QM_{2n+1} is obtained from the odd cycle on the vertices $v_0, \ldots, v_{2n}$ by joining each vertex v_i to the two antipodal vertices v_{i+n} and v_{i+n+1} (in this case indices are read modulo $2n + 1$). In both cases, the edges of the cycle are known as *rim edges*, and the edges between antipodal vertices are *spoke edges*. We note that $CM_4 \cong K_4$, $CM_6 \cong K_{3,3}$, and $QM_5 \cong K_5$. In addition, QM_3 is isomorphic to the graph obtained from K_3 by replacing each edge with a parallel pair.

3.1. Triangular Möbius matroids

Let r be an integer exceeding two and let $\{e_1, \ldots, e_r\}$ be the standard basis in the vector space of dimension r over GF(2). For $1 \leq i \leq r-1$ let a_i be the sum of e_i and e_r, and for $1 \leq i \leq r-2$ let b_i be the sum of e_i and e_{i+1}. Let b_{r-1} be the sum of e_1, e_{r-1}, and e_r. The *rank-r triangular Möbius matroid*, denoted by Δ_r, is represented over GF(2) by the set $\{e_1, \ldots, e_r, a_1, \ldots, a_{r-1}, b_1, \ldots, b_{r-1}\}$. Thus Δ_r has rank r and $|E(\Delta_r)| = 3r - 2$. We also take $\{e_1, \ldots, e_r, a_1, \ldots, a_{r-1}, b_1, \ldots, b_{r-1}\}$ to be the ground set of Δ_r.

Figure 3.1 shows a matrix A such that Δ_4 is represented over GF(2) by $[I_4|A]$, and also the fundamental graph $G_B(\Delta_4)$, where B is the basis $\{e_1, \ldots, e_4\}$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

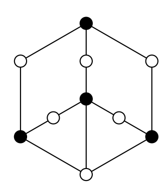

Figure 3.1: Matrix and fundamental graph representations of Δ_4.

Figure 3.2 shows geometric representations of Δ_4 and Δ_5. Since Δ_5 has rank 5 we cannot draw an orthodox representation, but Figure 3.2 gives an idea of its structure by displaying its triangles.

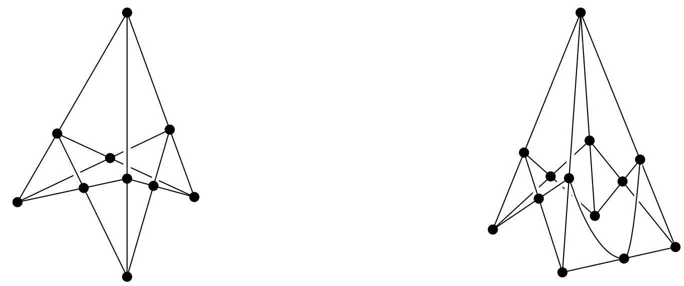

Figure 3.2: Geometric representations of Δ_4 and Δ_5.

Let $2n \geq 2$ be an even integer. For $i \in 1, \ldots, n$ let the edge of CM_{2n} that joins v_{i-1} to v_i be labeled e_i and let the edge that joins v_{i+n-1} to v_{i+n} be labeled a_i. Let the edge that joins v_i to v_{i+n} be labeled b_i. Figure 3.3 shows two drawings of CM_6 equipped with this labeling. Under this labeling $\Delta_r \backslash e_r = M^*(CM_{2r-2})$ for $r \geq 3$. For this reason we will refer to the elements $e_1, \ldots, e_{r-1}$, $a_1, \ldots, a_{r-1}$ as the *rim elements* of Δ_r, while referring to the elements $b_1, \ldots, b_{r-1}$ as *spoke elements*. We will also call e_r the *tip* of Δ_r.

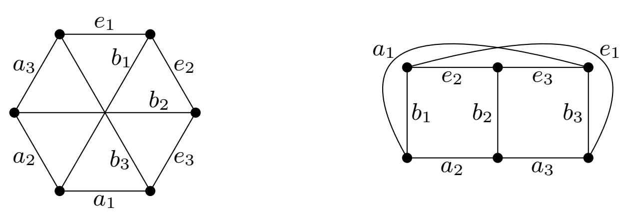

Figure 3.3: Two drawings of the cubic Möbius ladder on six vertices.

The triangular Möbius matroids do not contain any triads. When $r \geq 3$ the triangles of Δ_r are the sets $\{a_i, e_i, e_r\}$ for $1 \leq i \leq r-1$, the sets $\{a_i, a_{i+1}, b_i\}$ and $\{e_i, e_{i+1}, b_i\}$ for $1 \leq i \leq r-2$, and the sets $\{a_1, e_{r-1}, b_{r-1}\}$ and $\{a_{r-1}, e_1, b_{r-1}\}$.

It is easy to see that Δ_3 is isomorphic to F_7, the Fano plane. The rank-4 triangular Möbius matroid is known to Zhou as $\widetilde{K}_5$ [**Zho04**] and to Kung as C_{10} [**Kun86**]. Moreover, Δ_r is known to Kingan and Lemos as S_{3r-2} [**KL02**].

3.2. Triadic Möbius matroids

Let $r \geq 4$ be an even integer, and again let $\{e_1, \ldots, e_r\}$ be the standard basis of the vector space over GF(2) of dimension r. Let c_i be the sum of e_i, e_{i+1}, and e_r for $1 \leq i \leq r-2$. Let c_{r-1} be the sum of e_1, e_{r-1}, and e_r. The *rank-r triadic Möbius matroid*, denoted by Υ_r, is represented over GF(2) by the set $\{e_1, \ldots, e_r, c_1, \ldots, c_{r-1}\}$. Thus Υ_r has rank r and $|E(\Upsilon_r)| = 2r-1$. Again we take $\{e_1, \ldots, e_r, c_1, \ldots, c_{r-1}\}$ to be the ground set of Υ_r. Figure 3.4 shows a matrix A such that $[I_4|A]$ represents Υ_4

over GF(2) and the fundamental graph $G_B(\Upsilon_4)$, where $B = \{e_1, \ldots, e_4\}$. Figure 3.5 shows a geometric representation of Υ_4.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

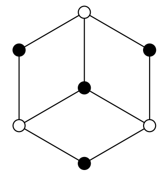

Figure 3.4: Matrix and fundamental graph representations of Υ_4.

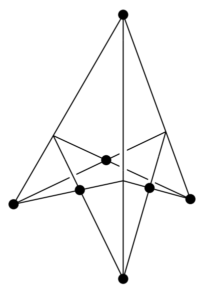

Figure 3.5: A geometric representation of Υ_4.

Let $2n+1 \geq 5$ be an odd integer. For $i \in \{1, \ldots, 2n+1\}$ let e_i be the edge of QM_{2n+1} that joins v_{ni} to v_{ni+1}. Let c_i be the edge that joins $v_{n(i-1)}$ to v_{ni}. Figure 3.6 shows two drawings of QM_7 labeled in this way. In addition we label the edges of QM_3 so that the parallel pairs are $\{c_1, e_3\}$, $\{c_2, e_1\}$, and $\{c_3, e_2\}$. Now $\Upsilon_r \backslash e_r = M^*(QM_{r-1})$ for any even integer $r \geq 4$. Thus we will refer to the elements $e_1, \ldots, e_{r-1}$ as the *rim elements* of Υ_r and the elements $c_1, \ldots, c_{r-1}$ as *spoke elements*. We will call e_r the *tip* of Υ_r.

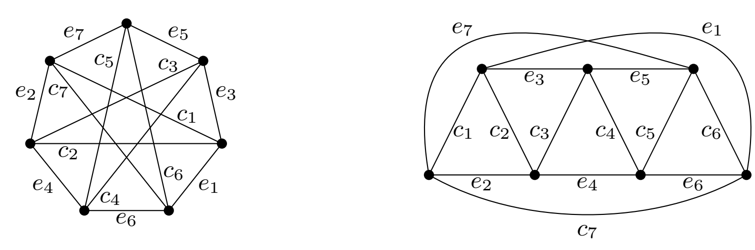

Figure 3.6: Two drawings of the quartic Möbius ladder on seven vertices.

The triadic Möbius matroids contain no triangles. For an even integer $r \geq 4$ the triads of Υ_4 are the sets $\{c_i, c_{i+1}, e_{i+1}\}$ for $1 \leq i \leq r-2$ and $\{c_1, c_{r-1}, e_1\}$. When $r = 4$ the sets $\{e_1, e_2, e_3\}$, $\{c_1, e_3, e_4\}$, $\{c_2, e_1, e_4\}$, and $\{c_3, e_2, e_4\}$ are also triads. The rank-4 triadic Möbius matroid is isomorphic to the Fano dual, F_7^*. The rank-6 triadic Möbius matroid is isomorphic to any single element deletion of the matroid T_{12}, introduced by Kingan [**Kin97**]. In addition Υ_r^* is known to Kingan and Lemos as F_{2r-1} [**KL02**].

The following proposition is not difficult to confirm.

PROPOSITION 3.1. *Every Möbius matroid is internally* 4*-connected.*

3.3. Minors of Möbius matroids

If $r \geq 4$ then we can obtain Δ_{r-1} from Δ_r by contracting a spoke element and deleting an element from each of the two parallel pairs that result. This is made more formal in the next (easily proved) result.

PROPOSITION 3.2. *Let $r \geq 4$. We can obtain a minor of Δ_r isomorphic to Δ_{r-1} by:*

(i) contracting b_i, deleting an element from $\{e_i, e_{i+1}\}$ and an element from $\{a_i, a_{i+1}\}$ where $1 \leq i \leq r-2$; or,
(ii) contracting b_{r-1}, deleting an element from $\{a_1, e_{r-1}\}$ and an element from $\{a_{r-1}, e_1\}$.

There are two ways to obtain an Υ_{r-2}-minor from Υ_r, where $r \geq 6$ is an even integer. In the first we contract two consecutive spoke elements. This produces a triangle that contains e_r and whose closure contains a parallel pair. We delete the element of this triangle that is not e_r and is not in the parallel pair, and then we delete an element from the parallel pair.

PROPOSITION 3.3. *Let $r \geq 6$ be an even integer. We can obtain a minor of Υ_r isomorphic to Υ_{r-2} by:*

(i) contracting c_i and c_{i+1}, deleting e_{i+1} and an element from $\{e_i, e_{i+2}\}$ where $1 \leq i \leq r-3$;
(ii) contracting c_1 and c_{r-1}, deleting e_1 and an element from $\{e_2, e_{r-1}\}$; or,
(iii) contracting c_{r-2} and c_{r-1}, deleting e_{r-1} and an element from $\{e_1, e_{r-2}\}$.

The other method involves contracting two spoke elements that are separated by one other spoke element, and then deleting the rim elements that lie "between" them.

PROPOSITION 3.4. *Let $r \geq 6$ be an even integer. We can obtain a minor of Υ_r isomorphic to Υ_{r-2} by:*

(i) contracting c_i and c_{i+2} and deleting e_{i+1} and e_{i+2}, where $1 \leq i \leq r-3$;
(ii) contracting c_1 and c_{r-2} and deleting e_1 and e_{r-1}; or,
(iii) contracting c_2 and c_{r-1} and deleting e_1 and e_2.

PROPOSITION 3.5. *Suppose that $(e_1, \ldots, e_4)$ is a fan of the matroid M. If $M \backslash e_1$ is cographic then M is cographic.*

PROOF. Let $X = \{e_1, e_2, e_3, e_4\}$. First suppose that $X - e_1$ is not a triad in $M\backslash e_1$. Then $e_1 \in \mathrm{cl}^*_M(X - e_1)$, so there is a cocircuit $C^* \subseteq X$ such that $e_1 \in C^*$. It follows from cocircuit exchange that e_1 is contained in a series pair in M. The result follows easily.

Suppose that $X - e_1$ is a triad in $M\backslash e_1$. Let G be a graph such that $M\backslash e_1 = M^*(G)$. Then $X - e_1$ is the edge set of a triangle in G. Let u be the vertex of G incident with both e_2 and e_3. Let G' be the graph formed from G by deleting u and replacing it with v and w, where v is incident with e_1, e_2, and e_3, and w is incident with e_1 and all edges incident with u other than e_2 and e_3. Then $M = M^*(G')$. $\square$

PROPOSITION 3.6. *Suppose $r \geq 3$. Any minor produced from Δ_r by one of the following operations is cographic.*

(i) Deleting or contracting e_r;
(ii) Contracting e_i or a_i for $1 \leq i \leq r-1$; or,
(iii) Deleting b_i for $1 \leq i \leq r-1$.

PROOF. We have already noted (and it is easy to confirm) that $\Delta_r\backslash e_r \cong M^*(CM_{2r-2})$. It is also easy to confirm that Δ_r/e_r is isomorphic to a matroid obtained from the rank-$(r-1)$ wheel by adding $r-1$ parallel elements. Thus Δ_r/e_r is cographic. This completes the proof of (i).

It follows from (i) that $\Delta_r\backslash e_r/e_i$ is cographic, where $1 \leq i \leq r-1$. But Δ_r/e_i is obtained from $\Delta_r\backslash e_r/e_i$ by adding e_r in parallel to a_i. Thus Δ_r/e_i is cographic, and the same argument shows that Δ_r/a_i is also cographic.

If $1 \leq i \leq r-2$ then $(e_r, a_{i+1}, e_{i+1}, b_{i+1})$ is a fan of $\Delta_r\backslash b_i$. By (i) we know that $\Delta_r\backslash b_i\backslash e_r$ is cographic. Proposition 3.5 now implies that $\Delta_r\backslash b_i$ is cographic. The case when $i = r-1$ is similar. $\square$

PROPOSITION 3.7. *Suppose $r \geq 4$ is an even integer. Any minor produced from Υ_r by one of the following operations is cographic.*

(i) Deleting or contracting e_r;
(ii) Contracting e_i for $1 \leq i \leq r-1$; or,
(iii) Deleting c_i for $1 \leq i \leq r-1$.

PROOF. We have noted that $\Upsilon_r\backslash e_r \cong M^*(QM_{r-1})$. It is easy to see that Υ_r/e_r is isomorphic to the rank-$(r-1)$ wheel.

If $1 \leq i \leq r-2$ then $(e_r, c_i, e_{i+1}, c_{i+1})$ is a fan of Υ_r/e_i. Since $\Upsilon_r/e_i\backslash e_r$ is cographic it follows that Υ_r/e_i is cographic by Proposition 3.5. A similar argument holds when $i = r-1$.

Finally, for $1 \leq i \leq r-1$, the minor $\Upsilon_r\backslash e_r\backslash c_i$ is isomorphic to $M^*(G)$ where G is shown in Figure 3.7. It is now easy to confirm that $\Upsilon_r\backslash c_i \cong M^*(G')$. $\square$

LEMMA 3.8. *Suppose that M is a Möbius matroid and N is an internally 4-connected non-cographic minor of M. Then N is also a Möbius matroid.*

PROOF. Suppose that the lemma fails, so that M is a Möbius matroid, and N is an internally 4-connected non-cographic minor of M such that N

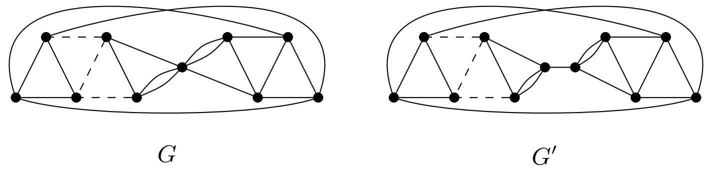

Figure 3.7: $\Upsilon_r\backslash e_r\backslash c_i \cong M^*(G)$ and $\Upsilon_r\backslash c_i \cong M^*(G')$.

is not a Möbius matroid. Let C and D be independent and coindependent subsets of $E(M)$ respectively such that $N = M/C\backslash D$. We will assume that the lemma does not fail for any Möbius matroid smaller than M.

We first suppose that M is the triangular Möbius matroid Δ_r. The result is true if $r = 3$, so we assume that $r \geq 4$. It follows from Proposition 3.6 that C can contain only spoke elements. However, if we contract a spoke element of M then we obtain two parallel pairs, and at least one element from each must be deleted to obtain N. Thus it follows from Proposition 3.2 that N is a minor of Δ_{r-1}, and this contradicts our assumption on the minimality of M. Thus we assume that C is empty.

Proposition 3.6 shows that D contains only rim elements of M. The set $\{b_{i-1}, b_i\}$ is a series pair in $M\backslash e_i\backslash a_i$ for $1 \leq i \leq r-1$ (henceforth subscripts are to be read modulo $r-1$; we identify the subscript $r-1$ with zero), so if D contains both e_i and a_i, then at least one of b_{i-1} or b_i is contained in C, contrary to our earlier conclusion. Thus D contains at most one element from each set $\{e_i, a_i\}$.

By relabeling we can assume that $e_1 \in D$. But (a_2, b_1, a_1, b_{r-1}) is a fan of $M\backslash e_1$, so
$$\lambda_{M\backslash e_1}(\{a_1, a_2, b_1, b_{r-1}\}) \leq 2.$$
Suppose that $C \cup D$ contains none of the elements $\{a_1, a_2, b_1, b_{r-1}\}$. Since N is internally 4-connected it follows from Proposition 2.2 that the complement of $\{a_1, a_2, b_1, b_{r-1}\}$ in N contains at most three elements. Thus $|E(N)| \leq 7$, and since N is a non-cographic matroid this means that N is isomorphic to either F_7 or F_7^*. Both of these are Möbius matroids, so this is a contradiction. Thus we must either delete or contract at least one element from $\{a_1, a_2, b_1, b_{r-1}\}$ to obtain N.

By the above discussion and Proposition 3.6 we must delete a_2. But (e_3, b_2, e_2, b_1) is a fan of $M\backslash a_2$, so using the same argument as before we see that we are forced to delete or contract an element in $\{b_1, b_2, e_2, e_3\}$ to obtain N. Since C is empty and D does not contain $\{a_2, e_2\}$ we must delete e_3. Continuing in this way we see that the elements $e_1, a_2, e_3, a_4, \ldots$ are all contained in D.

If r is odd then $a_{r-1} \in D$. However $(e_{r-1}, b_{r-1}, a_1, b_1)$ is a fan of $M\backslash e_1$, so by using the earlier arguments we can show that $e_{r-1} \in D$. Thus D contains both e_{r-1} and a_{r-1}, contrary to our earlier conclusion.

Hence r is even. It is easy to check that the matroid obtained from Δ_r by deleting $\{e_1, a_2, \ldots, a_{r-2}, e_{r-1}\}$ is isomorphic to Υ_r. Thus N is a minor of Υ_r, and this contradicts our assumption on the minimality of M.

Now we suppose that M is the triadic Möbius matroid Υ_r. The result holds if $r = 4$, so we assume that $r \geq 6$. Assume that C is empty. By Proposition 3.7 we have to delete an element e_i. But $\{c_{i-1}, c_i\}$ is a series pair of $M \backslash e_i$, so one of these elements is contracted. Thus C is not empty.

Claim 3.8.1. If $c_i, c_{i+1} \in C$, but $c_{i+2} \notin C$, then $c_{i+3} \in C$.

Proof. Assume that the claim is false, so $c_i, c_{i+1} \in C$, but $c_{i+2}, c_{i+3} \notin C$. Now $(e_{i+1}, c_{i+2}, e_{i+3}, c_{i+3})$ is a fan of M/c_{i+1}, so one of the elements in $\{e_{i+1}, e_{i+3}, c_{i+2}, c_{i+3}\}$ is deleted or contracted to obtain N. By our assumption either e_{i+1} or e_{i+3} is in D. But $\{c_{i+2}, c_{i+3}\}$ is a series pair in $M \backslash e_{i+3}$, so if $e_{i+3} \in D$ then either c_{i+2} or c_{i+3} is in C, a contradiction. Thus $e_{i+1} \in D$. But $\{e_i, e_{i+2}\}$ is a parallel pair in $M/c_i/c_{i+1}$, so one of these elements belongs to D. Now it follows from Proposition 3.3 that N is a minor of Υ_{r-2}, and this contradicts our assumption on the minimality of M. □

Claim 3.8.2. If $c_i \notin C$ then c_{i-1} and c_{i+1} are both in C.

Proof. Let us assume that the claim fails. Since $C \neq \varnothing$, we can assume by symmetry, relabeling if necessary, that $c_i, c_{i+1} \notin C$, but that $c_{i+2} \in C$. It cannot be the case that $c_{i+3} \in C$, for then, by Claim 3.8.1 and symmetry it follows that $c_i \in C$. Since $\{c_i, c_{i+1}\}$ is a series pair of $M \backslash e_{i+1}$ it follows that $e_{i+1} \notin D$. Note that $(e_{i+3}, c_{i+1}, e_{i+1}, c_i)$ is a fan of M/c_{i+2}. Since $e_{i+1} \notin D$ it follows that $e_{i+3} \in D$.

The set $\{c_{i+1}, c_{i+3}\}$ is a series pair of $M \backslash e_{i+3} \backslash e_{i+2}$. Since $e_{i+3} \in D$, but neither c_{i+1} nor c_{i+3} is in C this means that $e_{i+2} \notin D$.

Now $(e_{i+4}, c_{i+3}, e_{i+2}, c_{i+1})$ is a fan of $M/c_{i+2} \backslash e_{i+3}$, and since $e_{i+2} \notin D$ it follows that $e_{i+4} \in D$. But $\{c_{i+3}, c_{i+4}\}$ is a series pair of $M \backslash e_{i+4}$, and since $c_{i+3} \notin C$ this implies that $c_{i+4} \in C$.

We have shown that $c_{i+2}, c_{i+4} \in C$ and that $e_{i+3}, e_{i+4} \in D$. It now follows from Proposition 3.4 that N is a minor of Υ_{r-2}. This contradiction proves the claim. □

Claim 3.8.2 implies that of any pair of consecutive spoke elements at least one belongs to C. Now it is an easy matter to check that M/C is isomorphic to a matroid obtained from a triangular Möbius matroid by (possibly) adding parallel elements to rim elements. Thus N is a minor of a triangular Möbius matroid that is in turn a proper minor of M. This contradiction completes the proof of Lemma 3.8. □

Corollary 3.9. *No Möbius matroid has an $M(K_{3,3})$-minor.*

Proof. In the light of Lemma 3.8, to prove the corollary we need only check that $M(K_{3,3})$ is not a Möbius matroid. This is trivial. □

CHAPTER 4

From internal to vertical connectivity

The purpose of this chapter is to develop the machinery we will need to show that a minimal counterexample to Theorem 1.1 can be assumed to be vertically 4-connected. Much of the material we use has already been introduced in Section 2.6 of Chapter 2.

Recall that a matroid is almost vertically 4-connected if it is vertically 3-connected, and whenever (X, Y) is a vertical 3-separation, then either X contains a triad that spans X, or Y contains a triad that spans Y. Throughout this chapter we will suppose that M is an internally 4-connected non-cographic member of $\mathcal{EX}(M(K_{3,3}))$. Then M is almost vertically 4-connected by definition. If M has no triads then M is vertically 4-connected, so we assume that T is a triad of M. Since M is non-cographic it has rank at least three, and thus, as M is almost vertically 4-connected, it follows that T is independent. Lemmas 2.31 and 2.33 and Proposition 2.34 imply that $\nabla_T(M)$ is an almost vertically 4-connected non-cographic member of $\mathcal{EX}(M(K_{3,3}))$. Proposition 2.35 implies that $\nabla_T(M)$ has strictly fewer triads than M. Thus we can repeat this process until we obtain a matroid M_0 that has no triads, and which is therefore vertically 4-connected. By then deleting all but one element from every parallel class of M_0 we obtain a simple vertically 4-connected matroid.

Suppose that while reducing M to a vertically 4-connected matroid we perform the Y-Δ operation on the triads $T_1, \ldots, T_n$ in that order. Let $\mathcal{T}_0 = \{T_1, \ldots, T_n\}$.

Claim 4.1. The triads in $\mathcal{T}_0$ are pairwise disjoint.

Proof. Let $N_0 = M$, and for $1 \le i \le n$ let N_i be the matroid obtained by performing Y-Δ operations on the triads $T_1, \ldots, T_i$. Suppose that T_i and T_j have a non-empty intersection for $1 \le i < j \le n$. Repeated application of Proposition 2.35 shows that T_j is a triad of N_i. Let e be an element in $T_i \cap T_j$. Then $N_i \backslash e$ contains a series pair. But $N_i = \nabla_{T_i}(N_{i-1})$, and $\nabla_{T_i}(N_{i-1}) \backslash e$ is isomorphic to N_{i-1}/e by Proposition 2.27. Hence N_{i-1} contains a series pair. But this is a contradiction as the matroids $N_0, \ldots, N_n$ are all almost vertically 4-connected. $\square$

Each member of $\mathcal{T}_0$ is a triangle of M_0. We can recover M from M_0 by performing a Δ-Y operation on each of these triangles in turn. Proposition 2.25 and Claim 4.1 tell us that the order in which we perform the

Δ-Y operations is immaterial. Obviously each triangle in $\mathcal{T}_0$ can be identified with a triangle of $\mathrm{si}(M_0)$. Let $\mathcal{T}$ be the set of triangles in $\mathrm{si}(M_0)$ that corresponds to the set of triangles $\mathcal{T}_0$ in M_0.

We may not be able to recover M from $\mathrm{si}(M_0)$ by performing Δ-Y operations on the triangles of $\mathcal{T}$, because these triangles may not be disjoint. Thus recovering M from $\mathrm{si}(M_0)$ involves one more step, namely adding parallel elements to reconstruct M_0. The next two results show how we can perform this step using a knowledge of $\mathcal{T}$.

Claim 4.2. Let T and T' be triangles of M_0 such that $T, T' \in \mathcal{T}_0$. Then $r_{M_0}(T \cup T') > 2$.

Proof. Let us assume otherwise. Then $r_{M_0}(T \cup T') = 2$. Because the order in which we applied the Y-Δ operation to the members of $\mathcal{T}_0$ while obtaining M_0 is immaterial, we can assume that $T = T_{n-1}$ and $T' = T_n$. Since M_0 is non-cographic by repeated application of Lemma 2.33, it follows that the rank and corank of M_0 are at least three. Now it follows easily from Lemma 2.7 that M_0 has a minor N such that N is isomorphic to the matroid obtained from $M(K_4)$ by adding a point in parallel to each element of a triangle, and both T and T' are triangles of N. Proposition 2.26 tells us that $\Delta_T(\Delta_{T'}(N))$ is a minor of $\Delta_T(\Delta_{T'}(M_0))$. But $\Delta_T(\Delta_{T'}(M_0))$ is obtained from M by performing Y-Δ operations on the triads $T_1, \ldots, T_{n-2}$ in turn. Proposition 2.34 implies that this matroid has no $M(K_{3,3})$-minor. However $\Delta_T(\Delta_{T'}(N))$ is isomorphic to $M(K_{3,3})$, so we have a contradiction. $\square$

Claim 4.2 shows that distinct triangles in $\mathcal{T}_0$ correspond to distinct triangles in $\mathcal{T}$. Therefore the number of triangles in $\mathcal{T}$ is exactly equal to the number of triangles in $\mathcal{T}_0$.

Claim 4.3. Let F be a rank-one flat of M_0 such that $|F| > 1$. The size of F is exactly equal to the number of triangles in $\mathcal{T}_0$ that have a non-empty intersection with F.

Proof. Suppose that no member of $\mathcal{T}_0$ has a non-empty intersection with F. Since M is recovered from M_0 by performing Δ-Y operations on the triangles in $\mathcal{T}_0$ it follows from Proposition 2.27 that M contains a parallel pair. This is a contradiction as M is internally 4-connected. Therefore there is an element $e \in F$ and a member T of $\mathcal{T}_0$ such that $e \in T$.

Suppose that $e' \in F$ and e' is contained in no member of $\mathcal{T}_0$. Clearly T is a triad of $\Delta_T(M_0)$ and it is easy to see that $(T - e) \cup e'$ is a triangle of $\Delta_T(M_0)$. Moreover, it follows from Proposition 2.27 that T is a triad and $(T - e) \cup e'$ is a triangle in M. Thus $\lambda_M(T \cup e') \leq 2$. Since M is internally 4-connected the complement of $T \cup e'$ in M contains at most three elements, so $|E(M)| \leq 7$. Since M is non-cographic this means that M is isomorphic to either F_7 or F_7^*. However F_7 has no triads and F_7^* has no triangles, which is a contradiction as M has both a triangle and a triad. Therefore each element of F is contained in at least one member of $\mathcal{T}_0$, and by Claim 4.1

each element of F is contained in exactly one member of $\mathcal{T}_0$. The result follows. $\square$

Using Claims 4.2 and 4.3 we can recover M_0 from $\mathrm{si}(M_0)$ as follows: For each element $e \in E(\mathrm{si}(M_0))$ let t_e be the number of triangles in $\mathcal{T}$ that contain e. If $t_e > 1$ then add $t_e - 1$ parallel elements to e. The resulting matroid is isomorphic to M_0. Now we can find a set of pairwise disjoint triangles that correspond to the triangles in $\mathcal{T}$ and perform Δ-Y operations on each of them. In other words, M is isomorphic to the matroid $\Delta(\mathrm{si}(M_0); \mathcal{T})$ described in Section 2.6 of Chapter 2.

Next we examine what restrictions apply to the set $\mathcal{T}$. Let T be a triangle of $\mathrm{si}(M_0)$ such that $T \in \mathcal{T}$. We can assume that T is also a triangle of M_0. Thus $T \in \mathcal{T}_0$ and we will assume that $T = T_n$. The matroid $\Delta_T(M_0)$ is equal to that obtained from M by performing Y-Δ operations on the triads $T_1, \ldots, T_{n-1}$, and therefore has no $M(K_{3,3})$-minor by Proposition 2.34. But $\mathrm{si}(M_0)$ is a minor of M_0, so Proposition 2.26 implies the following fact.

Claim 4.4. $\Delta_T(\mathrm{si}(M_0))$ has no $M(K_{3,3})$-minor.

Recall that T is an allowable triangle of a matroid $M \in \mathcal{EX}(M(K_{3,3}))$ if $\Delta_T(M)$ has no $M(K_{3,3})$-minor. Claim 4.4 asserts that $\mathcal{T}$ contains only allowable triangles of $\mathrm{si}(M_0)$.

Suppose that T, $T' \in \mathcal{T}_0$ are triangles of M_0, and that $T \cup T'$ contains a cocircuit of size four. Proposition 2.29 tells us that M contains a series pair, which is a contradiction. Therefore, if T, $T' \in \mathcal{T}_0$ are triangles of M_0 then $T \cup T'$ does not contain a cocircuit of size four. Suppose that T, $T' \in \mathcal{T}$ are triangles of $\mathrm{si}(M_0)$ such that $T \cup T'$ does contain a cocircuit C^* of size four in $\mathrm{si}(M_0)$. It is easy to see that $T \cap C^*$ and $T' \cap C^*$ are disjoint sets of size two. It cannot be the case that C^* is a cocircuit in M_0. Thus some element in C^* is in a parallel pair in M_0. Now it follows easily from Claim 4.3 that there is some triangle $T'' \in \mathcal{T}$ such that T'' meets both $T \cap C^*$ and $T' \cap C^*$.

The restrictions that apply to the set $\mathcal{T}$ are summarized here.

(i) Every triangle in $\mathcal{T}$ is an allowable triangle of $\mathrm{si}(M_0)$.
(ii) If T and T' are in $\mathcal{T}$ and C^* is a four-element cocircuit of $\mathrm{si}(M_0)$ contained in $T \cup T'$, then $\mathcal{T}$ contains a triangle T'' that meets both $T \cap C^*$ and $T' \cap C^*$.

Any set of triangles that obeys these two conditions will be called a *legitimate set*.

We first consider the case that $\mathrm{si}(M_0)$ is a sporadic matroid.

Lemma 4.5. *Suppose that M is an internally 4-connected non-cographic matroid in $\mathcal{EX}(M(K_{3,3}))$ and that M has at least one triad. Let M_0 be the matroid obtained from M by repeatedly performing Y-Δ operations until the resulting matroid has no triads. If $\mathrm{si}(M_0)$ is isomorphic to any of the sporadic matroids listed in Appendix* B *then $\mathrm{si}(M_0)$ is isomorphic to $M_{4,11}$, and M is isomorphic to $M_{5,11}$.*

Proof. Appendix B lists 18 sporadic matroids. Each is simple and vertically 4-connected, except for $M_{5,11}$, which is not vertically 4-connected. For each of the remaining 17 matroids M we will find all possible non-empty legitimate sets of triangles, and for each such set $\mathcal{T}$, we will construct $\Delta(M; \mathcal{T})$. By the preceding discussion, if the lemma fails then this procedure will uncover at least one internally 4-connected non-cographic member of $\mathcal{EX}(M(K_{3,3}))$ other than $M_{5,11}$. By the results of Appendix C, there are only six sporadic matroids that contain allowable triangles, so we need only consider these six.

Performing a Δ-Y operation on any one of the allowable triangles in
✠3 $M_{4,11}$ produces a matroid isomorphic to $M_{5,11}$. If $\mathcal{T}$ is a legitimate set of triangles in $M_{4,11}$ that contains more than one triangle then $\Delta(M_{4,11}; \mathcal{T})$
✠4 has an $M(K_{3,3})$-minor.

Since the allowable triangles in $M^a_{5,12}$, $M_{6,13}$, $M_{7,15}$, $M_{9,18}$, and $M_{11,21}$ are pairwise disjoint and any pair contains a cocircuit of size four it follows that a non-empty legitimate set contains exactly one allowable triangle. The following argument demonstrates that performing a Δ-Y operation on a single allowable triangle in any of these matroids produces a matroid that is not internally 4-connected.

Suppose that M_1 is one of the five matroids listed in the previous paragraph. Then $M_1 \cong \nabla(F_7^*; \mathcal{T})$, where $\mathcal{T}$ is a set of at least four triangles in F_7. Let T be an allowable triangle in M_1. We will show that $(\Delta_T(M_1))^*$ is not internally 4-connected. Since internal 4-connectivity is preserved by duality this will suffice.

Let $M = F_7$, so that $M_1^* \cong \Delta(M; \mathcal{T})$. Let M' be the matroid obtained by adding parallel elements to M in the appropriate way so that we can find a set $\mathcal{T}_0 = \{T_1, \ldots, T_n\}$ of disjoint triangles in M' such that each of these corresponds to a triangle in $\mathcal{T}$. Thus M_1^* is isomorphic to the matroid obtained from M' by performing Δ-Y operations on each of the triangles in $\mathcal{T}_0$ in turn. Since T is an allowable triangle of M_1 it is a member of $\mathcal{T}_0$ by Proposition 2.37 and the results in Appendix C. Since the triangles in $\mathcal{T}_0$ are disjoint we can assume that $T = T_n$. Note that $(\Delta_T(M_1))^* = \nabla_T(M_1^*)$ is isomorphic to $\nabla_{T_n}(\Delta(M; \mathcal{T}))$ and this matroid is equal to the matroid obtained from M' by performing Δ-Y operations on the triangles $T_1, \ldots, T_{n-1}$. For any integer $i \in \{1, \ldots, n-1\}$ there is a parallel class F of M' such that T_i contains an element $e \in F$ and T_n contains an element $e' \in F$. Now T_i is a triad in $\nabla_T(\Delta(M; \mathcal{T}))$, and it is easy to see that $(T_i - e) \cup e'$ is a triangle. Therefore $\lambda(T_i \cup e') = 2$, and it follows easily that $\nabla_T(\Delta(M; \mathcal{T}))$ is not internally 4-connected. Thus $(\Delta_T(M_1))^*$ is not internally 4-connected. This completes the proof. $\square$

Next we consider the case that $\mathrm{si}(M_0)$ is isomorphic to a triangular Möbius matroid. Recall that the triangles of Δ_r are exactly the sets $\{a_i, e_i, e_r\}$ where $1 \leq i \leq r-1$, the sets $\{a_i, a_{i+1}, b_i\}$ and

$\{e_i, e_{i+1}, b_i\}$ where $1 \leq i \leq r-2$, along with the sets $\{a_1, e_{r-1}, b_{r-1}\}$ and $\{a_{r-1}, e_1, b_{r-1}\}$.

CLAIM 4.6. Let $r \geq 4$ be an integer. The triangle $\{a_i, e_i, e_r\}$ is not allowable in Δ_r for $1 \leq i \leq r-1$.

PROOF. Let $T = \{a_i, e_i, e_r\}$. There is an automorphism of Δ_r taking T to $\{a_1, e_1, e_r\}$, so we will assume that $i = 1$. If $r \geq 5$ then Proposition 3.2 implies that by contracting the elements $b_3, \ldots, b_{r-2}$ from Δ_r we obtain a minor isomorphic to Δ_4 (up to the addition of parallel elements) in which T is a triangle. Proposition 2.26 shows that $\Delta_T(\Delta_4)$ is a minor of $\Delta_T(\Delta_r)$. Figure 4.1 gives matrix and fundamental graph representations of $\Delta_T(\Delta_4)$. This matroid has an $M(K_{3,3})$-minor so we are done. $\square$ ✠5

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

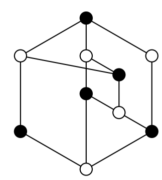

Figure 4.1: Matrix and fundamental graph representations of $\Delta_T(\Delta_4)$.

CLAIM 4.7. Let $r \geq 4$ be an integer. Suppose that N is obtained from Δ_r by adding an element b_i' in parallel to b_i, where $i \in \{1, \ldots, r-1\}$. Let T and T' be the disjoint triangles of N that contain b_i and b_i' respectively. Then $\Delta_T(\Delta_{T'}(N))$ has an $M(K_{3,3})$-minor.

PROOF. As in the previous proof we can assume that $T = \{b_1, e_1, e_2\}$ and that $T' = \{a_1, a_2, b_1'\}$. If $r \geq 5$ then we again obtain N' by contracting the elements $b_3, \ldots, b_{r-2}$, so that N' is isomorphic to Δ_4 up to the addition of parallel elements. It remains to consider $\Delta_T(\Delta_{T'}(N'))$. Figure 4.2 shows matrix and fundamental graph representations of this matroid. It has an $M(K_{3,3})$-minor, so by Proposition 2.26 we are done. $\square$ ✠6

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

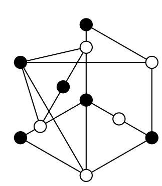

Figure 4.2: Matrix and fundamental graph representations of $\Delta_T(\Delta_{T'}(N'))$.

For any positive integer $r \geq 3$ let Δ_r° be the matroid obtained by adding a parallel point to each rim element of Δ_r. Suppose that the ground set of

Δ_r° is $E(\Delta_r) \cup \{a'_1, \ldots, a'_{r-1}\} \cup \{e'_1, \ldots, e'_{r-1}\}$, where a'_i is parallel to a_i and e'_i is parallel to e_i for $1 \le i \le r-1$.

LEMMA 4.8. *Let $r \geq 3$ be an integer and let T be a triangle of Δ_r° that contains a spoke element b. Then $\Delta_T(\Delta_r^\circ)$ is isomorphic to a restriction of Δ_{r+1}°, and this isomorphism takes spoke elements of Δ_r° other than b to spoke elements of Δ_{r+1}°.*

PROOF. It follows from Proposition 3.2 (and is easy to confirm) that we can obtain an isomorphic copy of Δ_r° from Δ_{r+1}° by deleting two elements from $\{a_r, a'_r, e_1, e'_1\}$, two elements from $\{a_1, a'_1, e_r, e'_r\}$, and then contracting b_r. In particular, if we delete $\{a'_r, e'_1, e_r, e'_r\}$, contract b_r, and then perform the following relabeling of elements we obtain a matroid that is actually equal to Δ_r°.

$$e_{r+1} \to e_r \quad a_r \to e'_1$$

Let $N = \Delta_{r+1}^\circ \backslash a'_r \backslash e'_1 \backslash e_r \backslash e'_r$ and let $T' = \{a_r, b_{r-1}, b_r\}$. Now T' is a triad of N. It follows from the discussion following Proposition 2.27 that N/b_r is isomorphic to $\nabla_{T'}(N)\backslash b_r$, where the isomorphism involves switching the labels on a_r and b_{r-1}. From this we derive another way to obtain Δ_r° from Δ_{r+1}°: Delete $\{a'_r, e'_1, e_r, e'_r\}$, then perform a Y-Δ operation on $\{a_r, b_{r-1}, b_r\}$. Then delete b_r and perform the following relabeling.

$$e_{r+1} \to e_r \quad a_r \to b_{r-1} \quad b_{r-1} \to e'_1$$

Note that $\{a'_{r-1}, a_r, b_{r-1}\}$ is a triangle of N. It is easy to see that it is also a triangle in $\nabla_{T'}(N)$. However, $T' = \{a_r, b_{r-1}, b_r\}$ is a triangle in $\nabla_{T'}(N)$, so it follows that a'_{r-1} and b_r are parallel in $\nabla_{T'}(N)$. Thus we can delete a'_{r-1} from $\nabla_{T'}(N)$ instead of b_r. But $\nabla_{T'}(N)\backslash a'_{r-1} = \nabla_{T'}(N\backslash a'_{r-1})$ since $a'_{r-1} \notin T'$. This provides us with yet another way of deriving Δ_r° from Δ_{r+1}°: Delete $\{a'_{r-1}, a'_r, e'_1, e_r, e'_r\}$. Perform a Y-Δ operation on $\{a_r, b_{r-1}, b_r\}$ and then perform the following relabeling.

$$e_{r+1} \to e_r \quad a_r \to b_{r-1} \quad b_{r-1} \to e'_1 \quad b_r \to a'_{r-1}$$

Suppose that T is a triangle of Δ_r° that contains a spoke element. There is an automorphism of Δ_r° that takes T to $\{a'_{r-1}, b_{r-1}, e'_1\}$, and this automorphism takes spoke elements to spoke elements. Therefore we will assume that T is equal to $\{a'_{r-1}, b_{r-1}, e'_1\}$.

By reversing the procedure discussed above, we see that if we relabel e_r with e_{r+1}, b_{r-1} with a_r, e'_1 with b_{r-1}, and a'_{r-1} with b_r, then perform a Δ-Y operation on T, then we obtain the matroid

$$\Delta_{r+1}^\circ \backslash a'_{r-1} \backslash a'_r \backslash e'_1 \backslash e_r \backslash e'_r.$$

Thus $\Delta_T(\Delta_r^\circ)$ is isomorphic to Δ_{r+1}° restricted to the set $E(\Delta_{r+1}^\circ) - \{a'_{r-1}, a'_r, e'_1, e_r, e'_r\}$, and the isomorphism is determined naturally by the relabeling. Any spoke element other than the one in T is taken to another spoke element in this procedure, so we are done. $\square$

LEMMA 4.9. *Suppose that M is an internally 4-connected non-cographic matroid in $\mathcal{EX}(M(K_{3,3}))$ and that M has at least one triad. Let M_0 be the matroid obtained from M by repeatedly performing Y-Δ operations until the resulting matroid has no triads. If $\mathrm{si}(M_0)$ is isomorphic to a triangular Möbius matroid then M is also a Möbius matroid.*

PROOF. Suppose that M_0 is obtained from M by performing Y-Δ operations on the triads $T_1, \ldots, T_n$ in turn. Let $\mathcal{T}_0 = \{T_1, \ldots, T_n\}$, and let $\mathcal{T}$ be the set of triangles in $\mathrm{si}(M_0)$ that correspond to the triangles in $\mathcal{T}_0$. Thus $M \cong \Delta(\mathrm{si}(M_0); \mathcal{T})$ and $\mathrm{si}(M_0) \cong \Delta_r$ for some $r \geq 3$.

We start by assuming that $r = 3$. This means that $\mathrm{si}(M_0) \cong \Delta_3 \cong F_7$. Up to relabeling there is only one legitimate set of triangles $\mathcal{T}$ in $\mathrm{si}(M_0)$ such that $|\mathcal{T}| = 1$. Moreover, if $|\mathcal{T}| = 1$ then $\Delta(\mathrm{si}(M_0); \mathcal{T}) \cong F_7^*$. The Fano dual is also a Möbius matroid, so in the case that $n = 1$ we are done.

Since any pair of triangles in F_7 contains a cocircuit of size four we see that there is no legitimate set of triangles in $\mathrm{si}(M_0)$ containing exactly two triangles. Therefore we assume that $n \geq 3$.

It is easy to see that since $\mathcal{T}$ is a legitimate set it contains a set of three triangles having no common point of intersection. Let $\mathcal{T}_3$ be such a set of three triangles in $\mathrm{si}(M_0)$. It follows from Lemma 2.36 that $\Delta(\mathrm{si}(M_0); \mathcal{T})$ has a minor isomorphic to $\Delta(F_7; \mathcal{T}_3)$. However this last matroid is isomorphic to the dual of Δ_4, and it has an $M(K_{3,3})$-minor, so we have a contradiction. ✠7

Therefore we assume that $r \geq 4$. For $1 \leq i \leq n$ let M_i be the matroid obtained from M_0 by performing Δ-Y operations on the triangles $T_1, \ldots, T_i$. Thus M_n is equal to M.

CLAIM 4.9.1. For $0 \leq i \leq n$ there is an isomorphism ψ_i between M_i and a restriction of Δ°_{r+i}. Moreover, if $i < j \leq n$, then T_j is a triangle of M_i containing an element b such that $\psi_i(b)$ is a spoke element of Δ°_{r+i}.

PROOF. It follows from Claim 4.6 that the only allowable triangles in Δ_r are triangles that contain spoke elements. Moreover, it is not difficult to use Claim 4.7 to show that any spoke element is contained in at most one triangle of $\mathcal{T}$. Therefore if a pair of triangles in $\mathcal{T}$ have a non-empty intersection, they meet in a rim element of $\mathrm{si}(M_0) \cong \Delta_r$. It follows from these facts that M_0 is obtained from Δ_r by replacing certain rim elements with parallel pairs. Thus the claim holds when $i = 0$.

Suppose that $i > 0$, and the claim holds for M_{i-1}. Clearly T_i is a triangle in M_{i-1}. Let $T = \psi_{i-1}(T_i)$. Then T contains a spoke element of Δ°_{r+i-1} by the inductive hypothesis. It follows easily from Proposition 2.17 that $\psi_{i-1}(\Delta_{T_i}(M_{i-1}))$ is a restriction of $\Delta_T(\Delta^\circ_{r+i-1})$. But Lemma 4.8 tells us that $\Delta_T(\Delta^\circ_{r+i-1})$ is isomorphic to a restriction of Δ°_{r+i} and this isomorphism takes spoke elements of Δ°_{r+i-1} (other than the spoke element contained in T) to spoke elements of Δ°_{r+i}. We let ψ'_i be the restriction of this isomorphism to the ground set of $\psi_{i-1}(\Delta_{T_i}(M_{i-1}))$. Since $M_i = \Delta_{T_i}(M_{i-1})$, if we act upon $\Delta_{T_i}(M_{i-1})$ first with ψ_{i-1} and then with ψ'_i, we obtain an isomorphism between M_i and a restriction of Δ°_{r+i} that takes spoke elements other than the spoke element in T_i to spoke elements. We let this isomorphism be ψ_i. Now it is easy to see that if $i < j \leq n$ then T_j contains an element b such that $\psi_i(b)$ is a spoke element of Δ°_{r+i}. $\square$

Since $M = M_n$ it follows from Claim 4.9.1 that M is isomorphic to a restriction of Δ°_{r+n}. In fact M is a restriction of Δ_{r+n}, since M is internally 4-connected and therefore simple. However Lemma 3.8 tells us that any internally 4-connected non-cographic minor of Δ_{r+n} is in fact a Möbius matroid. Thus M is a Möbius matroid, so Lemma 4.9 holds. $\square$

CHAPTER 5

An R_{12}-type matroid

The matroid R_{12} plays an important role in Seymour's decomposition of regular matroids. He shows that any regular matroid with an R_{12}-minor cannot be internally 4-connected. In this chapter we introduce a matroid that plays a similar role in our proof.

Consider the single-element coextension Δ_4^+ of Δ_4 by the element e_5 represented over GF(2) by $[I_5|A]$ (the matrix A is displayed in Figure 5.1, along with the fundamental graph $G_B(\Delta_4^+)$ where $B = \{e_1, \ldots, e_5\}$). We can check that Δ_4^+ has no $M(K_{3,3})$-minor. The set $\{a_1, a_2, b_1, e_5\}$ is a four- ✠8
element circuit-cocircuit of Δ_4^+.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

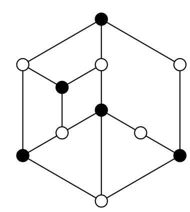

Figure 5.1: Matrix and fundamental graph representations of Δ_4^+.

Lemma 5.1. *If $M \in \mathcal{EX}(M(K_{3,3}))$ and M has a Δ_4^+-minor, then M is not internally 4-connected.*

Proof. Suppose that the lemma fails, and that M is an internally 4-connected member of $\mathcal{EX}(M(K_{3,3}))$ with a Δ_4^+-minor. By Proposition 2.10 there is a 3-connected single-element extension or coextension N' of Δ_4^+ such that $\{a_1, a_2, b_1, e_5\}$ is not a circuit-cocircuit of N' and N' is a minor of M. But every 3-connected binary single-element extension or coextension of Δ_4^+ in which $\{a_1, a_2, b_1, e_5\}$ is not a circuit-cocircuit has an $M(K_{3,3})$-minor. This contradiction completes the proof. $\square$ ✠9

Lemma 5.2. *Suppose that M and N are 3-connected binary matroids such that $|E(N)| > 7$. Assume that M contains a four-element circuit-cocircuit C^* and an N-minor. Then M has a 3-connected minor M' such that C^* is a four-element circuit-cocircuit of M', furthermore M' has an N-minor, but if $e \in E(M') - C^*$ then neither $M' \backslash e$ nor M'/e has an N-minor.*

Proof. Assume that M is a minimal counterexample. This means that $M \neq N$, so let e be an element of M such that either $M\backslash e$ or M/e has an N-minor. If the only such elements are contained in C^* then the result holds for M so we are done. Therefore we assume that $e \notin C^*$.

Let us assume that $M\backslash e$ has an N-minor. Then $\mathrm{co}(M\backslash e)$ has an N-minor. Suppose that $e \notin \mathrm{cl}^*_M(C^*) \cup \mathrm{cl}_M(C^*)$. Then it is easy to see that C^* is a four-element circuit-cocircuit of $\mathrm{co}(M\backslash e)$. Thus if $\mathrm{co}(M\backslash e)$ is 3-connected we have a contradiction to the minimality of M. Therefore we assume that $\mathrm{co}(M\backslash e)$ is not 3-connected, so Proposition 2.5 implies that $\mathrm{si}(M/e)$ is 3-connected.

Since $e \notin \mathrm{cl}_M(C^*)$ it follows that C^* is a four-element circuit-cocircuit in $\mathrm{si}(M/e)$. If M/e has an N-minor then so does $\mathrm{si}(M/e)$, and in this case we again have a contradiction to the minimality of M. Therefore we must assume that M/e does not have an N-minor.

Since $\mathrm{co}(M\backslash e)$ is not 3-connected there is a 2-separation (X_1, X_2) of $M\backslash e$ such that $|X_1|, |X_2| \geq 3$. If both $r(X_1), r(X_2) \geq 3$ then M/e has an N-minor by Proposition 2.22, a contradiction. Thus we will assume that $r(X_1) \leq 2$. Since M has no parallel pairs it follows that X_1 is a triangle in M.

As (X_1, X_2) is a 2-separation of $M\backslash e$ we conclude that $r(X_2) = r(M)-1$. Thus X_1 contains a cocircuit of $M\backslash e$, and in fact it must contain a series pair. Let $X_1 = \{x, y, z\}$, and suppose that x and y are in series in $M\backslash e$. Since N has no series pairs $M\backslash e/x$ has an N-minor. But y and z are in parallel in $M\backslash e/x$, so $M\backslash e/x\backslash y$ has an N-minor. However since M has no series pairs $\{e, x, y\}$ must be a triad of M. Thus x is a coloop in $M\backslash e\backslash y$, so $M\backslash e/x\backslash y = M\backslash e\backslash x\backslash y$. But e is a coloop in $M\backslash x\backslash y$, so $M\backslash e\backslash x\backslash y = M/e\backslash x\backslash y$. Thus $M/e\backslash x\backslash y$, and hence M/e, has an N-minor, a contradiction.

Now we must assume that $e \in \mathrm{cl}^*_M(C^*) \cup \mathrm{cl}_M(C^*)$. Suppose that $e \in \mathrm{cl}_M(C^*)$. Then $e \notin \mathrm{cl}^*_M(C^*)$, for otherwise $(C^* \cup e, E - (C^* \cup e))$ is a 2-separation of M. Thus C^* is a four-element cocircuit in $\mathrm{co}(M\backslash e)$. Note that in M/e there are two parallel pairs in C^*, and deleting a single element from each of these produces a series pair. Thus $\mathrm{si}(M/e)$ is not 3-connected, so $\mathrm{co}(M\backslash e)$ is 3-connected. Since $\mathrm{co}(M\backslash e)$ has an N-minor we have a contradiction to the minimality of M.

Therefore we suppose that $e \notin \mathrm{cl}_M(C^*)$, so that $e \in \mathrm{cl}^*_M(C^*)$. Now C^* contains two series pairs in $M\backslash e$ and contracting an element from each produces a parallel pair. Thus $\mathrm{co}(M\backslash e)$ is not 3-connected, and hence $\mathrm{si}(M/e)$ is 3-connected. Moreover $(C^*, E - (C^* \cup e))$ is a 2-separation of $M\backslash e$ and $r(C^*) \geq 3$. As $E - (C^* \cup e)$ contains at least four elements of $E(N)$ it also follows that $r(E - (C^* \cup e)) \geq 3$. Since $M\backslash e$ has an N-minor Proposition 2.22 tells us that M/e, and hence $\mathrm{si}(M/e)$, has an N-minor. But C^* is a four-element circuit-cocircuit in $\mathrm{si}(M/e)$, so we again have a contradiction to the minimality of M.

We have shown that $M\backslash e$ cannot have an N-minor. Therefore M/e has an N-minor. But M^* and N^* also provide a minimal counterexample to the problem, and C^* is a four-element circuit-cocircuit of M^*. Since $M^*\backslash e$ has an N^*-minor we can use exactly the same arguments as before to find a contradiction. Thus the result holds. $\square$

COROLLARY 5.3. *Suppose $M \in \mathcal{EX}(M(K_{3,3}))$ is a 3-connected matroid that has a Δ_4-minor and a four-element circuit-cocircuit. Then M has a Δ_4^+-minor.*

PROOF. Lemma 5.2 implies that M has a 3-connected minor M' such that M' contains a four-element circuit-cocircuit C^*, and M' has a Δ_4-minor, but if $e \in E(M') - C^*$ then neither $M'\backslash e$ nor M'/e has a Δ_4-minor. The rest of the proof is a straightforward case-check, an outline of which is given in Proposition A.1. $\square$

The following corollary of Lemma 5.1 and Corollary 5.3 is the main result of this chapter.

COROLLARY 5.4. *Suppose that $M \in \mathcal{EX}(M(K_{3,3}))$ is internally 4-connected and has a Δ_4-minor. If M' is a 3-connected minor of M and M' has a Δ_4-minor then M' has no four-element circuit-cocircuit.*

CHAPTER 6

A connectivity lemma

Connectivity results such as Seymour's Splitter Theorem are essential tools for inductive proofs in structural matroid theory. Theorem 2.12 implies that if N is a 3-connected matroid such that $|E(N)| \geq 4$ and N is not a wheel or whirl, and M has a proper N-minor, then M has a proper 3-connected minor M_0 such that M_0 has an N-minor and $|E(M)| - |E(M_0)| = 1$.

Because we are considering matroids that exhibit higher connectivity than 3-connectivity we need a new set of inductive tools.

Theorem 6.1. *Suppose that M and N are simple vertically 4-connected binary matroids such that N is a proper minor of M and $|E(N)| \geq 10$. Suppose also that whenever M' is a 3-connected minor of M and M' has a minor isomorphic to N then M' has no four-element circuit-cocircuit. Then M has a proper internally 4-connected minor M_0 such that M_0 has an N-minor and $|E(M)| - |E(M_0)| \leq 4$.*

The inductive step of our proof would be infeasible if we had to search through all extensions and coextensions on up to four elements. We need a refined version of Theorem 6.1 that tells us more about the way in which M_0 is derived from M. The next result is a step in this direction.

Theorem 6.2. *Under the hypotheses of Theorem 6.1 one of the following cases holds.*

(i) There exists an element $x \in E(M)$ such that $M\backslash x$ is internally 4-connected with an N-minor;

(ii) There exists an element $x \in E(M)$ such that $\mathrm{si}(M/x)$ is internally 4-connected, has an N-minor, and $|E(M)| - |E(\mathrm{si}(M/x))| \leq 3$;

(iii) There exist elements $x, y \in E(M)$ such that $M/x/y$ is vertically 4-connected, has an N-minor, and $|E(M)| - |E(\mathrm{si}(M/x/y))| \leq 3$; or,

(iv) There exist elements $x, y, z \in E(M)$ such that $M/x/y/z$ is vertically 4-connected, has an N-minor, and $|E(M)| - |E(\mathrm{si}(M/x/y/z))| \leq 4$.

Clearly Theorem 6.2 implies Theorem 6.1.

Theorem 6.2 is also too coarse a tool for our inductive proof: we need yet another refinement. Theorem 6.2 follows from Lemma 6.7, which is the main result of this chapter.

Before proving Lemma 6.7 we discuss some preliminary ideas. Suppose that M is a vertically 3-connected matroid and that N is an internally 4-connected minor of M with $|E(N)| \geq 7$. Suppose that (X_1, X_2) is a 3-separation of M. It follows from Proposition 2.2 that either $|E(N) \cap X_1| \leq 3$ or

$|E(N) \cap X_2| \le 3$. If $|E(N) \cap X_i| \le 3$ then we say that X_i is a *small* 3-*separator*. Since $|E(N)| \ge 7$ exactly one of X_1 and X_2 is a small 3-separator. If X is a small 3-separator and X is not properly contained in any other small 3-separator then we shall say that X is a *maximal small* 3-*separator*. A *small vertical* 3-*separator* is a small 3-separator that is also vertical, and a *maximal small vertical* 3-*separator* is a maximal small 3-separator that is also vertical.

PROPOSITION 6.3. *Suppose that M is a vertically 3-connected matroid on the ground set E and that N is an internally 4-connected minor of M with $|E(N)| \ge 10$. If X_1 and X_2 are maximal small 3-separators of M such that $X_1 \ne X_2$ and $r_M(E - X_1)$, $r_M(E - X_2) \ge 2$ then*

$$r_M(X_1 \cap X_2) \le 1.$$

Suppose that $X_1 \cap X_2 = F$ where $r_M(F) = 1$. If, in addition, $r_M(X_1)$, $r_M(X_2) \ge 2$ and both X_1 and X_2 contain at least three rank-one flats then either

(i) $F \subseteq \mathrm{cl}_M(E - X_1) \cap \mathrm{cl}_M(E - X_2)$; or,
(ii) $F \cap \mathrm{cl}_M(X_1 - F) = \varnothing$ and $F \cap \mathrm{cl}_M(X_2 - F) = \varnothing$.

PROOF. Suppose that $r_M(X_1 \cap X_2) \ge 2$. Since $\lambda_M(X_i) \le 2$ for all $i \in \{1, 2\}$ it follows from the submodularity of the connectivity function that

$$\lambda_M(X_1 \cup X_2) + \lambda_M(X_1 \cap X_2) \le 4.$$

Since $r_M(X_1 \cap X_2) \ge 2$ and $r_M(E - (X_1 \cap X_2)) \ge 2$ it cannot be the case that $\lambda_M(X_1 \cap X_2) \le 1$, for then M would have a vertical 2-separation. Thus $\lambda_M(X_1 \cup X_2) \le 2$. Suppose that $X_1 \cup X_2$ is not a 3-separator. This implies that $|E(M) - (X_1 \cup X_2)| \le 2$. Since $E(M) - X_1$ contains at least seven elements of $E(N)$ it follows that $X_2 - X_1$ contains at least five elements of $E(N)$, which contradicts the fact that X_2 is a small 3-separator. Therefore $X_1 \cup X_2$ is a 3-separator.

As X_1 and X_2 are distinct maximal 3-separators they are each properly contained in $X_1 \cup X_2$, so $X_1 \cup X_2$ is not a small 3-separator. Therefore $E - (X_1 \cup X_2)$ is a small 3-separator, so

$$|E(N) \cap (E - (X_1 \cup X_2))| \le 3.$$

However $|E(N) \cap X_i| \le 3$ for $i = 1, 2$ so $|E(N) \cap (X_1 \cup X_2)| \le 6$. Since $|E(N)| \ge 10$ this leads to a contradiction. We have shown that $r_M(X_1 \cap X_2) \le 1$.

Now we suppose that $r_M(X_1)$, $r_M(X_2) \ge 2$, and both X_1 and X_2 contain at least three rank-one flats. Let $F = X_1 \cap X_2$, where $r_M(F) = 1$. Assume that $F \subseteq \mathrm{cl}_M(E - X_1)$. If F were not contained in $\mathrm{cl}_M(X_1 - F)$ then $(X_1 - F, (E - X_1) \cup F)$ would be a vertical 2-separation of M. Hence $F \subseteq \mathrm{cl}_M(X_1 - F)$, and this implies that $F \subseteq \mathrm{cl}_M(E - X_2)$.

Next assume that $F \not\subseteq \mathrm{cl}_M(E - X_1)$. This implies that $F \not\subseteq \mathrm{cl}_M(X_2 - F)$, so in fact $F \cap \mathrm{cl}_M(X_2 - F) = \varnothing$. If F were contained in $\mathrm{cl}_M(E - X_2)$ then

$(X_2-F, (E-X_2)\cup F)$ would be a vertical 2-separation of M, so $F \nsubseteq \mathrm{cl}_M(E-X_2)$, and this implies that $F \nsubseteq \mathrm{cl}_M(X_1 - F)$. Thus $F \cap \mathrm{cl}_M(X_1 - F) = \varnothing$ and this completes the proof. □

If M is a matroid and X is a subset of $E(M)$ then let $G_M(X)$ denote the set $X \cap \mathrm{cl}_M(E(M)-X)$. We use $G^*_M(X)$ to denote $X \cap \mathrm{cl}^*_M(E(M)-X)$. We will make use of the fact that if X is a subset of $E(M)$ and $e \in X$ then $e \in \mathrm{cl}^*_M(E(M) - X)$ if and only if $e \notin \mathrm{cl}_M(X - e)$.

If X_1 and X_2 are small 3-separators in a 3-connected matroid then they automatically satisfy the hypotheses of Proposition 6.3, so the next result follows as a corollary.

Corollary 6.4. *Suppose that M is a 3-connected matroid and that N is an internally 4-connected minor of M such that $|E(N)| \geq 10$. If X_1 and X_2 are distinct maximal small 3-separators of M then $|X_1 \cap X_2| \leq 1$, and if $e \in X_1 \cap X_2$ then either $e \in G_M(X_1) \cap G_M(X_2)$ or $e \in G^*_M(X_1) \cap G^*_M(X_2)$.*

Suppose that M is a matroid and X is a subset of $E(M)$. Let $\mathrm{int}_M(X)$ denote the set $X - G_M(X) = X - \mathrm{cl}_M(E(M) - X)$.

Proposition 6.5. *Suppose that M is a vertically 3-connected matroid and that X is a vertical 3-separator of M. Then*

(i) $r_M(\mathrm{int}_M(X)) = r_M(X)$*;*
(ii) $X \subseteq \mathrm{cl}_M(\mathrm{int}_M(X))$*; and,*
(iii) $\mathrm{int}_M(X)$ *is a vertical 3-separator of M.*

Proof. Let $Y = E(M) - X$ and let $Y' = \mathrm{cl}_M(Y)$. Also, let $X' = \mathrm{int}_M(X) = E(M) - Y'$. It cannot be the case that X' is empty, for that would imply that $r_M(Y) = r(M)$ and that $r_M(X) = 2$. If $r_M(X') < r_M(X)$ then (X', Y') is a vertical k-separation for some $k < 3$, so $r_M(\mathrm{int}_M(X)) = r_M(X)$. Statements (ii) and (iii) follow easily. □

Suppose that $A = (e_1, \ldots, e_4)$ is a cofan of a binary matroid M. (Note that in this case A is also a fan, with the elements taken in reverse order.) We define e_1 to be a *good element* of A. Similarly, if $A = (e_1, \ldots, e_5)$ is a fan then e_2 and e_4 are good elements of A, and if $A = (e_1, \ldots, e_5)$ is a cofan then e_1 and e_5 are good elements.

Proposition 6.6. *Suppose that N is a proper minor of the binary matroid M such that N has no triads, and is simple and cosimple. Suppose that A is a fan or cofan with length four or five in M. If x is a good element of A then M/x has an N-minor.*

Proof. Suppose that $A = (e_1, \ldots, e_4)$ is a cofan. Then $x = e_1$. Suppose that M/e_1 does not have an N-minor. Since $\{e_1, e_2, e_3\}$ is a triad of M and N is cosimple with no triads it follows that either M/e_2 or M/e_3 has an N-minor. But it is easy to check that both M/e_2 and M/e_3 can be obtained from M/e_1 by deleting an element and adding a point in parallel to an existing element. Since N is simple it follows that M/e_1 also has an N-minor.

The argument is similar when $|A| = 5$. $\square$

We are now ready to tackle Lemma 6.7.

LEMMA 6.7. *Suppose that M and N are simple vertically 4-connected binary matroids such that N is a proper minor of M and $|E(N)| \geq 10$. Suppose also that whenever M' is a 3-connected minor of M and M' has a minor isomorphic to N then M' has no four-element circuit-cocircuit. Then one of the following cases holds:*

(i) There is an element $x \in E(M)$ such that $M\backslash x$ is internally 4-connected with an N-minor;

(ii) There is an element $x \in E(M)$ such that M/x is simple and vertically 4-connected with an N-minor;

(iii) There is an element $x \in E(M)$ such that $\mathrm{si}(M/x)$ is internally 4-connected with an N-minor. Furthermore $\mathrm{si}(M/x)$ contains at least one triangle and at least one triad. Moreover M/x has no loops and exactly one parallel pair;

(iv) There is an element $x \in E(M)$ such that M/x is vertically 4-connected and has an N-minor. Furthermore, there is a triangle T of M/x such that x is in a four-element cocircuit C^* of M with the property that $|C^* \cap T| = 2$. Moreover x is in at most two triangles in M, and if x is in two triangles of M, then exactly one of these triangles contains an element of T;

(v) There is an element $x \in E(M)$ such that M/x is vertically 4-connected and has an N-minor. Furthermore, there exist triangles T_1 and T_2 in M/x such that $|T_1 \cap T_2| = 1$, there is a four-element cocircuit C^* of M such that $x \in C^*$, $|C^* \cap T_i| = 2$ for $i = 1, 2$, and $T_1 \cap T_2 \subseteq C^*$. Moreover, there are no loops and at most two parallel pairs in M/x, and x is not contained in a triangle of M with the element in $T_1 \cap T_2$.

(vi) There is a triangle $\{x, y, z\}$ of M such that $M/x/y$ is vertically 4-connected with an N-minor. Furthermore, there exist triangles T_1 and T_2 of $M/x/y$ such that $r_{M/x/y}(T_1 \cup T_2) = 4$, and x is in a triad of $M\backslash z$ with two elements of T_1, and y is in a triad of $M\backslash z$ with two elements of T_2. Moreover, $M/x/y$ has exactly one loop and no parallel elements;

(vii) There is a triangle $\{x, y, z\}$ of M such that $M/x/y$ is vertically 4-connected with an N-minor. Furthermore, there exist triangles T_1 and T_2 of $M/x/y$ such that $|T_1 \cap T_2| = 1$, and both $(T_1 - T_2) \cup x$ and $(T_2 - T_1) \cup y$ are triads of $M\backslash z$. Moreover, $M/x/y$ has exactly one loop and no parallel elements;

(viii) There is a triangle $\{x, y, z\}$ of M such that $M/x/y$ is vertically 4-connected with an N-minor. Furthermore, there exist triangles T_1 and T_2 of $M/x/y$ such that $|T_1 \cap T_2| = 1$, and in $M\backslash z$ the element x is in a triad with the single element from $T_1 \cap T_2$ and a single element from

$T_1 - T_2$, and $(T_2 - T_1) \cup y$ is a triad. Moreover, $M/x/y$ has exactly one loop and no parallel elements; or,

(ix) There is an element $x \in E(M)$ such that $M\backslash x$ contains three cofans, $(x_1, \ldots, x_5)$, $(y_1, \ldots, y_5)$, and $(z_1, \ldots, z_5)$, where $x_5 = y_1$, $y_5 = z_1$, and $z_5 = x_1$, and $M/x_1/y_1/z_1$ is vertically 4-connected with an N-minor. Furthermore, $\{x, x_1, y_1, z_1\}$ is a circuit of M, and $M/x_1/y_1/z_1$ has exactly one loop and no parallel elements.

Proof. Suppose that the lemma does not hold for the pair of matroids M and N. Since both M and N have ground sets of size at least ten, and both M and N are simple we deduce that $r(M), r(N) \geq 4$. It follows from Proposition 2.1 that neither M nor N has any triads.

6.7.1. *There exists an element $e \in E(M)$ such that $M\backslash e$ is $(4, 5)$-connected and has an N-minor.*

Proof. If the claim is false, then by Theorem 2.16 there is an element $e \in E(M)$ such that M/e is $(4, 5)$-connected with an N-minor. Since M has no triads M/e has no triads. Every fan or cofan with length four or five contains a triad and as M/e has no four-element circuit-cocircuit it follows from Proposition 2.6 that if (X, Y) is a 3-separation of M/e then either $r_{M/e}(X) \leq 2$ or $r_{M/e}(Y) \leq 2$. Thus M/e is vertically 4-connected, and since M/e is $(4, 5)$-connected it is simple, so statement (ii) of the lemma holds. This contradiction implies that the sublemma is true. □

Henceforth we will suppose that $e \in E(M)$ has been chosen so that $M\backslash e$ is $(4, 5)$-connected and has an N-minor. Let us fix a particular N-minor of $M\backslash e$, so that we can define small 3-separators of $M\backslash e$, just as in the introduction to this chapter.

Since statement (i) does not hold we deduce that $M\backslash e$ has a 3-separation (X, Y) such that $|X|, |Y| \geq 4$. Since $M\backslash e$ is simple $r(X), r(Y) \geq 3$, so (X, Y) is a vertical 3-separation. Therefore $M\backslash e$ has at least one small vertical 3-separator.

Since M has no vertical 3-separations the next result follows from Proposition 2.4.

6.7.2. *If (X, Y) is a vertical 3-separation of $M\backslash e$ then $e \notin \mathrm{cl}_M(X)$ and $e \notin \mathrm{cl}_M(Y)$.*

Suppose that A is a small vertical 3-separator of $M\backslash e$. Then $E(M\backslash e) - A$ contains at least seven elements of $E(N)$, and as $M\backslash e$ is $(4, 5)$-connected it follows that $|A| \leq 5$. Since $M\backslash e$ has no four-element circuit-cocircuit we conclude from Proposition 2.6 that A is either a triad (which is to say a cofan of length three), or a fan or cofan with length four or five.

Figure 6.1 shows the four possible small vertical 3-separators of $M\backslash e$. They are, in order, a triad, a cofan of length four, a fan of length five, and a cofan of length five. In each case a hollow square represents a point in the underlying projective space that may not be in $E(M\backslash e)$. In the non-triad cases, a hollow circle indicates a good element.

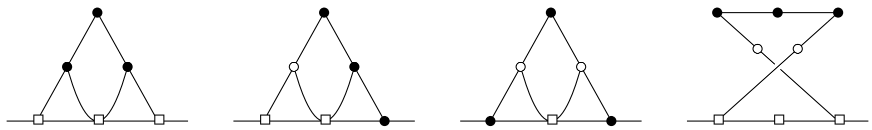

Figure 6.1: Small vertical 3-separators of $M\backslash e$.

6.7.3. *Suppose that A is a small vertical 3-separator of $M\backslash e$ and that x is a good element of A. Then $M\backslash e/x$ is vertically 3-connected.*

Proof. Suppose that the claim is false. Let (X, Y) be a vertical k-separation of $M\backslash e/x$ where $k < 3$. Since $M\backslash e$ is (vertically) 3-connected it is easy to see that both $(X\cup x, Y)$ and $(X, Y\cup x)$ are vertical 3-separations of $M\backslash e$ and that $x \in \mathrm{cl}_{M\backslash e}(X) \cap \mathrm{cl}_{M\backslash e}(Y)$. Suppose that $X \cup x$ is not a small 3-separator of $M\backslash e$. Then $|Y| \leq 5$, and hence $Y \cup x$ cannot contain seven elements of $E(N)$, so $Y \cup x$ is a small 3-separator. Therefore, by relabeling if necessary, we will assume that $X \cup x$ is a small 3-separator of $M\backslash e$. Let $\overline{X}$ be a maximal small 3-separator of $M\backslash e$ that contains $X \cup x$ and let $\overline{A}$ be a maximal small 3-separator that contains A.

First let us suppose that $\overline{X}$ and $\overline{A}$ are not equal. Then, since $M\backslash e$ is 3-connected, Corollary 6.4 tells us that $\overline{X}\cap\overline{A} = \{x\}$. But then $x \in \mathrm{cl}_{M\backslash e}(X)$ implies that $x \in G_{M\backslash e}(\overline{A})$, and therefore $x \in G_{M\backslash e}(A)$. However, it is easy to see that $x \in \mathrm{int}_{M\backslash e}(A)$, as x is a good element, so we have a contradiction. Hence we assume that $\overline{X} = \overline{A}$. This means that X is a vertical 3-separator of $M\backslash e$ contained in $\overline{A}$ such that $x \in \mathrm{cl}_{M\backslash e}(X)$, and $x \in \mathrm{cl}_{M\backslash e}(E(M\backslash e) - X)$, while $x \notin G_{M\backslash e}(A)$. An easy case-check confirms that A must be a cofan $(e_1, \ldots, e_5)$, and that x must be either e_2 or e_4, contradicting the fact that x is a good element of A. □

If A is a cofan or fan of $M\backslash e$ with length four or five then A contains a good element x by definition, and clearly $x \in \mathrm{int}_{M\backslash e}(A)$. The next result summarizes these observations and the consequences of Proposition 6.6 and 6.7.3.

6.7.4. *Let A be a small vertical 3-separator of $M\backslash e$ such that $|A| \geq 4$. Then A contains a good element x. Moreover, $x \in \mathrm{int}_{M\backslash e}(A)$, and $M\backslash e/x$ is vertically 3-connected with an N-minor.*

If A is a small vertical 3-separator of $M\backslash e$ such that $|A| \geq 4$ and $x \in A$ is a good element then, since $M\backslash e/x$ has an N-minor, we can define small 3-separators of $M\backslash e/x$ in the same way that we defined them for $M\backslash e$.

6.7.5. *Suppose that A is a maximal small vertical 3-separator of $M\backslash e$ and $|A| \geq 4$. Let $x \in A$ be a good element and let X be a small vertical 3-separator of $M\backslash e/x$. Then either*

(i) $X \subseteq A - x$ and X is a small vertical 3-separator of $M\backslash e$; or,

(ii) there exists a small vertical 3-separator X_0 of $M\backslash e$ such that $\mathrm{int}_{M\backslash e/x}(X) = \mathrm{int}_{M\backslash e}(X_0)$.

Proof. Let $X' = \mathrm{int}_{M\backslash e/x}(X)$. Proposition 6.5 tells us that X' is a vertical 3-separator of $M\backslash e/x$, and clearly it is a small vertical 3-separator of $M\backslash e/x$. Let us suppose that $x \notin \mathrm{cl}_{M\backslash e}(X')$. Then $r_{M\backslash e/x}(X') = r_{M\backslash e}(X')$ and X' is a 3-separator in $M\backslash e$. If X' is not a small 3-separator of $M\backslash e$, then the complement of X' in $M\backslash e$ is a small 3-separator, so this complement has size at most five. But X' is a small 3-separator of $M\backslash e/x$, so its complement in $M\backslash e/x$ contains at least seven elements of $E(N)$. Therefore X' is a small vertical 3-separator of $M\backslash e$, and it is easy to see that $\mathrm{int}_{M\backslash e}(X') = X'$, so we can take X_0 to be X'. Therefore we will assume that $x \in \mathrm{cl}_{M\backslash e}(X')$. This in turn implies that $X' \cap (A - x) \neq \varnothing$ for otherwise $x \in G_{M\backslash e}(A)$, and this contradicts 6.7.4.

It is a trivial exercise to verify that $A - x$ is a 3-separator of $M\backslash e/x$. Since $|A| \leq 5$ it is clear that $A - x$ is a small 3-separator of $M\backslash e/x$. Suppose that $A - x$ is properly contained in A', a small 3-separator of $M\backslash e/x$. Then $x \notin \mathrm{cl}_{M\backslash e}(E(M\backslash e) - A')$, so $A' \cup x$ is a 3-separator of $M\backslash e$. Since $A' \cup x$ properly contains A it cannot be a small 3-separator of $M\backslash e$. This easily leads to a contradiction, so $A - x$ is a maximal small 3-separator of $M\backslash e/x$.

Let $\overline{X}$ be a maximal small 3-separator of $M\backslash e/x$ that contains X. Suppose $\overline{X} = A - x$. Then $X \subseteq A - x$. Since $r_{M\backslash e/x}(X) \geq 3$ it follows that A has rank at least four in $M\backslash e$, so A is a cofan $(e_1, \ldots, e_5)$. By symmetry we can assume that $x = e_1$ and that $X \subseteq \{e_2, \ldots, e_5\}$. Now it is easy to check that X is a small vertical 3-separator of $M\backslash e$, so the first statement holds and we are done.

Thus we assume that $\overline{X}$ and $A - x$ are distinct maximal small 3-separators of $M\backslash e/x$. Since $X' \cap (A - x) \neq \varnothing$ it follows that $\overline{X}$ and $A - x$ are not disjoint.

Since the complements of $\overline{X}$ and $A - x$ in $M\backslash e/x$ each contain at least seven elements of $E(N)$, it is easy to see that they both have rank at least two in $M\backslash e/x$. Proposition 6.3 now tells us that $\overline{X}$ and $A - x$ meet in F, a rank-one flat of $M\backslash e/x$. Note that, as $X' \cap (A - x) \neq \varnothing$, it follows that X' contains at least one element of F, and in fact X' contains F, for $E(M\backslash e/x) - X'$ is a flat of $M\backslash e/x$ by definition.

Claim 6.7.6. $F \cap G_{M\backslash e/x}(A - x) = \varnothing$.

Proof. By inspection $A - x$ has rank at least two, and contains at least three rank-one flats in $M\backslash e/x$. Clearly the same statement applies to $\overline{X}$, as it is a vertical 3-separator of $M\backslash e/x$. Since F is contained in X' and the complement of X' is a flat in $M\backslash e/x$ we deduce that F is not contained in $\mathrm{cl}_{M\backslash e/x}(E(M\backslash e/x) - \overline{X})$, as $X' \subseteq \overline{X}$. Therefore statement (i) in Proposition 6.3 cannot hold. Hence statement (ii) holds, so $F \cap \mathrm{cl}_{M\backslash e/x}(A - (F \cup x)) = \varnothing$. Now, if F had a non-empty intersection with $\mathrm{cl}_{M\backslash e/x}(E(M\backslash e/x) - (A - x))$ then

$$(A - (F \cup x),\ (E(M\backslash e/x) - (A - x)) \cup F)$$

would be a vertical 2-separation of $M\backslash e/x$, contradicting 6.7.3. Therefore the claim holds. $\square$

We have assumed that $x \in \mathrm{cl}_{M\backslash e}(X')$. Let C be a circuit of $M\backslash e$ that is contained in $X' \cup x$ and that contains x. Since $x \notin G_{M\backslash e}(A)$ it follows that C contains an element f of $X' \cap (A - x) = F$. Since $x \in C$ and $F \cup x$ has rank two in $M\backslash e$ it follows that f is the only element of F in C. Now $C - x$ is a circuit of $M\backslash e/x$ which contains f, and $(C - x) - f$ is contained in $E(M\backslash e/x) - (A - x)$, so $f \in G_{M\backslash e/x}(A - x)$, contradicting Claim 6.7.6. This completes the proof of 6.7.5. $\square$

In the case that the second statement of 6.7.5 holds, $\mathrm{int}_{M\backslash e}(X_0)$ is a vertical 3-separator of $M\backslash e$ by Proposition 6.5 (iii), and is clearly a small vertical 3-separator. Therefore the next fact follows as a corollary of 6.7.5.

6.7.7. *Suppose that A is a maximal small vertical 3-separator of $M\backslash e$ and $|A| \geq 4$. Let $x \in A$ be a good element. If X is a small vertical 3-separator of $M\backslash e/x$ then X contains a small vertical 3-separator of $M\backslash e$.*

We have defined small 3-separators of $M\backslash e$, and in the case that A is a small vertical 3-separator $M\backslash e$ with $|A| \geq 4$ and $x \in A$ is a good element we have defined small 3-separators of $M\backslash e/x$. The fact that $M\backslash e/x$ is vertically 3-connected means that M/x is also vertically 3-connected. We next define small 3-separators of M/x. We do so in such a way that the definition is compatible with the definition for $M\backslash e/x$. Suppose that (X_1, X_2) is a partition of $E(M) - \{e, x\}$ such that either $(X_1 \cup e, X_2)$ or $(X_1, X_2 \cup e)$ is a vertical 3-separation of M/x. Since $M\backslash e/x$ is vertically 3-connected by 6.7.3 it follows that (X_1, X_2) is a vertical 3-separation of $M\backslash e/x$. Assume that X_i is a small 3-separator of $M\backslash e/x$, where $\{i, j\} = \{1, 2\}$. Then either $(X_i \cup e, X_j)$ or $(X_i, X_j \cup e)$ is a vertical 3-separation of M/x. In the first case we say that $X_i \cup e$ is a small 3-separator of M/x and in the second we say that X_i is. Thus if (X, Y) is a vertical 3-separation of M/x either X or Y is a small 3-separator. Note that we have not defined small 3-separators of M/x in full generality: we have only defined small vertical 3-separators.

6.7.8. *Suppose that A is a maximal small vertical 3-separator of $M\backslash e$ and $|A| \geq 4$. Let $x \in A$ be a good element. If X is a small vertical 3-separator of M/x then $e \in X$ and $e \in \mathrm{cl}_M((X \cup x) - e)$.*

PROOF. First suppose that $e \notin X$. Then $e \in \mathrm{cl}_{M/x}(Y - e)$, for otherwise $(X, Y - e)$ is a vertical 2-separation of $M\backslash e/x$. By definition X is a small 3-separator of $M\backslash e/x$, so it follows from 6.7.7 that X contains a small vertical 3-separator X' of $M\backslash e$. Let $Y' = E(M\backslash e) - X'$, so that (X', Y') is a vertical 3-separation of $M\backslash e$. Now $x \notin X'$ and $Y - e \subseteq Y'$, so $(Y - e) \cup x \subseteq Y'$. The fact that $e \in \mathrm{cl}_{M/x}(Y - e)$ implies that $e \in \mathrm{cl}_M(Y')$, which contradicts 6.7.2.

Therefore $e \in X$. Then $e \in \mathrm{cl}_{M/x}(X - e)$, for otherwise $(X - e, Y)$ is a vertical 2-separation of $M\backslash e/x$. But $e \in \mathrm{cl}_{M/x}(X - e)$ implies that $e \in \mathrm{cl}_M((X \cup x) - e)$, so we are done. □

We have assumed that $M\backslash e$ has at least one small vertical 3-separator. We next suppose that, in particular, $M\backslash e$ contains a cofan of length five.

6.7.9. *Suppose that $A = (e_1, \ldots, e_5)$ is a cofan of $M\backslash e$. Let X be a maximal small vertical 3-separator of M/e_1. Then $e \in X$, and $X - e = \{f_1, \ldots, f_5\}$, where $(f_1, \ldots, f_5)$ is a cofan of $M\backslash e$ such that*

(i) $f_1 = e_5$*;*
(ii) $(X - e) \cap A = \{f_1\}$*; and,*
(iii) $\{e, e_1, f_1, f_5\}$ *is a circuit of M.*

Proof. The fact that $e \in X$ follows from 6.7.8, as does the fact that $e \in \mathrm{cl}_{M/e_1}(X - e)$. Now $X - e$ is a vertical small 3-separator of $M\backslash e/e_1$ by definition, and since $e \in \mathrm{cl}_{M/e_1}(X - e)$ the next claim is easy to check.

Fact 1. $X - e$ is a maximal small 3-separator of $M\backslash e/e_1$.

Clearly $A - e_1$ is a fan of length four in $M\backslash e/e_1$, and hence a 3-separator. Obviously $A - e_1$ is a small 3-separator, and in fact the next result is easy to prove.

Fact 2. $A - e_1$ is a maximal small 3-separator of $M\backslash e/e_1$.

We know by 6.7.3 that $M\backslash e/e_1$ is vertically 3-connected. Suppose that it is not 3-connected. It follows $M\backslash e/e_1$ is not simple, and that therefore e_1 is contained in a triangle T of $M\backslash e$. Since $M\backslash e$ is binary T meets the triad $\{e_1, e_2, e_3\}$ in exactly two element. Let f be the element in $T - \{e_1, e_2, e_3\}$, so that $f \in \mathrm{cl}_{M\backslash e}(A)$. By inspection $f \notin A$, so $A \cup f$ is a 3-separator of $M\backslash e$ that properly contains A. This easily leads to a contradiction, so the next claim follows.

Fact 3. $M\backslash e/e_1$ is 3-connected.

It cannot be the case that $A - e_1 = X - e$, for, as $e \in \mathrm{cl}_M((X \cup e_1) - e)$, this would imply that $e \in \mathrm{cl}_M(A)$, in contradiction to 6.7.2. Thus $A - e_1$ and $X - e$ are distinct maximal small 3-separators of $M\backslash e/e_1$. Fact 3 and Corollary 6.4 imply that $A - e_1$ and $X - e$ meet in at most one element. Note that $T = \{e_3, e_4, e_5\}$ is a triad of $M\backslash e$. If $X - e$ is disjoint from T then 6.7.8 implies that $e \in \mathrm{cl}_M(E(M) - (T \cup e))$ and this implies that T is a triad of M, a contradiction. Thus $A - e_1$ and $X - e$ meet in exactly one element of $\{e_3, e_4, e_5\}$. Since none of these three elements is in $G_{M\backslash e/e_1}(A - e_1)$ we see that Corollary 6.4 also implies that the single element in $(A - e_1) \cap (X - e)$ is in $G^*_{M\backslash e/e_1}(A - e_1)$. This set contains only one element: e_5. Thus we have proved that $(X - e) \cap (A - e_1) = \{e_5\}$, and since $e_1 \notin X - e$ we have established the following fact.

Fact 4. $(X - e) \cap A = \{e_5\}$.

We next assume that $e_1 \in \mathrm{cl}_{M\backslash e}(X - e)$. Let C be a circuit of $M\backslash e$ contained in $(X - e) \cup e_1$ such that $e_1 \in C$. Fact 4 implies that C meets the triad $\{e_1, e_2, e_3\}$ in exactly one element, e_1, which is impossible. Thus $e_1 \notin \mathrm{cl}_{M\backslash e}(X - e)$. This implies that $X - e$ is a 3-separator of $M\backslash e$, and in fact the next statement is easy to confirm.

Fact 5. $X - e$ is a small vertical 3-separator of $M\backslash e$.

Since $e_5 \notin G_{M\backslash e}(A)$ we deduce that $e_5 \notin \mathrm{cl}_{M\backslash e}(X - \{e, e_5\})$. By examining the possible small vertical 3-separators of $M\backslash e$ we see that one of the following cases holds:

(i) $X - e$ is a triad of $M\backslash e$;
(ii) $X - e$ is a cofan $(f_1, \ldots, f_4)$ of $M\backslash e$, and $f_1 = e_5$; or,
(iii) $X - e$ is a cofan $(f_1, \ldots, f_5)$ of $M\backslash e$, and $f_1 = e_5$.

Let us assume that either (i) or (ii) applies. In either of these cases $X - e$ contains a triad T such that $e_5 \in T$. Moreover, $X - e \subseteq \mathrm{cl}_{M\backslash e}(T)$, so 6.7.8 implies that $e \in \mathrm{cl}_M(T \cup e_1)$. Let $C \subseteq T \cup \{e, e_1\}$ be a circuit of M that contains e.

If $e_5 \notin C$ then $e \in \mathrm{cl}_M(E(M) - \{e, e_3, e_4, e_5\})$, and this implies that $\{e_3, e_4, e_5\}$ is a triad (and hence a vertical 3-separator) of M. Similarly, if $e_1 \notin C$ then $T \cup e$ is a vertical 3-separator of M. Thus $\{e_1, e_5\} \subseteq C$. Now, since M has no triads, $T \cup e$ is a cocircuit of M, and therefore C meets $T \cup e$ in an even number of elements. We know that C contains at least two elements, e and e_5, of $T \cup e$, but it cannot be the case that $C = \{e, e_1, e_5\}$, for that would imply that $e \in \mathrm{cl}_M(A)$. Thus $C = T \cup \{e, e_1\}$. Fact 3 tells us that $M\backslash e/e_1$ is 3-connected. If M/e_1 were not 3-connected then e would have to be in a triangle of M with e_1. Such a triangle would necessarily meet the cocircuit $T \cup e$ in exactly one element, a contradiction. Thus M/e_1 is 3-connected. However, since $T \cup e$ is a cocircuit of M, and $T \cup \{e, e_1\}$ is a circuit, it follows that $T \cup e$ is a four-element circuit-cocircuit in M/e_1. Since M/e_1 has an N-minor this contradicts our hypotheses on M. Thus we have established the following fact.

Fact 6. $X - e$ is a cofan $(f_1, \ldots, f_5)$ of $M\backslash e$, and $f_1 = e_5$.

Let $C \subseteq X \cup e_1$ be a circuit of M that contains e. As before, we can argue that $\{e_1, e_5\} \subseteq C$, but that $C \neq \{e, e_1, e_5\}$. Let $T_1 = \{f_1, f_2, f_3\}$, and let $T_2 = \{f_3, f_4, f_5\}$. Since both $T_1 \cup e$ and $T_2 \cup e$ are cocircuits of M it follows that C meets both of these sets in an even number of elements. We know that C meets $T_1 \cup e$ in the two elements e and f_1. Suppose that C contains $T_1 \cup e$. Then C does not contain f_4, for then it would properly contain the triangle $\{f_2, f_3, f_4\}$. Nor can C contain f_5, for then C would meet $T_2 \cup e$ in an odd number of elements. Thus $C = T_1 \cup \{e, e_1\}$. But then we can again show that M/e_1 is 3-connected, and since $T_1 \cup e$ is a four-element circuit-cocircuit of M/e_1 we again have a contradiction to our hypotheses on M.

Therefore we conclude that $C \cap T_1 = \{f_1\}$. Since $C \neq \{e, e_1, e_5\}$ we see that C contains f_4 or f_5, and since C meets $T_2 \cup e$ in an even number of elements, C contains exactly one of f_4 or f_5. Suppose that $f_4 \in C$. Then $C = \{e, e_1, f_1, f_4\}$. By properties of circuits in binary matroids, the symmetric difference of C and $\{f_2, f_3, f_4\}$, which is $T_1 \cup \{e, e_1\}$, is a disjoint union of circuits. Since M has no circuits of size less than three this means that $T_1 \cup \{e, e_1\}$ is a circuit. But we have already shown that this leads to a contradiction to the hypotheses of the lemma. Thus we have proved the following fact, and hence have completed the proof of 6.7.9.

Fact 7. $\{e, e_1, f_1, f_5\}$ is a circuit of M. □

We have assumed that $M\backslash e$ contains a cofan $A_1 = (e_1, \ldots, e_5)$. We use this to prove that statement (ix) of Lemma 6.7 holds, and hence derive a contradiction. Since $M\backslash e/e_1$ is vertically 3-connected by 6.7.3, it follows that M/e_1 is also vertically 3-connected. Suppose that M/e_1 has no vertical 3-separations. In this case we will show that statement (iv) of Lemma 6.7 holds with $x = e_1$. By assumption M/e_1 is vertically 4-connected. Also M/e_1 has an N-minor by 6.7.4. The triangle T of statement (iv) is equal to $\{e_2, e_3, e_4\}$. Since $\{e_1, e_2, e_3\}$ is a triad of $M\backslash e$ it follows that $\{e, e_1, e_2, e_3\}$ is a cocircuit in M. Moreover, it is easy to see that e_1 is contained in no triangles in $M\backslash e$. Therefore any triangle of M that contains e_1 contains e, and hence e_1 is in at most one such triangle. Thus statement (iv) holds.

Therefore we assume that M/e_1 has at least one vertical 3-separation. Now 6.7.9 guarantees the existence of a cofan $A_2 = (f_1, \ldots, f_5)$ of $M\backslash e$ such that $f_1 = e_5$. Using an identical argument we see that M/f_1 has a vertical 3-separation, and by again applying 6.7.9 we see that there exists a cofan $A_3 = (g_1, \ldots, g_5)$ of $M\backslash e$ such that $g_1 = f_5$.

We know from 6.7.9 (ii) that $f_5 \notin A_1$, and therefore $g_1 \notin A_1$. Similarly $e_5 \notin A_3$. Now $G_{M\backslash e}(A_1) = G_{M\backslash e}(A_3) = \varnothing$, but $G^*_{M\backslash e}(A_1) = \{e_1, e_5\}$ and $G^*_{M\backslash e}(A_3) = \{g_1, g_5\}$. Since A_1 and A_3 are distinct maximal small 3-separators of $M\backslash e$, we can deduce from Corollary 6.4 that if A_1 and A_3 have a non-empty intersection, then they meet in the element $e_1 = g_5$.

Suppose that $A_1 \cap A_3 = \varnothing$. Now $T = \{e_1, e_2, e_3\}$ is a triad of $M\backslash e$. However 6.7.9 (iii) implies that e is in the closure of $\{f_1, g_1, g_5\}$ in M, and this set is contained in $E(M) - (T \cup e)$. Thus T is a triad of M. This contradiction shows that A_1 and A_3 have a non-empty intersection, and hence $e_1 = g_5$.

Let $M' = M/e_1/f_1/g_1$. We first show that M' has an N-minor. Since A_1 is a cofan of length five in $M\backslash e$ we see that $M\backslash e/e_1$ has an N-minor by Proposition 6.6. Now $A_1 - e_1$ is a cofan of length four in $M\backslash e/e_1$, and $e_5 = f_1$ is a good element of $A_1 - e_1$. Applying Proposition 6.6 again we conclude that $M\backslash e/e_1/f_1$ has an N-minor. Similarly, $A_2 - f_1$ is a cofan of length four in $M\backslash e/e_1/f_1$, and $f_5 = g_1$ is a good element, so $M\backslash e/e_1/f_1/g_1$, and hence M', has an N-minor.

Next we wish to show that M' is vertically 4-connected. Assume otherwise and let (X, Y) be a vertical k-separation of M', where $k < 4$. If T is a triangle of M' and $|T \cap X| \geq 2$ then we shall say that T is *almost contained* in X.

It is clear that $T_1 = \{e_2, e_3, e_4\}$, $T_2 = \{f_2, f_3, f_4\}$, and $T_3 = \{g_1, g_2, g_3\}$ are triangles of M'. By relabeling if necessary we can assume that both T_1 and T_2 are almost contained in X. Now it follows from Proposition 2.3 that $(X \cup T_1 \cup T_2, Y - (T_1 \cup T_2))$ is a vertical k'-separation of M', where $k' \leq k$. Therefore we may as well assume that X contains both T_1 and T_2.

Note that $\{e, e_1, f_1, g_1\}$ is a circuit of M by 6.7.9 (iii), so e is a loop in M'. Therefore we can also assume that $e \in X$. Now $\{e, e_1, e_2, e_3\}$, $\{e, f_1, f_2, f_3\}$, and $\{e, g_1, f_3, f_4\}$ are all cocircuits of M. If C is a circuit of M contained in $Y \cup \{e_1, f_1, g_1\}$ that has a non-empty intersection with $\{e_1, f_1, g_1\}$ then C meets at least one of these cocircuits in a single element, which is impossible. Therefore no such circuit can exist, and it follows that

$$r_M(Y \cup \{e_1, f_1, g_1\}) = r_M(Y) + 3.$$

Hence $r_M(Y) = r_{M'}(Y)$, and it now follows that $(X \cup \{e_1, f_1, g_1\}, Y)$ is a vertical k-separation of M for some $k < 4$. This is a contradiction because M is vertically 4-connected. Thus M' is vertically 4-connected.

Next we show that M' contains exactly one loop, e, and no parallel pairs. We have already noted that e is a loop in M'. Suppose that $e' \neq e$ is also a loop. Then there is a circuit $C \subseteq \{e', e_1, f_1, g_1\}$ of M such that $e' \in C$. There are no loops in M, so let us assume that C contains e_1 (the other cases are identical). Since neither e_2 nor e_3 is a loop in M' it follows that C is a circuit in $M \backslash e$ that meets the triad $\{e_1, e_2, e_3\}$ in exactly one element, a contradiction.

Now assume that $\{e', f'\}$ is a parallel pair in M'. There is a circuit C of M such that $C \subseteq \{e', f', e_1, f_1, g_1\}$ and $e', f' \in C$. As before, we will assume that $e_1 \in C$. As $\{e_1, e_2, e_3\}$ and $\{e_1, g_3, g_4\}$ are triads in $M \backslash e$ and C is a circuit of $M \backslash e$ it follows that (relabeling if necessary) $e' \in \{e_2, e_3\}$ and $f' \in \{g_3, g_4\}$. Suppose that $C = \{e', f', e_1\}$. Now, since $\{e_3, e_4, e_5\}$ and $\{g_1, g_2, g_3\}$ are also triads of $M \backslash e$, it follows that $e' = e_2$ and $f' = g_4$. This implies that $(g_2, g_3, g_4, e_1, e_2, e_3)$ is a fan of $M \backslash e$. As $M' \backslash e$ has an N-minor and $|E(N)| \geq 10$ it follows that the complement of $\{g_2, g_3, g_4, e_1, e_2, e_3\}$ in $M \backslash e$ contains at least six elements. Moreover $\{g_2, g_3, g_4, e_1, e_2, e_3\}$ is a 3-separator of $M \backslash e$ and it provides a contradiction to the fact that $M \backslash e$ is (4, 5)-connected. Therefore we assume that $|C| > 3$, so that $f_1 \in C$ (the case when $g_1 \in C$ is identical). Now C meets the triad $\{f_1, f_2, f_3\}$ in exactly one element. This contradiction shows that M' has no parallel pairs.

We have shown that M' is vertically 4-connected with an N-minor, and that M' has exactly one loop and no parallel pairs. Therefore statement (ix) holds with $x = e$, $x_i = e_i$, $y_i = f_i$, and $z_i = g_i$ for $1 \leq i \leq 5$. Thus our assumption that $M \backslash e$ contains a cofan of length five has lead to a contradiction. Hence we have established the following claim.

6.7.10. *There are no cofans of length five in* $M\backslash e$.

However, $M\backslash e$ does have at least one small vertical 3-separator with at least four elements. It is a consequence of 6.7.10 that any small vertical 3-separator of $M\backslash e$ has rank three.

6.7.11. *Let* A *be a small vertical* 3*-separator of* $M\backslash e$ *such that* $|A| \geq 4$, *and let* $x \in A$ *be a good element. Then* M/x *is not vertically* 4*-connected.*

PROOF. Suppose that M/x is vertically 4-connected. We will show that statement (iv) of Lemma 6.7 holds. We know that M/x has an N-minor by 6.7.4. Since A is a cofan of length four or a fan of length five in $M\backslash e$ we see that A contains a triad T_A and a triangle T of $M\backslash e$ such that $x \in T_A$, $x \notin T$, and $T \cap T_A = T_A - x$. Now T is a triangle of M/x, and since M has no triads $T_A \cup e$ is a cocircuit of M that meets T in two elements.

Next we show that x can be in at most two triangles of M. Any triangle of $M\backslash e$ that contains x contains a member of $T_A - x$, so x can be in at most two triangles in $M\backslash e$. If x is in two triangles then A is contained in a 3-separator $\overline{A}$ of $M\backslash e$ such that $|\overline{A}| = 6$. Since $M\backslash e$ is (4, 5)-connected this means that $\overline{A}$ is not a small 3-separator, so it contains at least seven elements of $E(N)$, an impossibility. Furthermore, x can be contained in at most one triangle of M that contains e, and any such triangle cannot contain an element of T, for otherwise $e \in \mathrm{cl}_M(T_A)$, contradicting 6.7.2. Thus x is in at most two triangles in M, and if x is in two triangles, then it is contained in exactly one triangle that contains e, and exactly one triangle that meets $T_A - x$. Therefore statement (iv) holds with $C^* = T_A \cup e$. $\square$

6.7.12. *Let* A *be a maximal small vertical* 3*-separator of* $M\backslash e$ *and suppose that* $|A| \geq 4$. *Let* $x \in A$ *be a good element. Suppose that* X *is a small vertical* 3*-separator of* M/x. *Then* $e \in X$, *and there is a triad* T *of* $M\backslash e$ *such that the following statements hold.*

(i) $T \subseteq X - e$.
(ii) $X - e \subseteq \mathrm{cl}_{M\backslash e}(T \cup x)$.
(iii) There is an element $y \in T$ *such that* $\{e, x, y\}$ *is a triangle of* M.

PROOF. By 6.7.3 we see that $M\backslash e/x$ is vertically 3-connected. This implies that M/x is also vertically 3-connected. We know that $e \in X$ by 6.7.8. Moreover, $X - e$ is a small vertical 3-separator of $M\backslash e/x$ by definition. It cannot be the case that $X - e \subseteq A - x$, for then A would have rank four in $M\backslash e$. Thus, by 6.7.5, there is some vertical 3-separator X_0 of $M\backslash e$ such that $\mathrm{int}_{M\backslash e/x}(X - e) = \mathrm{int}_{M\backslash e}(X_0)$. Since X_0 is a small 3-separator of $M\backslash e$ we deduce from 6.7.10 that $r_{M\backslash e}(X_0) = 3$ and X_0 contains a triad T of $M\backslash e$ such that $X_0 \subseteq \mathrm{cl}_{M\backslash e}(T)$. It is elementary to demonstrate that T is contained in $\mathrm{int}_{M\backslash e}(X_0)$, so T is contained in $X - e$.

Let $X' = \mathrm{int}_{M\backslash e/x}(X - e)$. Note that $X' \subseteq X_0$, so $X' \subseteq \mathrm{cl}_{M\backslash e}(T)$. Proposition 6.5 (ii) tells us that $X - e \subseteq \mathrm{cl}_{M\backslash e/x}(X')$, so $X - e \subseteq \mathrm{cl}_{M\backslash e}(X' \cup x)$. This combined with the fact that $X' \subseteq \mathrm{cl}_{M\backslash e}(T)$ shows that $X - e \subseteq \mathrm{cl}_{M\backslash e}(T \cup x)$.

It remains to show that there is an element $y \in T$ such that $\{e, x, y\}$ is a triangle of M. From 6.7.8 we see that $e \in \mathrm{cl}_M((X \cup x) - e)$. Since $X - e \subseteq \mathrm{cl}_{M\backslash e}(T \cup x)$ it follows that $e \in \mathrm{cl}_M(T \cup x)$. Let $C \subseteq T \cup \{e, x\}$ be a circuit of M that contains e. Then $x \in C$, for otherwise $T \cup e$ is a vertical 3-separator of M. Since $T \cup e$ is a cocircuit of M it follows that C meets $T \cup e$ in either two or four elements. Suppose that the latter case holds, so that $C = T \cup \{e, x\}$. This means that $T \cup e$ is a four-element circuit-cocircuit in M/x. Any triangle of M that contains x cannot contain an element of $T \cup e$, for if it did it would meet the cocircuit $T \cup e$ in exactly one element. Thus any parallel pair in M/x contains no element of $T \cup e$. It follows that $T \cup e$ is a four-element circuit-cocircuit in $\mathrm{si}(M/x)$. Moreover, $\mathrm{si}(M/x)$ is 3-connected since M/x is vertically 3-connected. We know that $M\backslash e/x$ has an N-minor by 6.7.4, so M/x, and hence $\mathrm{si}(M/x)$, has an N-minor. Thus $\mathrm{si}(M/x)$ provides a contradiction to the hypotheses of the lemma.

Therefore C does not meet $T \cup e$ in four elements, and hence there is an element $y \in T$ such that $\{e, x, y\}$ is a circuit of M. □

Assume that A is a maximal small vertical 3-separator of $M\backslash e$ and that $|A| \geq 4$. Let $x \in A$ be a good element. By inspection we see that A contains a triad T_A of $M\backslash e$ such that $x \in T_A$ and $A \subseteq \mathrm{cl}_{M\backslash e}(T_A)$. We know from 6.7.11 that M/x is not vertically 4-connected, so let X be a small vertical 3-separator of M/x. Let $\mathcal{T}_X$ be the set of triads of $M\backslash e$ satisfying 6.7.12. Let $\mathcal{T}$ be the union of the collections $\mathcal{T}_X$, taken over all small vertical 3-separators of M/x.

6.7.13. *If $T_B \in \mathcal{T}$ then $T_A \cap T_B = \varnothing$.*

Proof. Let T_B be a member of $\mathcal{T}$, and let X be a small vertical 3-separator of M/x such that $T_B \in \mathcal{T}_X$. Then T_B is contained in $\overline{T_B}$, a maximal small vertical 3-separator of $M\backslash e$. Clearly A and $\overline{T_B}$ are distinct maximal small 3-separators of $M\backslash e$. Since A is either a cofan of length four or a fan of length five in $M\backslash e$ it is easy to see that $G^*_M(A)$ is either empty or equal to $\{x\}$. Suppose that $\overline{T_B}$ contains x. Then as $T_B \subseteq \overline{T_B}$, and both T_B and $\overline{T_B}$ have rank three in $M\backslash e$ it follows that $r_{M\backslash e}(T_B \cup x) = 3$. Now $X - e \subseteq \mathrm{cl}_{M\backslash e}(T_B \cup x)$ by 6.7.12 (ii) and $e \in \mathrm{cl}_M((X \cup x) - e)$ by 6.7.8, so $X \subseteq \mathrm{cl}_M(T_B \cup x)$. Thus $r_M(X) \leq r_M(T_B \cup x) = 3$, so X has rank two in M/x. This is a contradiction, so $\overline{T_B}$ does not contain x and therefore $\overline{T_B}$ cannot meet $G^*_M(A)$. Since T_A has an empty intersection with $G_M(A)$ it follows from Corollary 6.4 that T_B cannot contain any element of T_A. □

The remainder of the proof is divided into two cases, each of which is then divided into various subcases and sub-subcases.

Case 1. There is a triad $T_B \in \mathcal{T}$ such that $T_A \cup T_B$ contains a circuit C of $M\backslash e$.

Let X be the small vertical 3-separator of M/x such that $T_B \subseteq X - e$ and $X - e \subseteq \mathrm{cl}_{M\backslash e}(T_B \cup x)$.

We need one more technical result before we proceed.

6.7.14. *There is at most one triangle of M contained in $T_A \cup T_B \cup e$ that is not also contained in $C \cup e$.*

PROOF. First note that any triangle of M contained in $T_A \cup T_B \cup e$ contains e, for otherwise it meets one of the cocircuits $T_A \cup e$ and $T_B \cup e$ in exactly one element. Similarly, any such triangle contains exactly one element from T_A and one element from T_B, for otherwise it meets either $T_A \cup e$ or $T_B \cup e$ in three elements.

Now suppose that $\{e, x, y\}$ and $\{e, x' y'\}$ are two distinct triangles of M such that $x, x' \in T_A$ and $y, y' \in T_B$, but that neither $\{e, x, y\}$ nor $\{e, x' y'\}$ is contained in $C \cup e$. Note that $x \neq x'$, for if x were equal to x' then y and y' are distinct elements (for $\{e, x, y\}$ and $\{e, x' y'\}$ are distinct triangles) and $\{y, y'\}$ is a parallel pair. But M is simple. Thus $x \neq x'$, and similarly $y \neq y'$. Thus $\{x, x', y, y'\}$ is a circuit of M.

Since T_A and T_B are disjoint triads of $M\backslash e$ by 6.7.13 we see that $|T_A \cap C| = |T_B \cap C| = 2$. Therefore we can assume by relabeling that $x, y' \in C$. Assume that $x' \in C$. Then the symmetric difference of C and $\{x, x', y, y'\}$ contains two elements, so M contains a circuit of size at most two, a contradiction. Thus $x' \notin C$, and by a symmetrical argument $y \notin C$.

Now it is easy to see that $T_A \cup e$ spans $T_A \cup T_B \cup e$ in M, so $r_M(T_A \cup T_B \cup e) = 4$. Also, both $T_A \cup e$ and $T_B \cup e$ are cocircuits in M, so $T_A \cup T_B$ is a disjoint union of cocircuits in M. Since M has no cocircuits of size less than four this means that $T_A \cup T_B$ is a cocircuit. Hence $r^*_M(T_A \cup T_B \cup e) = 5$.

For any subset $S \subseteq E(M)$ the equation $r^*_M(S) = |S| - r(M) + r_M(E(M) - S)$ holds. From this we deduce that

$$\lambda_M(S) = r_M(S) + r_M(E(M) - S) - r(M) = r_M(S) + r^*_M(S) - |S|. \tag{6.1}$$

Equation (6.1) implies that $\lambda_M(T_A \cup T_B \cup e) = 2$. Since M has no vertical 3-separations this means that the complement of $T_A \cup T_B \cup e$ in M has rank at most two. Therefore the complement of $T_A \cup T_B \cup e$ contains at most three elements, which means that $|E(M)| \leq 10$, a contradiction. $\square$

Now we proceed to analyze the subcases.

SUBCASE 1.1. $x \notin C$.

We will show that statement (vii) of Lemma 6.7 holds.

Let T_x be the triangle supplied by 6.7.12 (iii), so that $e, x \in T_x$, and T_x contains an element of T_B.

Since $|T_A \cap C| = 2$ it follows that $C \cap T_A = T_A - x$. Since x is a good element of A, and A is either a cofan of length four or a fan of length five in $M\backslash e$ there is a triangle T of $M\backslash e$ such that $T \subseteq A$ and $x \notin A$. Moreover, $T \cap T_A = C \cap T_A = T_A - x$. Let f be the element in $T - T_A$.

Since $(T_B \cap C) \cup f$ is the symmetric difference of the circuits C and T it is a triangle. Therefore $f \in \mathrm{cl}_{M\backslash e}(T_B)$, and it follows that $T_B \cup f$ is a small vertical 3-separator of $M\backslash e$ with four elements. Let $\overline{T_B}$ be a maximal small

3-separator of $M\backslash e$ that contains $T_B \cup f$. The single element in $T_B - C$ is a good element of $\overline{T_B}$. Let us call this element y.

By 6.7.11 we see that M/y is not vertically 4-connected, so let X' be a small vertical 3-separator of M/y. By applying 6.7.12 to $\overline{T_B}$ and X' we see that there is a triangle T_y of M such that $e, y \in T_y$ and T_y meets X' in the single element x'.

If $x' \notin T_A$ then e is in $\mathrm{cl}_M(E(M) - (T_A \cup e))$, implying that T_A is a triad of M. Thus $x' \in T_A$, so $T_y \subseteq T_A \cup T_B \cup e$. We already know that $T_x \subseteq T_A \cup T_B \cup e$. Since neither x nor y is contained in C, neither T_x nor T_y is contained in $C \cup e$. Now 6.7.14 implies that T_x and T_y are equal, so that $\{e, x, y\}$ is a triangle.

We know that $M\backslash e/x$ has an N-minor by 6.7.4. Moreover, $T_B \cup f$ is a cofan of length four in $M\backslash e/x$, and y is a good element of this cofan so $M\backslash e/x/y$ has an N-minor by Proposition 6.6.

Next we show that $M/x/y$ is vertically 4-connected. Suppose otherwise and let (X', Y') be a vertical k-separation of $M/x/y$ where $k < 4$. By relabeling if necessary we can assume that at least two elements of the triangle $(T_B \cap C) \cup f$ are contained in X'. Then Proposition 2.3 implies that

$$(X' \cup (T_B \cap C) \cup f,\, Y' - ((T_B \cap C) \cup f))$$

is a vertical k'-separation of $M/x/y$ where $k' \leq k$, so we may as well assume that $(T_B \cap C) \cup f$ is contained in X'. Since e is a loop in $M/x/y$ we can assume that it is in X'. Now $T_B \cup e$ is a cocircuit of M/x, and since $(T_B \cup e) - y$ is contained in X' this means that $y \notin \mathrm{cl}_{M/x}(Y')$. Therefore $(X' \cup y, Y')$ is a vertical k-separation of M/x, and k is three. If Y' is a small vertical 3-separator of M/x, then by 6.7.12 there is an element $y' \in Y'$ such that $\{e, x, y'\}$ is a triangle of M. Since y' cannot be equal to y this means that y' and y are parallel in M, a contradiction. Thus $X' \cup y$ is a small vertical 3-separator of M/x. But it follows from 6.7.12 (ii) and 6.7.8 that $X' \cup x \cup y$ is spanned in M by the union of x with a triad of $M\backslash e$. Therefore $X' \cup x \cup y$ has rank at most four in M and X' has rank at most two in $M/x/y$, a contradiction. Therefore $M/x/y$ is vertically 4-connected.

Finally we show that $M/x/y$ has exactly one loop and no parallel pairs. We have noted that e is a loop in $M/x/y$. If $M/x/y$ has more than one loop then M has a circuit of size at most two, a contradiction.

Suppose that $\{e', f'\}$ is a parallel pair in $M/x/y$. Note that neither e' nor f' is equal to e. Then there is a circuit $C' \subseteq \{e', f', x, y\}$ of M such that $e', f' \in C'$. Assume that $|C'| = 4$. By relabeling if necessary we can assume that $e' \in T_A - x$ and $f' \in T_B - y$, for otherwise C' meets one of the cocircuits $T_A \cup e$ or $T_B \cup e$ in exactly one element. Now we see that $T_A \cup e$ spans $T_A \cup T_B \cup e$ in M, so $r_M(T_A \cup T_B \cup e) = 4$, and since $T_A \cup T_B$ is a cocircuit we deduce that $r^*_M(T_A \cup T_B \cup e) = 5$. From this we can obtain a contradiction, as in the proof of 6.7.14.

Therefore we will assume that $|C'| = 3$. We will assume that $x \in C'$, since the case when $y \in C'$ is identical. Since C' meets $T_A \cup e$ in two elements we will assume that $e' \in T_A$ while $f' \notin T_A$. Now $T_A \cup \{f, f'\}$ is a fan of length five in $M\backslash e$ and since A is a maximal small 3-separator of $M\backslash e$ this means that $A = T_A \cup \{f, f'\}$. Thus A contains a good element distinct from x. Let this element be x'. By 6.7.11 we see that M/x' contains a small vertical 3-separator, and 6.7.12 implies the existence of a triangle $T_{x'}$ that contains e and x'. Then $T_{x'}$ contains an element y' of T_B, for otherwise T_B is a triad of M. Note that y' is not equal to y, for if it were then x and x' would be parallel in M. Thus $\{x, y, x', y'\}$ is a circuit of M. From this we again conclude that $r_M(T_A \cup T_B \cup e) = 4$ while $r^*_M(T_A \cup T_B \cup e) = 5$ and hence derive a contradiction.

Now if we let $z = e$, $T_1 = T$, and $T_2 = (T_B - y) \cup f$ we see that statement (vii) of the lemma holds. Our assumption that $x \notin C$ has lead us to a contradiction, so we consider the next subcase.

Subcase 1.2. $x \in C$.

We have already noted that C contains exactly two elements of T_A and two elements of T_B. Let c_0 be the single element other than x contained in $T_A \cap C$. Let c_1 and c_2 be the two elements in $T_B \cap C$, and let y be the single element in $T_B - \{c_1, c_2\}$.

Now $\{c_0, c_1, c_2\}$ is a triangle of $M\backslash e/x$, and it intersects the triad T_B in two elements. Thus (c_0, c_1, c_2, y) is a four-element fan in $M\backslash e/x$ and y is a good element of this fan. We know that $M\backslash e/x$ has an N-minor by 6.7.4, so it follows from Proposition 6.6 that $M\backslash e/x/y$, and hence M/y, has an N-minor.

By 6.7.12 there is a triangle of M that contains e and x and an element of T_B.

Sub-subcase 1.2.1. $\{e, x, y\}$ is not a triangle of M.

By relabeling if necessary we can assume that $\{e, x, c_1\}$ is a triangle. We will show that statement (v) of the lemma holds.

The set $\{e, c_0, c_2\}$ is the symmetric difference of the circuits C and $\{e, x, c_1\}$, and therefore is a triangle of M.

Suppose that M/y is not vertically 4-connected, and let (X', Y') be a vertical k-separation of M/y, where $k < 4$. There is a triangle T contained in A that contains two elements of T_A, but does not contain x. In particular, T contains c_0. By relabeling if necessary we can assume that at least two elements of T are contained in X'. Then by Proposition 2.3 we can assume that T is contained in X'.

Suppose that one of e or c_2 is contained in $\mathrm{cl}_{M/y}(X')$. Then the entire triangle $\{e, c_0, c_2\}$ is contained in $\mathrm{cl}_{M/y}(X')$, and we can again apply Proposition 2.3 and assume that X' contains $\{e, c_0, c_2\}$. In this case X' contains three elements of the cocircuit $T_A \cup e$, so in particular $x \in \mathrm{cl}^*_M(X')$. Suppose that $(X' \cup x, Y' - x)$ is not a vertical k'-separation of M/y for some $k' \leq k$. As M/y is vertically 3-connected and (X', Y') is a vertical k-separation of M/y for some $k < 4$ it follows that $r_{M/y}(Y') = 3$ and $r_{M/y}(Y' - x) = 2$.

But it is easy to see that this implies that Y' contains a triad of M/y, and hence of M. This is a contradiction, so we can now assume that (X', Y') is a vertical k-separation of M/y such that X' contains $T \cup T_A \cup \{c_2, e\}$. Then X' contains two elements of the triangle $\{e, x, c_1\}$, so we assume that X' contains the entire triangle. This means that X' contains $(T_B - y) \cup e$. Since $T_B \cup e$ is a cocircuit of M we conclude that $y \notin \mathrm{cl}_M(Y')$. This means that $(X' \cup y, Y')$ is a vertical k-separation of M, a contradiction.

Therefore we assume that neither e nor c_2 is contained in $\mathrm{cl}_{M/y}(X')$, and in particular $e, c_2 \in Y'$. If c_1 were in Y', then Y' would contain $(T_B - y) \cup e$, which would imply that $y \notin \mathrm{cl}_M(X')$ and that $(X', Y' \cup y)$ is a vertical k-separation of M. Thus $c_1 \in X'$. This implies that $x \in Y'$, for $\{e, x, c_1\}$ is a triangle in M/y, and we have concluded that $e \notin \mathrm{cl}_{M/y}(X')$. But now, since both x and e are in Y', we see that c_1 is in $\mathrm{cl}_{M/y}(Y')$, and therefore $(X' - c_1, Y' \cup c_1)$ is a vertical k'-separation for some $k' \leq k$. Since $Y' \cup c_1$ contains $(T_B - y) \cup e$ we easily derive a contradiction.

Therefore M/y is vertically 4-connected with an N-minor. The set $T_B \cup e$ is a cocircuit of M, and any triangle of M that contains y contains exactly two members of this cocircuit. Suppose that there is a triangle T of M that contains both y and e. Since $T_A \cup e$ is also a cocircuit of M, it follows that the single element in $T - \{e, y\}$ is contained in T_A. Now we see that $T_A \cup e$ spans $T_A \cup T_B \cup e$ in M, so as before we deduce that $r_M(T_A \cup T_B \cup e) = 4$ and $r^*_M(T_A \cup T_B \cup e) = 5$. This again leads to a contradiction.

It follows from these observations that y is in at most two triangles in M and hence M/y has at most two parallel pairs. Clearly M/y has no loops, since M has no parallel pairs. Now, by relabeling y with x we see that statement (v) holds with $C^* = T_B \cup e$, $T_1 = \{e, c_0, c_2\}$, and $T_2 = \{e, x, c_1\}$. The assumption that $\{e, x, y\}$ is not a triangle of M has lead to a contradiction, so we are forced into the next subcase.

SUB-SUBCASE 1.2.2. $\{e, x, y\}$ is a triangle of M.

We will show in this sub-subcase that statement (viii) of the lemma holds.

We noted at the beginning of Subcase 1.2 that $M\backslash e/x/y$ has an N-minor. Suppose that $M/x/y$ is not vertically 4-connected and that (X', Y') is a vertical k-separation of $M/x/y$, where $k < 4$. The set $\{c_0, c_1, c_2\}$ is a triangle of $M/x/y$. By relabeling if necessary we assume that it is contained in X'. Since e is a loop in $M/x/y$ we assume that it too is contained in X'. Then X' contains three elements of the cocircuit $T_B \cup e$ of M/x, and it follows that $y \notin \mathrm{cl}_{M/x}(Y')$. Thus $(X' \cup y, Y')$ is a vertical k-separation of M/x and $k = 3$ since M/x is vertically 3-connected by 6.7.4. If Y' is a small vertical 3-separator of M/x then 6.7.12 implies that there is a triangle of M that contains e, x, and an element of Y'. Since $y \notin Y'$ this implies that y is parallel to an element of Y' in M, a contradiction. Therefore $X' \cup y$ is a small vertical 3-separator of M/x. But then 6.7.12 tells us that $X' \cup x \cup y$ has rank at most four in M and hence X' has rank at most two in $M/x/y$, contradicting the fact that (X', Y') is a vertical 3-separation of $M/x/y$. Thus $M/x/y$ is vertically 4-connected.

Next we show that $M/x/y$ has exactly one loop and no parallel pairs. Clearly e is a loop of $M/x/y$, and the presence of any other loop implies a circuit of a size at most two in M. Suppose that $\{e', f'\}$ is a parallel pair in $M/x/y$. Let $C' \subseteq \{e', f', x, y\}$ be a circuit of M that contains e' and f'. Suppose that $|C'| = 4$. We can assume that $e' \in T_A$ and that $f' \in T_B$, for otherwise C' intersects $T_A \cup e$ or $T_B \cup e$ in a single element. Now we see that $T_A \cup e$ spans $T_A \cup T_B \cup e$ in M. As before we can obtain a contradiction.

Therefore we assume that $|C'| = 3$. Let us suppose that $C' = \{e', f', x\}$. We can assume that $e' \in T_A$ while $f' \notin T_A$. Then $A \cup f'$ contains a five-element fan of $M\backslash e$, so in fact f' is contained in A, since A is a maximal small 3-separator of $M\backslash e$. Now A contains a good element $x' \neq x$. Then 6.7.11 implies that M/x' contains a small vertical 3-separator, and 6.7.12 tells us that there is a triangle $T_{x'}$ of M that contains e and x'. There must be an element y' of T_B in $T_{x'}$, and this element cannot be equal to y. Thus $\{x, y, x', y'\}$ is a circuit of M and we can again obtain a contradiction.

If $C' = \{e', f', y\}$ then we assume that $e' \in T_B$ while $f' \notin T_B$. Then $T_B \cup f'$ is a four-element cofan in $M\backslash e$, and it contains a good element y' not equal to y. Then 6.7.11 and 6.7.12 imply the existence of a triangle $T_{y'}$ containing e and y', and $T_{y'}$ contains an element $x' \in T_A$ not equal to x, so $\{x, y, x', y'\}$ is a circuit in M. If $x' \notin C$ then we can obtain a contradiction as before. If x' is in C then the symmetric difference of C and $\{x, y, x', y'\}$ contains at most two elements, leading to a contradiction.

Thus statement (viii) of the lemma holds with $z = e$, T_1 equal to the triangle in A that meets T_A in $T_A - x$, and $T_2 = \{c_0, c_1, c_2\}$.

This contradiction means that we must now consider Case 2.

Case 2. There is no triad $T_B \in \mathcal{T}$ such that $T_A \cup T_B$ contains a circuit of $M\backslash e$.

Subcase 2.1. No small vertical 3-separator of M/x contains more than four elements.

Our aim in this subcase is to show that statement (iii) of Lemma 6.7 holds. We know that M/x has an N-minor by 6.7.4. Next we show that $\mathrm{si}(M/x)$ is internally 4-connected. Suppose otherwise and let (X, Y) be a 3-separation of $\mathrm{si}(M/x)$ such that $|X|, |Y| \geq 4$. Then both X and Y have rank at least three in $\mathrm{si}(M/x)$. The separation (X, Y) naturally induces a vertical 3-separation of M/x. Let us call this vertical 3-separation (X', Y'). By relabeling assume that X' is a small vertical 3-separator of M/x. Then by 6.7.12 there is an element $y \in X'$ such that $\{e, x, y\}$ is a triangle of M. Therefore X' contains a parallel pair in M/x. But because $|X'| \leq 4$ by assumption, and X is contained in X' this means that $|X| \leq 3$, contrary to hypothesis. Thus $\mathrm{si}(M/x)$ is internally 4-connected.

Next we show that M/x has no loops and exactly one parallel pair. Clearly M/x cannot have a loop, since M has no parallel pairs. We know from 6.7.11 that M/x has a small vertical 3-separator X. Let T_B be the triad of $M\backslash e$ contained in $X - e$ that 6.7.12 supplies. Then there is an element $y \in T_B$ such that $\{e, x, y\}$ is a triangle of M. Thus M/x has at least one parallel pair. Suppose that M/x has another parallel pair. Obviously x can be contained in only one triangle of M that also contains e. It follows that x is contained in a triangle of $M\backslash e$. Therefore that A is a five-element fan, so A contains a good element $x' \neq x$.

Now $\{e, x'\}$ is contained in a triangle $T_{x'}$ of M by 6.7.11 and 6.7.12. Moreover, the other element in $T_{x'}$ is an element y' of T_B, for otherwise T_B is a triad of M. Also, y' is not equal to y, for that would imply that x and x' are parallel in M. Thus $\{x, y, x', y'\}$ is a circuit of $M\backslash e$ contained in $T_A \cup T_B$, contrary to our assumption. Thus M/x has exactly one parallel pair.

It remains to show that $\mathrm{si}(M/x)$ has at least one triangle and at least one triad. There is a triangle of A that does not contain x, and this is also a triangle of $\mathrm{si}(M/x)$. We know that M/x has a small vertical 3-separator X and that X contains a triad T_B of $M\backslash e$. By the above discussion $\mathrm{si}(M/x)$ is isomorphic to $M/x\backslash e$, and T_B is a triad of $M/x\backslash e$.

Therefore statement (iii) of the lemma holds, so we are forced to consider the final subcase.

Subcase 2.2. There is a small vertical 3-separator X of M/x such that $|X| > 4$.

We will show that statement (vi) holds. Let T_B be the triad of $M\backslash e$ supplied by 6.7.12, so that $T_B \in \mathcal{T}_X$. We know from 6.7.12 that $X - e \subseteq \mathrm{cl}_{M\backslash e}(T_B \cup x)$. Assume that $f \in X - e$ is not contained in $\mathrm{cl}_{M\backslash e}(T_B)$. Then there is a circuit $C \subseteq (T_B \cup x) \cup f$ that contains f and x. Since C meets T_A in two elements we deduce from 6.7.13 that $f \in T_A$. But now C is contained in $T_A \cup T_B$, contrary to our assumption.

Therefore $X - e \subseteq \mathrm{cl}_{M\backslash e}(T_B)$, and since $|X - e| \geq 4$, it follows that $X - e$ is a fan or cofan with length four or five in $M\backslash e$. Let $y \in X - e$ be a good element. Let $y' \in T_B$ be the element such that $\{e, x, y'\}$ is a triangle of M, and suppose that $y' \neq y$.

By 6.7.11 and 6.7.12 there is a triangle T_y of M such that $e, y \in T_y$. There exists an element $x' \in T_y \cap T_A$, for otherwise T_A is a triad of M. It cannot be the case that $x' = x$ for that would imply that y and y' are parallel in M. Thus $\{x, y, x', y'\}$ is a circuit of $M\backslash e$ contained in $T_A \cup T_B$, contrary to our assumption.

It follows that $y = y'$, so $\{e, x, y\}$ is a triangle of M. As before, we can show that $M/x/y$ has an N-minor. Next we show that $M/x/y$ is vertically 4-connected. Suppose that (X', Y') is a vertical k-separation of $M/x/y$ where $k < 4$. There is an element $f \in X - e$ such that $(T_B - y) \cup f$ is a triangle of $M\backslash e$ and of $M/x/y$. We assume that $(T_B - y) \cup f$ is contained in X'. We can also assume that $e \in X'$ as it is a loop of $M/x/y$. Since $T_B \cup e$ is a cocircuit in M/x and $(T_B - y) \cup e$ is contained in X' it follows that

$y \notin \mathrm{cl}_{M/x}(Y')$ so $(X' \cup y, Y')$ is a vertical 3-separation of M/x, and hence $k = 3$ by 6.7.4. Assuming that Y' is a small vertical 3-separator of M/x implies that there is a triangle of M containing e, x, and an element of Y', and hence y is in parallel with an element of Y' in M. Therefore $X' \cup y$ is a small vertical 3-separator of M/x. But then $X' \cup x \cup y$ has rank at most four in M and X' has rank at most two in $M/x/y$, contradicting the fact that (X', Y') is a vertical 3-separation. Thus $M/x/y$ is vertically 4-connected.

It is easy to see that $M/x/y$ has exactly one loop. Suppose that $\{e', f'\}$ is a parallel pair in $M/x/y$. There is a circuit $C' \subseteq \{e', f', x, y\}$ that contains both e' and f'. If $|C'| = 4$ then $C' \subseteq T_A \cup T_B$, contrary to assumption. If $C' = \{e', f', x\}$, then A is a five-element fan, so there is a good element $x' \neq x$ in A. Then x' is contained in a triangle $T_{x'}$ of M along with e and an element of T_B. This again implies the existence of a circuit contained in $T_A \cup T_B$. We can obtain a similar contradiction if $C' = \{e', f', y\}$. This shows that there are no parallel pairs in $M/x/y$.

Let T_1 be the triangle contained in A that avoids x, and T_2 be the triangle contained in X that avoids y. Suppose that T_1 and T_2 have a non-empty intersection. Then they meet in an element in neither T_A nor T_B, for otherwise one of these triangles meets either T_A or T_B in one element in $M\backslash e$. Now the symmetric difference of T_1 and T_2 is a circuit contained in $T_A \cup T_B$, a contradiction. Thus T_1 and T_2 are disjoint triangles. Since there are no parallel pairs in $M/x/y$ this means that $r_{M/x/y}(T_1 \cup T_2) = 4$. Hence statement (vi) holds with $z = e$,

This exhausts the possible cases, so no counterexample to Lemma 6.7 can exist. □

Theorem 6.2, and hence Theorem 6.1, follows from Lemma 6.7.

CHAPTER 7

Proof of the main result

In this chapter we prove Theorem 1.1. In Section 7.1 we assemble a collection of lemmas, and in Section 7.2 we move to the proof of the theorem.

7.1. A cornucopia of lemmas

LEMMA 7.1. *Let $r \geq 4$ be an even integer. If M is a simple vertically 4-connected single-element extension of Υ_r such that $M \in \mathcal{EX}(M(K_{3,3}))$, then $r = 6$ and M is isomorphic to T_{12}.*

PROOF. Let M be a simple vertically 4-connected binary single-element extension of Υ_r by the element e. Consider the unique circuit C of M contained in $B \cup e$, where B is the basis $\{e_1, \ldots, e_r\}$ of M. If $e_i \notin C$ for some $i \in \{1, \ldots, r-1\}$ then $e \in \mathrm{cl}_M(B - e_i)$, and therefore the triad $\{c_{i-1}, c_i, e_i\}$ of Υ_r is also a triad of M (subscripts are to be read modulo $r-1$). Since M is vertically 4-connected this contradicts Proposition 2.1. Therefore C is equal to either B or to $B - e_r$, and M can be represented over GF(2) by the set of vectors $\{e_1, \ldots, e_r, c_1, \ldots, c_{r-1}, e\}$, where, by an abuse of notation, e stands for either the column of all ones, or the column which has ones in every position other than that corresponding to e_r.

Using a computer we can check the following facts: If $r = 4$ then neither of the matroids represented by this set of vectors is vertically 4-connected. If $r = 6$ and e is the column of all ones then the resulting matroid has an $M(K_{3,3})$-minor. If $r = 6$ and e is the column containing all ones except in its last position then the resulting matroid is isomorphic to T_{12}. ✠10

Suppose that $r \geq 8$. We prove by induction that the matroids represented over GF(2) by $\{e_1, \ldots, e_r, c_1, \ldots, c_{r-1}, e\}$ have $M(K_{3,3})$-minors. Clearly this will complete the proof of the lemma. Here e is either the sum of the vectors $e_1, \ldots, e_{r-1}$ or the sum of $e_1, \ldots, e_r$.

A computer check shows that this claim is true when $r = 8$. ✠11
Suppose that $r > 8$. Let M be the matroid represented over GF(2) by $\{e_1, \ldots, e_r, c_1, \ldots, c_{r-1}, e\}$. It is elementary to check that $M/c_{r-2}/c_{r-1}\backslash e_{r-2}\backslash e_{r-1}$ is isomorphic to the matroid represented by $\{e_1, \ldots, e_{r-2}, c_1, \ldots, c_{r-3}, e\}$, where e is the sum of either $e_1, \ldots, e_{r-3}$ or of $e_1, \ldots, e_{r-2}$. By the inductive assumption $M/c_{r-2}/c_{r-1}\backslash e_{r-2}\backslash e_{r-1}$ has an $M(K_{3,3})$-minor, so we are done. $\square$

Lemma 7.2. *Let $r \geq 4$ be an integer. If M is a 3-connected binary single-element extension of Δ_r such that M has no $M(K_{3,3})$-minor, then $r = 4$ and M is isomorphic to either C_{11} or $M_{4,11}$.*

Proof. The proof is by induction on r. A computer check shows that up to isomorphism Δ_4 has only two 3-connected single-element extensions in $\mathcal{EX}(M(K_{3,3}))$, and that these are isomorphic to C_{11} and $M_{4,11}$; moreover,
✠12 Δ_5 has no 3-connected single-element extensions in $\mathcal{EX}(M(K_{3,3}))$. Suppose that $r > 5$ and that Δ_{r-1} has no 3-connected single-element extensions in $\mathcal{EX}(M(K_{3,3}))$. Assume that the lemma fails for Δ_r. Let M be a 3-connected binary extension of Δ_r by the element e such that M has no $M(K_{3,3})$-minor.

Proposition 3.2 tells us that $M/b_1\backslash a_1\backslash e_1$ is isomorphic to a single-element extension of Δ_{r-1}. By the inductive hypothesis $M/b_1\backslash a_1\backslash e_1$ cannot be 3-connected. Thus e is in a parallel pair in $M/b_1\backslash a_1\backslash e_1$ and this means that e is in a triangle of M with b_1. Let x_1 be the element of this triangle that is not equal to b_1 or e. Note that x_1 cannot be equal to a_1 or e_1, for then e would be parallel to a_2 or e_2 in M.

Since $r > 5$ the sets $\{a_3, b_2, b_3, e_3\}$ and $\{a_{r-1}, b_{r-2}, b_{r-1}, e_{r-1}\}$ are disjoint. We assume that x_1 is not in the first of these two sets. The argument when x_1 is not in the second set is identical. Now $M/b_2\backslash a_3\backslash e_3$ is isomorphic to a single-element extension of Δ_{r-1}, so by using the same argument as before we see that e is in a triangle of M that contains b_2. Let x_2 be the element in this triangle not equal to b_2 or e. It cannot be the case that x_2 is equal to a_3 or e_3. Nor can it be the case that $x_1 = x_2$, for that would imply that b_1 and b_2 are parallel.

Since $\{b_1, b_2, x_1, x_2\}$ is the symmetric difference of two triangles it is a circuit of $M\backslash e = \Delta_r$. But $\{a_3, b_2, b_3, e_3\}$ is a cocircuit of Δ_r, and if $x_2 \notin \{a_3, b_2, b_3, e_3\}$ then $\{b_1, b_2, x_1, x_2\}$ meets this cocircuit in one element, a contradiction. It follows that x_2 is equal to b_3. The set $\{a_1, b_1, b_{r-1}, e_1\}$ is also a cocircuit in Δ_r, and by using the same argument we see that x_1 is equal to b_{r-1}. Now $\{b_1, b_2, b_3, b_{r-1}\}$ is a circuit, which is a contradiction, as this set is independent in Δ_r. □

Lemma 7.3. *Let $r \geq 5$ be an integer. Suppose that M is a 3-connected coextension of Δ_r by the element e such that $M \in \mathcal{EX}(M(K_{3,3}))$. Let B be the basis $\{e_1, \ldots, e_r, e\}$ of M. Let C^* be the unique cocircuit of M contained in $(E(M) - B) \cup e$. Then $C^* - e$ is equal to one of the following sets.*

- *(i) A subset of two or three elements from the set $\{a_i, a_{i+1}, b_i\}$ where $1 \leq i \leq r-2$;*
- *(ii) The set $\{a_{i+1}, a_{i+2}, b_i, b_{i+2}\}$ where $1 \leq i \leq r-3$, or the set $\{a_1, a_2, b_2, b_{r-1}\}$;*
- *(iii) The set $\{a_{i+1}, a_{i+2}, b_i, b_{i+1}, b_{i+2}\}$ where $1 \leq i \leq r-3$, or the set $\{a_1, a_2, b_1, b_2, b_{r-1}\}$;*
- *(iv) One of the sets $\{a_1, b_{r-1}\}$ or $\{a_{r-1}, b_{r-1}\}$;*
- *(v) One of the sets $\{a_1, a_{r-1}, b_1\}$ or $\{a_1, a_{r-1}, b_{r-2}\}$; or,*
- *(vi) One of the sets $\{a_1, a_{r-1}, b_1, b_{r-1}\}$ or $\{a_1, a_{r-1}, b_{r-2}, b_{r-1}\}$.*

Proof. The proof is by induction on r. To check the base case we need to generate all the 3-connected binary single-element coextensions of Δ_5 that have no $M(K_{3,3})$-minor. There are 24 such coextensions (ignoring isomorphisms). In each case one of the statements of the lemma holds. ✠13

Suppose that $r > 5$ and that the lemma holds for Δ_{r-1}. Let $M \in \mathcal{EX}(M(K_{3,3}))$ be a 3-connected coextension of Δ_r by the element e such that $M \in \mathcal{EX}(M(K_{3,3}))$.

For $1 \leq i \leq r-2$ let E_i be the set $\{a_i, a_{i+1}, b_i, e_i, e_{i+1}\}$ and let $E_{r-1} = \{a_1, a_{r-1}, b_{r-1}, e_1, e_{r-1}\}$. We consider the fundamental graph $G_B(M)$.

Claim 7.3.1. There is an integer $i \in \{1, \ldots, r-1\}$ such that e is adjacent in $G_B(M)$ to at most one element in E_i and e is not adjacent to b_i.

Proof. Recall from Proposition 3.2 that $M_0 = M/b_{r-2}\backslash a_{r-1}\backslash e_{r-1}$ is isomorphic to a single-element coextension of Δ_{r-1}. Indeed, we will relabel the ground set of M_0 by giving e_r the label e_{r-1} and b_{r-1} the label b_{r-2}; under this relabeling $M_0/e = \Delta_{r-1}$.

Let B_0 be the basis $\{e_1, \ldots, e_{r-1}, e\}$ of M_0. Let G' be the graph obtained from $G_B(M)$ by pivoting on the edge $b_{r-2}e_{r-1}$. Then the fundamental graph $G_{B_0}(M_0)$ can be obtained from G' by deleting the vertices a_{r-1}, b_{r-2}, and e_{r-1}, and then relabeling e_r with e_{r-1} and b_{r-1} with b_{r-2}.

Suppose that M_0 is not 3-connected. Then e is either a coloop in M_0 or is in series with some element in M_0. If e is a coloop of M_0 then it is an isolated vertex in $G_{B_0}(M_0)$. Thus e is adjacent to at most two vertices in G': a_{r-1} and e_{r-1}. Since $G_B(M)$ can be obtained from G' by pivoting on the edge $b_{r-2}e_{r-1}$ it follows that e is adjacent to at most three vertices in $G_B(M)$: a_{r-1}, b_{r-2}, and b_{r-1}. The claim follows easily.

Next we suppose that e is in a series pair in M_0. In this case either: (i) e has degree one in $G_{B_0}(M_0)$, or (ii) there is some element $e' \in B_0 - e$ such that e and e' are adjacent to exactly the same vertices in $G_{B_0}(M_0)$. Suppose that case (i) holds. Then e is adjacent to exactly one vertex x in $G_{B_0}(M_0)$, and is therefore adjacent to at most three vertices in G': x, a_{r-1}, and e_{r-1}. Thus e is adjacent in $G_B(M)$ to at most four vertices: x, a_{r-1}, b_{r-1}, and b_{r-2}. The claim follows.

Suppose that case (ii) hold. If e has exactly the same set of neighbors in $G_{B_0}(M_0)$ as the vertex e_i, where $1 \leq i \leq r-2$, then e has degree three in $G_{B_0}(M_0)$. Since e_i is adjacent to vertices in exactly two sets E_{i-1} and E_i, by again pivoting on the edge $b_{r-2}e_{r-1}$ in G' we can see that the claim holds.

Next we assume that e has exactly the same set of neighbors in $G_{B_0}(M_0)$ as e_{r-1}. Thus e and e_{r-1} are in series in $M_0 = M/b_{r-2}\backslash a_{r-1}\backslash e_{r-1}$. Thus $\{a_{r-1}, e, e_{r-1}, e_r\}$ contains a cocircuit C^* in M such that $e, e_r \in C^*$. Since M is 3-connected $|C^*| \geq 3$. Suppose that $|C^*| = 3$. Then M is a coextension of Δ_r by the element e such that e is in a triad of M with two elements from the triangle $T = \{a_{r-1}, e_{r-1}, e_r\}$ of Δ_r. Corollary 2.28 implies that

$\Delta_T(\Delta_r)$ is a minor of M. But $\Delta_T(\Delta_r)$ has an $M(K_{3,3})$-minor by Claim 4.6, so we have a contradiction.

Suppose that $C^* = \{a_{r-1}, e, e_{r-1}, e_r\}$. Then C^* is also a circuit, for otherwise $\{a_{r-1}, e_{r-1}, e_r\}$ is a circuit of M that meets C^* in an odd number of elements. Then the fundamental graph $G_B(M)$ is obtained from $G_{B-e}(\Delta_r)$ by adding the vertex e and making it adjacent to a_{r-1} and every vertex that is adjacent to exactly one of e_{r-1} and e_r. Now it is easy to confirm that $M\backslash a_{r-1}$ is isomorphic to $\Delta_T(\Delta_r)$, which has an $M(K_{3,3})$-minor. Thus we have a contradiction.

Therefore we must assume that M_0 is 3-connected. By the inductive assumption Lemma 7.3 holds for M_0. If C^* is the unique cocircuit contained in $(E(M_0)-B_0)\cup e$ then the elements of C^*-e are exactly the neighbors of e in $G_{B_0}(M_0)$. Thus e has degree at most five in $G_{B_0}(M_0)$ and degree at most eight in $G_B(M)$. A straightforward case-check confirms that if the claim is false, then r is equal to six, and C^*-e is the union of $\{a_2, a_3, b_1, b_3, b_4, b_5\}$ with some subset of $\{a_5, b_2\}$. We can eliminate this case by considering the four corresponding coextensions of Δ_6. They all have $M(K_{3,3})$-minors, so
✠14 this completes the proof of the claim. □

Let i be the integer supplied by Claim 7.3.1. We will assume that $1 \leq i \leq r-2$, for the proof when $i = r-1$ is similar. It follows from Claim 7.3.1 that either e is adjacent in $G_B(M)$ to no vertex in $\{a_i, b_i, e_i\}$, or e is adjacent to no vertex in $\{a_{i+1}, b_i, e_{i+1}\}$. We will assume the former; the argument in the latter case is similar.

Consider $M_1 = M/b_i\backslash a_i\backslash e_i$. We relabel the ground set of M_1 so that every element in M_1 with an index $j > i$ is relabeled with the index $j-1$. Then $M_1/e = \Delta_{r-1}$. Let $B_1 = \{e_1, \ldots, e_{r-1}, e\}$ and let G'' be the fundamental graph derived from $G_B(M)$ by pivoting on the edge b_ie_i. Then $G_{B_1}(M_1)$ is obtained from G'' by deleting a_i, b_i, and e_i, and then doing the appropriate relabeling.

Since e is not adjacent to any of the vertices in $\{a_i, b_i, e_i\}$ in $G_B(M)$ it is not adjacent to any of them in G''. From this observation it follows that if e is a coloop or in a series pair in M_1 then it is in a coloop or series pair in M, a contradiction. Thus M_1 is 3-connected.

The inductive hypothesis implies that the lemma holds for M_1. Therefore in $G_{B_1}(M_1)$ the vertex e is adjacent to vertices with at most three different indices. Now the only case it which it is not immediate that the lemma holds for M is the case in which e is adjacent to both at least one vertex with the index $i-1$, and at least one with the index $i+1$. It is not difficult to see that if this is the case then the hypotheses of Claim 7.3.1 apply to either E_{i-2} or E_{i+2}. Assuming the former, either $M/b_{i-2}\backslash a_{i-2}\backslash e_{i-2}$ or $M/b_{i-2}\backslash a_{i-1}\backslash e_{i-1}$ provides a contradiction to the inductive hypothesis. The second case is similar. □

Recall that if $M \in \mathcal{EX}(M(K_{3,3}))$, then an allowable triangle of M is a triangle T such that $\Delta_T(M)$ has no $M(K_{3,3})$-minor. Claim 4.6 tells us that the allowable triangles in Δ_r are exactly those triangles that contain spoke elements.

COROLLARY 7.4. *Suppose that $r \geq 4$ is an integer. If M is a 3-connected coextension of Δ_r by the element e such that $M \in \mathcal{EX}(M(K_{3,3}))$ then either:*

(i) *$r = 4$ and M is isomorphic to $M_{5,11}$; or,*

(ii) *there exists an allowable triangle T of Δ_r such that either $T \cup e$ is a four-element circuit-cocircuit of M, or e is in a triad of M with two elements of T.*

PROOF. To prove the lemma when $r = 4$ we consider all the 3-connected single-element coextensions of Δ_4 that belong to $\mathcal{EX}(M(K_{3,3}))$. There are 21 such coextensions (ignoring isomorphisms). The lemma holds for each of these. ✠15

When $r \geq 5$ we apply Lemma 7.3. The set $\{a_i, a_{i+1}, b_i\}$ is a triangle of Δ_r for $1 \leq i \leq r-2$, so the lemma holds if statement (i) of Lemma 7.3 applies. If statement (ii) holds then $\{e, e_i, e_{i+1}\}$ is a triad of M for some value of $i \in \{1, \ldots, r-2\}$. Thus the lemma holds. Similarly, if statement (iii) holds, then $\{e, b_i, e_i, e_{i+1}\}$ is a four-element circuit-cocircuit for some value of i. The sets $\{a_1, b_{r-1}, e_{r-1}\}$ and $\{a_{r-1}, b_{r-1}, e_1\}$ are triangles of Δ_r, so the lemma holds if statement (iv) applies. If statement (v) holds then either $\{e, a_1, b_{r-1}, e_{r-1}\}$ or $\{e, a_{r-1}, b_{r-1}, e_1\}$ is a four-element circuit-cocircuit. In case (vi) either $\{e, a_1, e_{r-1}\}$ or $\{e, a_{r-1}, e_1\}$ is a triad. □

LEMMA 7.5. *Let T_0 be a triangle of Δ_4 containing the spoke element b. Suppose that the binary matroid M_1 is a coextension of Δ_4 by the element e such that e is in a triad T_1 of M_1 with two elements of T_0. Suppose that the binary matroid M_2 is an extension of M_1 by the element f so that there is a triangle T_2 of M_2 that contains e and f. Then either:*

(i) *M_2 has an $M(K_{3,3})$-minor;*

(ii) *M_2 is isomorphic to $M^a_{5,12}$ or $M^b_{5,12}$;*

(iii) *$T_2 \subseteq T_0 \cup \{e, f\}$; or,*

(iv) *T_1 contains b, and the single element in $T_2 - \{e, f\}$ comes from the unique allowable triangle T of Δ_4 such that $T \cup (T_1 - e)$ contains a cocircuit of size four in Δ_4.*

PROOF. Because the automorphisms of Δ_4 act transitively upon the allowable triangles we can assume that $T_0 = \{a_1, a_2, b_1\}$, so that $b = b_1$. First suppose that $T_1 = \{e, a_1, a_2\}$. Let x be the single element in $T_2 - \{e, f\}$. If $x \in T_0$ then statement (iii) holds, so we assume that $x \notin T_0$. If x is equal to b_2, b_3, e_1, or e_2, then M_2 has an $M(K_{3,3})$-minor. If $x = e_4$ then M_2 is isomorphic to $M^a_{5,12}$ and if x is equal to a_3 or e_3 then M_2 is isomorphic to $M^b_{5,12}$. Thus the result holds in this case. ✠16

Next we consider the case that $T_1 = \{e, a_1, b_1\}$ is a triad of M_1. We again let x be the single element in $T_2 - \{e, f\}$. If $x \in T_0$ then we are done, so we assume that $x \notin T_0$. The unique triangle T of Δ_4 such that

$T \cup \{a_1, b_1\}$ contains a cocircuit of size four is $\{a_3, b_3, e_1\}$, so if x is one of these elements statement (iv) holds. Thus we need only check the case that $x \in \{b_2, e_2, e_3, e_4\}$. If x is equal to b_2, e_2, or e_3 then M_2 has an
✠17 $M(K_{3,3})$-minor. If $x = e_4$ then M_2 is isomorphic to $M^a_{5,12}$.

The final case is that in which $\{e, a_2, b_1\}$ is a triad of M_1. We can use the symmetries of Δ_4 to show that the result also holds in this case. □

LEMMA 7.6. *Suppose that $r \geq 5$ is an integer. Let T_0 be a triangle of Δ_r containing the spoke element b. Suppose that the binary matroid M_1 is a coextension of Δ_r by the element e such that e is in a triad T_1 of M_1 with two elements of T_0. Suppose that the binary matroid M_2 is an extension of M_1 by the element f so that there is a triangle T_2 of M_2 that contains e and f. Then either:*

(i) M_2 has an $M(K_{3,3})$-minor;
(ii) $T_2 \subseteq T_0 \cup \{e, f\}$; or,
(iii) T_1 contains b, and the single element in $T_2 - \{e, f\}$ comes from the unique allowable triangle T of Δ_r such that $T \cup (T_1 - e)$ contains a cocircuit of size four in Δ_r.

PROOF. Assume that the lemma fails, and that $r \geq 5$ is the smallest value for which there is a counterexample M_2. Suppose that $r = 5$. Since the automorphism group of Δ_5 is transitive on allowable triangles we will assume that $T_0 = \{a_1, a_2, b_1\}$, so that $b = b_1$. Suppose that $T_1 = \{e, a_1, a_2\}$. Let x be the single element in $T_2 - \{e, f\}$. It is easily checked that if $x \notin T_0$
✠18 then M_2 has an $M(K_{3,3})$-minor.

Suppose that $T_1 = \{e, a_1, b_1\}$. Again let x be the single element in $T_2 - \{e, f\}$. The unique allowable triangle T of Δ_5 such that $T \cup \{a_1, b_1\}$ contains a cocircuit of size four is $\{a_4, b_4, e_1\}$. If $x \notin T_0$ and $x \notin T$ then M_2
✠19 has an $M(K_{3,3})$-minor. We can see by symmetry that the result also holds if $T_1 = \{e, a_2, b_1\}$. Thus the lemma holds when $r = 5$, so we must assume that $r > 5$.

Again we assume that $T_0 = \{a_1, a_2, b_1\}$. Let x be the element in $T_2 - \{e, f\}$. Suppose that $x \in \{a_{r-2}, e_{r-2}\}$. Consider the minor $N_1 = M_2/b_{r-2}\backslash a_{r-1}\backslash e_{r-1}$ and relabel the ground set of N_1 so that b_{r-1} receives the label b_{r-2} and e_r receives the label e_{r-1}. Proposition 3.2 implies that $N_1/e\backslash f = \Delta_{r-1}$. Since a_{r-1} and e_{r-1} are in $\mathrm{cl}_{M_2}(E(M_2) - (T_1 \cup \{a_{r-1}, e_{r-1}\}))$ it follows that T_1 is a triad in N_1. Moreover b_{r-2} cannot be parallel to any element in T_2, so T_2 is a triangle in N_1. Therefore the lemma holds for N_1.

If N_1 has an $M(K_{3,3})$-minor, then so does M_2, a contradiction. Thus statement (i) does not apply to N_1. It is easy to see that if statement (ii) holds for N_1 then it also holds for M_2, a contradiction. The only way that statement (iii) can hold is if $T_1 - e = \{a_1, b_1\}$ and $x = a_{r-2}$, for the unique triangle T of N_1 specified in statement (iii) consists of the elements $\{a_{r-2}, b_{r-2}, e_1\}$. Let us assume that this is the case. Then we consider the minor $N_2 = M_2/b_{r-3}\backslash a_{r-3}\backslash e_{r-3}$ and relabel the ground set of N_2 so that any element with the index $j \geq r-2$ is relabeled with the index $j-1$. Then $N_2/e\backslash f = \Delta_{r-1}$ and we again conclude that the lemma holds for N_2. It is not difficult to demonstrate the neither statement (ii) nor (iii) holds, so N_2 has an $M(K_{3,3})$-minor. Thus M_2 has an $M(K_{3,3})$-minor, a contradiction.

Therefore we will assume that $x \notin \{a_{r-2}, e_{r-2}\}$. Thus x is in at most one of the sets $\{a_{r-2}, b_{r-3}, e_{r-2}\}$ and $\{a_{r-2}, b_{r-2}, e_{r-2}\}$. Let us assume that x is not contained in the second of these sets (the argument is similar in either case). Let N_3 be the minor $M_2/b_{r-2}\backslash a_{r-2}\backslash e_{r-2}$, relabeled so that any element with the index $j \geq r-1$ receives the index $j-1$. Then $N_3/e\backslash f = \Delta_{r-1}$ and the lemma holds for N_3.

If N_3 has an $M(K_{3,3})$-minor then we are done. Similarly, if $x \in T_0$ in N_3 then the lemma holds. Thus we will assume that statement (iii) of the lemma holds in N_3. Then either $x \in \{b_2, e_2, e_3\}$, or $x \in \{a_{r-2}, b_{r-1}, e_1\}$. In either case we can see that statement (iii) also holds in M_2. This contradiction completes the proof. □

COROLLARY 7.7. *Let $r \geq 4$ be an integer. Suppose that T_0 is a triangle of Δ_r that contains a spoke element. Let M_1 be a binary coextension of Δ_r by the element e such that e is in a triad T_1 of M_1 with two elements of T_0. Suppose that M_2 is an extension of M_1 by the element f such that there is a triangle T_2 of M_2 that contains e and f. If M_2 is vertically 4-connected then either it contains a minor isomorphic to $M(K_{3,3})$, or $r = 4$ and M_2 is isomorphic to $M^a_{5,12}$ or $M^b_{5,12}$.*

PROOF. Suppose that M_2 is a counterexample to the corollary. Thus M_2 is vertically 4-connected. We apply Lemmas 7.5 and 7.6. It is easy to demonstrate that if $T_2 \subseteq T_0 \cup \{e, f\}$ then M_2 is not vertically 4-connected. Therefore there is a triangle T of Δ_r such that $T \cup (T_1 - e)$ contains a four-element cocircuit in Δ_r, and the single element in $T_2 - \{e, f\}$ is contained in T. We will show that $T \cup T_2$ is a vertical 3-separator of M_2, and this will provide a contradiction that completes the proof. Now T is a triangle of M_1, for otherwise $T \cup e$ is a circuit that meets the cocircuit T_1 in one element. Thus T is a triangle in M_2, and the rank of $T \cup T_2$ in M_2 is three. Moreover, because $T \cup (T_1 - e)$ contain a cocircuit of size four in Δ_r, and T_1 is a triad in M_1, by the properties of cocircuits in binary matroids it follows that $T \cup e$ contains a triad in M_1. Thus $T \cup \{e, f\}$ contains a cocircuit in M_2, so the complement of $T \cup T_2$ has rank at most $r(M_2) - 1$. The result follows. □

LEMMA 7.8. *Suppose that T_0 is a triangle of Δ_4 containing a spoke element. Let M_1 be a binary coextension of Δ_4 by the element e such that e is contained in a triad T_1 of M_1, where T_1 contains two elements of T_0. Suppose that M_2 is a binary extension of M_1 by the elements f and g such that M_2 contains triangles T_f and T_g, where $\{e, f\} \subseteq T_f$ and $\{e, g\} \subseteq T_g$. If M_2 is vertically 4-connected and contains no minor isomorphic to $M(K_{3,3})$, $M^a_{5,12}$, or $M^b_{5,12}$, then M_2 is isomorphic to Δ_5 or $M_{5,13}$.*

Proof. By the symmetries of Δ_4 we can assume that $T_0 = \{a_1, a_2, b_1\}$. If $T_1 = \{e, a_1, a_2\}$ then Lemma 7.5 implies that $T_f \subseteq T_0 \cup \{e, f\}$ and $T_g \subseteq T_0 \cup \{e, g\}$. In this case it is easy to see that $T_0 \cup \{e, f, g\}$ is a vertical 3-separator of M_2. Therefore, by the symmetries of Δ_4, we can assume that $T_1 = \{e, a_1, b_1\}$. Let x_f and x_g be the single elements in $T_f - \{e, f\}$ and $T_g - \{e, g\}$ respectively. By Lemma 7.5 the elements x_f and x_g are contained in T_0 or in $\{a_3, b_3, e_1\}$. If both are contained in T_0 then $T_0 \cup \{e, f, g\}$ is a vertical 3-separator of M_2. Similarly, if both x_f and x_g are contained in $\{a_3, b_3, e_1\}$, then, as in the proof of Corollary 7.7, the set $\{e, f, g, a_3, b_3, e_1\}$ is a vertical 3-separator.

Thus we will assume that $x_f \in T_0$, while $x_g \in \{a_3, b_3, e_1\}$. A computer check verifies that if $x_f = a_1$ while $x_g = e_1$ then M_2 is isomorphic to Δ_5. If $x_f = b_1$ and $x_g = b_3$ then M_2 is isomorphic to $M_{5,13}$. In each of the seven
✠20 remaining cases M_2 has an $M(K_{3,3})$-minor. □

Lemma 7.9. *Let $r \geq 5$ be an integer. Suppose that T_0 is a triangle of Δ_r containing a spoke element. Let M_1 be a binary coextension of Δ_r by the element e such that e is contained in a triad T_1 of M_1, where T_1 contains two elements of T_0. Suppose that M_2 is a binary extension of M_1 by the elements f and g such that M_2 contains triangles T_f and T_g, where $\{e, f\} \subseteq T_f$ and $\{e, g\} \subseteq T_g$. If M_2 is vertically 4-connected and has no $M(K_{3,3})$-minor then M_2 is isomorphic to Δ_{r+1}.*

Proof. We suppose that M_2 is vertically 4-connected with no $M(K_{3,3})$-minor. We can assume that $T_0 = \{a_1, a_2, b_1\}$. If $T_1 = \{e, a_1, a_2\}$ then Lemma 7.6 implies that $T_f \subseteq T_0 \cup \{e, f\}$ and $T_g \subseteq T_0 \cup \{e, g\}$. In this case $T_0 \cup \{e, f, g\}$ is a vertical 3-separator of M_2, so we will assume that $T_1 = \{e, a_1, b_1\}$. Let x_f and x_g be the single elements in $T_f - \{e, f\}$ and $T_g - \{e, g\}$ respectively. By Lemma 7.6 the elements x_f and x_g are contained in either T_0 or in $\{a_{r-1}, b_{r-1}, e_1\}$. As before, if both are contained in either T_0 or $\{a_{r-1}, b_{r-1}, e_1\}$ then M_2 is not vertically 4-connected. Thus we assume that $x_f \in T_0$ and $x_g \in \{a_{r-1}, b_{r-1}, e_1\}$.

Claim 7.9.1. $x_f = a_1$ and $x_g = e_1$.

Proof. Suppose $r = 5$. There are eight cases in which the claim is false.
✠21 In each of these M_2 has an $M(K_{3,3})$-minor. This provides the base case for an inductive argument.

Suppose that $r > 5$ and that the claim holds for $r - 1$. Consider $M_2' = M_2/b_{r-2}\backslash a_{r-2}\backslash e_{r-2}$. Then $M_2'/e\backslash f\backslash g$ is isomorphic to Δ_{r-1} under the relabeling that reduces by one the index of any element with an index that exceeds $r - 2$. It is easy to see that T_1 is a triad of M_2' and both T_f and T_g are triangles. Thus we can apply the inductive hypothesis, and the claim follows. □

It remains to show that M_2 is isomorphic to Δ_{r+1}. This can be accomplished by considering the basis graph $G_{B_2}(M_2)$, where $B_2 = \{e_1, \ldots, e_r, e\}$. This graph can be obtained from the basis graph $G_B(\Delta_r)$ (where $B =$

$\{e_1, \ldots, e_r\}$) by adding the vertex e so that it is adjacent to a_1 and b_1, the vertex f so that it is adjacent to e_1 and e_r, and the vertex g so that it is adjacent to e and e_1. Now consider the graph obtained from $G_{B_2}(M_2)$ by pivoting on the edges ea_1, fe_1, and ga_1. This produces a graph isomorphic to $G_{B'}(\Delta_{r+1})$ (where $B' = \{e_1, \ldots, e_{r+1}\}$) under the relabeling that takes e to b_r, f to e_r, g to e_1, and e_r to e_{r+1}. Thus M_2 is isomorphic to Δ_{r+1} and the lemma is true. □

LEMMA 7.10. *Let T_1 and T_2 be allowable triangles of Δ_4 such that $r_{\Delta_4}(T_1 \cup T_2) = 4$. Suppose that M_1 is the binary matroid obtained from Δ_4 by coextending with the elements e and f so that in M_1 the element e is in a triad with two elements from T_1, and f is in a triad with two elements from T_2. Let M_2 be the binary matroid obtained from M_1 by extending with the element g so that $\{e, f, g\}$ is a triangle of M_2. If M_2 is vertically 4-connected then it is isomorphic to $M_{6,13}$.*

PROOF. By the symmetries of Δ_4 we can assume that $T_1 = \{b_1, e_1, e_2\}$ while $T_2 = \{a_2, a_3, b_2\}$. The element e may be in a triad of M_1 with any two elements of T_1, and f may be in a triad with any two elements of T_2. It follows that there are nine cases to check. If $\{e, b_1, e_2\}$ and $\{f, a_2, b_2\}$ are triads of M_1 then e and f are in series in M_1. In this case M_2 is not vertically 4-connected. If $\{e, e_1, e_2\}$ and $\{f, a_2, a_3\}$ are triads of M_1, then M_2 is isomorphic to $M_{6,13}$. A computer check reveals that in the seven remaining cases M_2 is not vertically 4-connected. □ ✠22

LEMMA 7.11. *Let $r \geq 5$ be an integer. Suppose that T_1 and T_2 are disjoint allowable triangles of Δ_r such that either $b_i \in T_1$ and $b_{i+1} \in T_2$ for some $i \in \{1, \ldots, r-2\}$, or $b_{r-1} \in T_1$ and $b_1 \in T_2$. Suppose that M_1 is the binary matroid obtained from Δ_r by coextending with the elements e and f so that in M_1 the element e is in a triad T_e with two elements from T_1, and f is in a triad T_f with two elements from T_2. Let M_2 be the binary matroid obtained from M_1 by extending with the element g so that $\{e, f, g\}$ is a triangle of M_2. If M_2 is vertically 4-connected then it has an $M(K_{3,3})$-minor.*

PROOF. We will assume that $T_1 = \{e_1, e_2, b_1\}$ and that $T_2 = \{a_2, a_3, b_2\}$.

We first suppose that $T_e = \{e, b_1, e_2\}$. We claim that $T_2 \cup \{e, f, g\}$ is a vertical 3-separator of M_2. Note that $r_{M_2}(T_2 \cup \{e, f, g\}) = 4$ and $T_f \cup g$ is a cocircuit in M_2. If we can show that $T_2 \cup \{e, f, g\}$ contains a cocircuit distinct from $T_f \cup g$ then we will have shown that the complement of $T_2 \cup \{e, f, g\}$ has rank at most $r(M_2) - 2$, and this will prove the claim. Since $C^* = \{a_2, b_1, b_2, e_2\}$ is a cocircuit of M_1, and g is in the closure of the complement of C^* in M_2 it follows that C^* is also a cocircuit of M_2. Since $T_e \cup g$ is a cocircuit of M_2 the symmetric difference of C^* with $T_e \cup g$ contains a cocircuit in M_2. This symmetric difference is $\{e, g, a_2, b_2\}$. Any cocircuit contained in this set is different from $T_f \cup g$, since it cannot contain f. This proves the claim.

If $T_f = \{f, a_2, b_2\}$ then we can give a symmetric argument and show that $T_1 \cup \{e, f, g\}$ is a vertical 3-separator of M_2. Thus there are only four cases left to consider: T_e is equal to either $\{e, b_1, e_1\}$ or $\{e, e_1, e_2\}$, and T_f is equal to either $\{f, a_2, a_3\}$ or $\{f, a_3, b_2\}$.

We prove by induction on r that in any of these four cases M_2 has an
✠23 $M(K_{3,3})$-minor. A computer check confirms that this is the case when $r = 5$. Assume that claim holds for $r - 1$. Let $M' = M_2/b_{r-1}\backslash a_{r-1}\backslash e_{r-1}$. Then $M'/e/f\backslash g$ is equal to Δ_{r-1} once e_r is relabeled as e_{r-1}. Moreover it is easy to confirm that T_e and T_f are triads in $M'\backslash g$, and that $\{e, f, g\}$ is a triangle of M'. Thus M has an $M(K_{3,3})$-minor by the inductive hypothesis. $\square$

Lemma 7.12. *Let $r \geq 5$ be an integer. Suppose that T_1 and T_2 are disjoint allowable triangles of Δ_r. Suppose that M_1 is the binary matroid obtained from Δ_r by coextending with the elements e and f so that in M_1 the element e is in a triad T_e with two elements from T_1, and f is in a triad T_f with two elements from T_2. Let M_2 be the binary matroid obtained from M_1 by extending with the element g so that $\{e, f, g\}$ is a triangle of M_2. If M_2 is vertically 4-connected then it has an $M(K_{3,3})$-minor.*

Proof. Since the automorphism group of Δ_r is transitive on the allowable triangles we assume that $T_1 = \{b_1, e_1, e_2\}$. Lemma 7.11 and symmetry show that the lemma is true when $b_2 \in T_2$ or $b_{r-1} \in T_2$, so henceforth we will assume that T_2 contains neither b_2 nor b_{r-1}.

Claim 7.12.1. If M_2 does not have an $M(K_{3,3})$-minor, then either:

(i) $T_2 = \{b_3, e_3, e_4\}$, $T_e = \{e, b_1, e_2\}$, $T_f = \{f, b_3, e_3\}$; or,

(ii) $T_2 = \{a_{r-2}, a_{r-1}, b_{r-2}\}$, $T_e = \{e, b_1, e_1\}$, $T_f = \{f, a_{r-1}, b_{r-2}\}$.

Proof. We start by confirming that the claim holds when $r = 5$. Since T_2 does not contain b_2 or b_3 it follows that T_2 can only be $\{a_3, a_4, b_3\}$ or $\{b_3, e_3, e_4\}$. In either case e can be in a triad of M_1 with any two elements of T_1 and f can be in a triad with any two elements of T_2. Thus there are 18 cases to check. A computer check reveals that in only two of these cases does M_2 not have an $M(K_{3,3})$-minor: $T_2 = \{b_3, e_3, e_4\}$ and $T_e = \{e, b_1, e_2\}$ while $T_f = \{f, b_3, e_3\}$; or $T_2 = \{a_3, a_4, b_3\}$ and $T_e = \{e, b_1, e_1\}$ while
✠24 $T_f = \{f, a_4, b_3\}$.

Let us assume the claim is false, and let r be the smallest value for which it fails. The discussion above shows that $r > 5$. Suppose that M_2 is constructed from Δ_r as described in the hypotheses of the lemma and that M_2 provides a counterexample to the claim.

We first suppose that $b_{r-2} \in T_2$. Then either $T_2 = \{a_{r-2}, a_{r-1}, b_{r-2}\}$ or $\{b_{r-2}, e_{r-2}, e_{r-1}\}$. We consider $M' = M_2/b_2\backslash a_3\backslash e_3$. Then $M'/e/f\backslash g$ is equal to Δ_{r-1} after all the elements with index $j > 3$ are relabeled with the index $j - 1$. Furthermore $\{e, f, g\}$ is a triangle in M', and T_e and T_f are triads in $M'\backslash g$. Since M_2 has no $M(K_{3,3})$-minor, it follows that M'

has no $M(K_{3,3})$-minor. By the inductive hypothesis there are two cases to consider: In the first case $T_2 = \{a_{r-2}, a_{r-1}, b_{r-2}\}$ and $T_e = \{e, b_1, e_1\}$ while $T_f = \{f, a_{r-1}, b_{r-2}\}$. But this cannot occur since then M_2 would not be a counterexample to the claim. Thus we consider the other case: $r = 6$, and $T_2 = \{b_4, e_4, e_5\}$, while $T_e = \{e, b_1, e_2\}$ and $T_f = \{f, b_4, e_4\}$. But a computer check shows that if this is the case then M_2 has an $M(K_{3,3})$-minor. ✠25
Thus we conclude that T_2 does not contain b_{r-2}.

We consider the minor $M' = M_2/b_{r-1}\backslash a_{r-1}\backslash e_{r-1}$. Then $M'/e/f\backslash g$ is equal to Δ_{r-1} after e_r is relabeled with e_{r-1}. We can apply the inductive hypothesis to M'. Since M' does not have an $M(K_{3,3})$-minor we are forced to consider two possible cases: In the first $T_2 = \{b_3, e_3, e_4\}$, and $T_e = \{e, b_1, e_2\}$ while $T_f = \{f, b_3, e_3\}$. However in this case M_2 is not a counterexample to the claim. Therefore we assume that the other case holds: $T_2 = \{a_{r-3}, a_{r-2}, b_{r-3}\}$, and $T_e = \{e, b_1, e_1\}$ while $T_f = \{f, a_{r-2}, b_{r-3}\}$. A computer check reveals that if $r = 6$ then M_2 has an $M(K_{3,3})$-minor, contrary to hypothesis, so $r > 6$. But then $M_2/b_2\backslash a_3\backslash e_3$ is a counterexample to ✠26
the claim, contradicting our assumption on the minimality of r. $\square$

We complete the proof of Lemma 7.12 by showing that in either of the two cases in the statement of Claim 7.12.1 we can find a vertical 3-separation of M_2. If the first case holds, we show that $X = \{e, f, g, a_2, a_3, b_2\}$ is a vertical 3-separator. Note that X is the union of the two triangles $\{e, f, g\}$ and $\{a_2, a_3, b_2\}$, so $r_{M_2}(X) = 4$. Next we observe that $\{a_2, b_1, b_2, e_2\}$ is a cocircuit of Δ_r, and also of M_2. Furthermore $T_e \cup g$ is a cocircuit of M_2. Thus the symmetric difference of these two sets, $\{e, g, a_2, b_2\}$, contains a cocircuit of M_2. By using a symmetric argument on $T_f \cup g$ and $\{a_3, b_2, b_3, e_3\}$ we can show that $\{f, g, a_3, b_2\}$ contains a cocircuit. Since M_2 contains no series pairs it follows that X contains two distinct cocircuits. Thus the complement of X has rank at most $r(M_2) - 2$, and X is a vertical 3-separator of M_2. A similar argument shows that if the second case of Claim 7.12.1 holds then $\{e, f, g, a_1, b_{r-1}, e_{r-1}\}$ is a vertical 3-separator. This completes the proof of the lemma. $\square$

Lemma 7.13. *Suppose that $r \geq 4$ and that T_1 and T_2 are distinct allowable triangles of Δ_r such that $T_1 \cap T_2 \neq \varnothing$. Suppose that M_1 is the binary matroid obtained from Δ_r by coextending with the elements e and f so that $(T_1 - T_2) \cup e$ and $(T_2 - T_1) \cup f$ are triads of M_1. Let M_2 be the binary matroid obtained from M_1 by extending with the element g so that $\{e, f, g\}$ is a triangle of M_2. Then M_2 has an $M(K_{3,3})$-minor.*

Proof. First suppose that T_1 and T_2 meet in the spoke element b. Let $T_1' = (T_1 - T_2) \cup e$, and let $T_2' = (T_2 - T_1) \cup f$. It is not difficult to see that $\nabla_{T_1'}(\nabla_{T_2'}(M_1))$ is obtained from Δ_r by adding e and f as parallel elements to b. It follows that $M_1\backslash b$ is isomorphic to the matroid obtained from Δ_r by adding an element b' in parallel to b, and then performing Δ-Y operations on T_1 and $(T_2 - b) \cup b'$. Thus $M_1\backslash b$ has an $M(K_{3,3})$-minor, by Claim 4.7, so we are done. Therefore we now assume that T_1 and T_2 meet in a rim element.

The rest of the proof is by induction on r. Suppose that $r = 4$. By the symmetries of Δ_r we can assume that $T_1 = \{b_1, e_1, e_2\}$ and $T_2 = \{b_2, e_2, e_3\}$. Then M_2 is obtained by coextending with e and f so that $\{e, b_1, e_1\}$ and $\{f, b_2, e_3\}$ are triads, and then extending with g so that
✠27 $\{e, f, g\}$ is a triangle. The resulting matroid has an $M(K_{3,3})$-minor. This establishes the base case of our inductive argument.

Suppose that $r > 4$, and that the result holds for Δ_{r-1}. By the symmetries of Δ_4 we assume that $T_1 = \{b_1, e_1, e_2\}$ and $T_2 = \{b_2, e_2, e_3\}$. Let M_2 be the matroid obtained from Δ_r as described in the hypotheses of the lemma.

We consider the minor $M' = M_2/b_{r-1}\backslash a_{r-1}\backslash e_{r-1}$. Note that $\{e, g, b_1, e_1\}$ and $\{f, g, b_2, e_3\}$ are cocircuits in M_2, so it cannot be the case that g and b_{r-1} are in parallel in M_2, for then $\{e, f, b_{r-1}\}$ would be a triangle meeting these cocircuits in one element. Thus $\{e, f, g\}$ is a triangle in M'. Moreover, $\{e, b_1, e_1\}$ and $\{f, b_2, e_3\}$ are triads in $M_2\backslash g$. Since $\{a_{r-2}, a_{r-1}, b_{r-2}\}$ and $\{e_{r-2}, e_{r-1}, b_{r-2}\}$ are triangles in $M_2\backslash g$, it follows that a_{r-1} and e_{r-1} are not in $\mathrm{cl}^*_{M\backslash g}(\{e, b_1, e_1\})$ or $\mathrm{cl}^*_{M\backslash g}(\{f, b_2, e_3\})$, so $\{e, b_1, e_1\}$ and $\{f, b_2, e_3\}$ are triads in $M'\backslash g$. Using Proposition 3.2 we can now see that M' is constructed from Δ_{r-1} in the way described in the statement of the lemma. By induction M' has an $M(K_{3,3})$-minor, so we are done. □

LEMMA 7.14. *Suppose that $r \geq 4$ and that T_1 and T_2 are distinct allowable triangles of Δ_r such that $T_1 \cap T_2 \neq \varnothing$. Let M_1 be the binary matroid obtained from Δ_r by coextending with the element e so that e is in triad with the element in $T_1 \cap T_2$ and a single element from $T_1 - T_2$, and let M_2 be obtained from M_1 by coextending with the element f so that $(T_2 - T_1) \cup f$ is a triad of M_2. Finally, let M_3 be obtained from M_2 by extending with the element g so that $\{e, f, g\}$ is a triangle in M_3. Then M_3 has an $M(K_{3,3})$-minor.*

PROOF. Suppose that T_1 and T_2 meet in the spoke element b. Let T_1' and T_2' be the triads of M_2 that contain e and f respectively, and let x be the element in $T_1 - T_1'$. Then $\nabla_{T_1'}(\nabla_{T_2'}(M_2))$ is obtained from Δ_r by adding f as a parallel element to b, and adding e in parallel to x. Thus $M_2\backslash x$ is isomorphic to the matroid obtained from Δ_r by adding an element b' in parallel to b, and then performing Δ-Y operations on T_1 and $(T_2 - b) \cup b'$. Therefore $M_2\backslash x$ has an $M(K_{3,3})$-minor by Claim 4.7. Henceforth we assume that T_1 and T_2 meet in a spoke element.

We complete the proof by induction on r. Suppose that $r = 4$. By the symmetries of Δ_4 we can assume that $T_1 = \{b_1, e_1, e_2\}$ and $T_2 = \{b_2, e_2, e_3\}$. There are two cases to check. In the first case $\{e, e_1, e_2\}$ is a triad of M_1, and in the second $\{e, b_1, e_2\}$ is a triad. In either case M_3 has
✠28 an $M(K_{3,3})$-minor.

Suppose that $r > 4$, and that the result holds for Δ_{r-1}. By the symmetries of Δ_r we assume that $T_1 = \{b_1, e_1, e_2\}$ and $T_2 = \{b_2, e_2, e_3\}$.

Let M' be the minor $M_3/b_{r-1}\backslash a_{r-1}\backslash e_{r-1}$. It cannot be the case that g and b_{r-1} are in parallel in M_2 so $\{e, f, g\}$ is a triangle in M'. Furthermore, as in the proof of the previous lemma, we can show that $\{f, b_2, e_3\}$ is a triad of $M'\backslash g$, and that either $\{e, e_1, e_2\}$ or $\{e, b_1, e_2\}$ is a triad of $M'\backslash g$. It follows by induction that M' has an $M(K_{3,3})$-minor. □

7.2. The coup de grâce

Finally we move to the proof of the main theorem.

THEOREM 1.1. *An internally* 4*-connected binary matroid has no* $M(K_{3,3})$*-minor if and only if it is either:*

(i) cographic;
(ii) isomorphic to a triangular or triadic Möbius matroid; or,
(iii) isomorphic to one of the 18 *sporadic matroids listed in Appendix* B*.*

PROOF. A simple computer check shows that none of the sporadic matroids in Appendix B has an $M(K_{3,3})$-minor. It follows easily from this fact ✠29
and Corollary 3.9 that the matroids listed in the statement of the theorem do not have $M(K_{3,3})$-minors.

To prove the converse we start by showing that if the theorem fails, then there is a vertically 4-connected counterexample. Suppose that the theorem is false, and that M' is a minimum-sized counterexample. If M' is not vertically 4-connected, then we can reduce it to a vertically 4-connected matroid M'' by repeatedly performing Y-Δ operations, as shown in Chapter 4. Let $M = \mathrm{si}(M'')$. Lemma 2.33 shows that M'', and hence M, is not cographic. Certainly M must contain at least one triangle, so if M is not a counterexample to the theorem then M is either a triangular Möbius matroid or one of the sporadic matroids in Appendix B. But in this case Lemmas 4.5 and 4.9 show that M' is not a counterexample to Theorem 7.2. Thus M is simple and vertically 4-connected and a counterexample to Theorem 7.2. Moreover, since $|E(M)| \leq |E(M')|$ it follows that M too is a minimum-sized counterexample.

We note that F_7 is a triangular Möbius matroid, that F_7^* and $T_{12}\backslash e$ are not vertically 4-connected, and that $M(K_5)$, T_{12}/e, and T_{12} are all listed as sporadic matroids in Appendix B. It follows from Corollary 2.15 that M has a Δ_4-minor. Consider the vertically 4-connected proper minors of M that have Δ_4-minors, and among such minors let N be as large as possible.

Corollary 5.4 implies that M has no 3-connected minor M' having both a N-minor and a four-element circuit-cocircuit. Thus we can apply Lemma 6.7 to M. Therefore M has a proper minor M_0 such that M_0 is internally 4-connected with an N-minor, and M_0 is obtained from M by one of the operations described in Lemma 6.7. We assume that amongst such proper minors of M the size of M_0 is as large as possible. Since M_0 is internally

4-connected and $|E(M_0)| < |E(M)|$ it follows that M_0 obeys Theorem 7.2. It cannot be the case that M_0 is cographic, for M_0 has an N-minor, N has a Δ_4-minor, and Δ_4 is non-cographic. Therefore M_0 is either a Möbius matroid, or is isomorphic to one of the sporadic matroids. We divide what follows into various cases and subcases.

Case 1. M_0 is a triadic Möbius matroid.

Since the triadic Möbius matroids are not vertically 4-connected, it follows that the only statements in Lemma 6.7 that could apply are (i) or (iii). We can quickly eliminate the case that statement (iii) applies, since no Möbius matroid has both a triangle and a triad. If statement (i) applies then M is a vertically 4-connected single-element extension of a triadic Möbius matroid, so Lemma 7.1 implies that M is isomorphic to T_{12}, a contradiction.

This disposes of the case that M_0 is a triadic Möbius matroid.

Case 2. M_0 is a triangular Möbius matroid.

Claim 7.15. If statement (iv) of Lemma 6.7 applies then there is an allowable triangle T of M_0 such that one of the following cases applies.

(i) There is an element $x \in E(M)$ such that $M_0 = M/x$;
(ii) There are elements $x, y \in E(M)$ such that $M_0 \cong M/x\backslash y$ and x is contained in a triad of $M\backslash y$ with two elements of T. Moreover, $M\backslash y$ is 3-connected; or,
(iii) There are elements $x, y, z \in E(M)$ and triangles T_{xy} and T_{xz} of M such that $M_0 \cong M/x\backslash y\backslash z$ and $x, y \in T_{xy}$ while $x, z \in T_{xz}$. Moreover, $T_{xy} \cap T = \varnothing$ while $T_{xz} \cap T \neq \varnothing$, and x is contained in a triad of $M\backslash y\backslash z$ with two elements of T. Furthermore $M\backslash y\backslash z$ and $M\backslash z$ are both 3-connected.

Proof. Statement (iv) asserts that there is a triangle T of M/x and a four-element cocircuit C^* of M such that $x \in C^*$ and $|C^* \cap T| = 2$. If x is in no triangles of M then $M_0 = M/x$, so (i) holds. Suppose that x is in exactly one triangle T_{xy} of M. Clearly T_{xy} meets C^* in exactly two elements. Let y be the element in $(T_{xy} \cap C^*) - x$.

It cannot be the case that $T = T_{xy}$, for T_{xy} is not a triangle in M/x. Now we see that $y \notin T$, for otherwise there would be a parallel element in M. Therefore T is a triangle of $M/x\backslash y \cong M_0$. Moreover $C^* - y$ is a triad of $M\backslash y$ which contains x and two elements of T, as desired. If T is not an allowable triangle of M_0, then Corollary 2.28 implies that $M\backslash y$ has an $M(K_{3,3})$-minor, a contradiction.

It remains to show only that $M\backslash y$ is 3-connected. If this is not the case then x is in a cocircuit of size at most two in $M\backslash y$, implying that M contains a cocircuit of size at most three, a contradiction as M is vertically 4-connected.

Next we assume that x is contained in exactly two triangles, T_{xy} and T_{xz}, in M. Statement (iv) of Lemma 6.7 asserts that exactly one of T_{xy} and

T_{xz} meets T. Let us assume without loss of generality that $T_{xy} \cap T = \varnothing$ while $T_{xz} \cap T \neq \varnothing$. Obviously T_{xy} and T_{xz} meet C^* in exactly two elements each. Let y be the element in $(T_{xy} \cap C^*) - x$, and let z be the element in $T_{xz} - C^*$. Then $M_0 \cong M/x\backslash y\backslash z$. Neither T_{xy} nor T_{xz} is equal to T, and it follows that neither y nor z is contained in T, so T is a triangle of $M/x\backslash y\backslash z$.

It cannot be the case that z is contained in $\mathrm{cl}^*_M(C^*)$, for if this were the case then properties of cocircuits in binary matroids would imply that M contains a cocircuit of size at most three, a contradiction. Therefore C^* is a cocircuit in $M\backslash z$, so $C^* - y$ is a triad in $M\backslash y\backslash z$ and $C^* - y$ contains both x and two elements of T, as desired. Corollary 2.28 again implies T is an allowable triangle of M_0.

Suppose that $M\backslash y\backslash z$ is not 3-connected. Then x is contained in a cocircuit of size at most two in $M\backslash y\backslash z$. In fact it must be in a cocircuit of size exactly two, for otherwise M contains a cocircuit of size at most three. Let x' be the other element in this cocircuit. Since $C^* - y$ is a triad in $M\backslash y\backslash z$ it follows that $(C^* - \{x, y\}) \cup x'$ is a triad in $M\backslash y\backslash z$, and hence in $M/x\backslash y\backslash z$. This is a contradiction, as $M/x\backslash y\backslash z$ is vertically 4-connected.

Therefore $M\backslash y\backslash z$ is 3-connected. If $M\backslash z$ is not 3-connected then y is in a circuit of size at most two in $M\backslash z$ and hence in M, a contradiction. This completes the proof of the claim. □

Claim 7.16. If statement (v) of Lemma 6.7 holds then:

(i) there is an element $x \in E(M)$ such that $M_0 = M/x$;
(ii) there are elements $x, y \in E(M)$ such that $M_0 \cong M/x\backslash y$ and $M\backslash y$ are 3-connected; or,
(iii) there are elements $x, y, z \in E(M)$ such that $M_0 \cong M/x\backslash y\backslash z$, and $M\backslash y\backslash z$ and $M\backslash z$ are both 3-connected.

Proof. Statement (v) asserts that there is an element x such that M/x is vertically 4-connected, and a cocircuit C^* such that $|C^*| = 4$ and $x \in C^*$. Moreover x is in at most two triangles in M.

If x is in no triangles then (i) applies and we are done. Suppose that x is in exactly one triangle T. Let y be an element of $T - x$. Then $M_0 \cong M/x\backslash y$. If $M\backslash y$ is not 3-connected then x is contained in a cocircuit of size at most two in $M\backslash y$, and M contains a cocircuit of size at most three, a contradiction.

Suppose x is in exactly two triangles, T_y and T_z, in M. Let y be the single element in $(C^* \cap T_y) - x$, and let z be the single element in $T_z - C^*$. Then $M_0 \cong M/x\backslash y\backslash z$. If $M\backslash y\backslash z$ is not 3-connected then x is contained in a series pair with some element, x', in $M\backslash y\backslash z$. In this case $(C^* - \{x, y\}) \cup x'$ is a triad of $M\backslash y\backslash z$, and hence of $M/x\backslash y\backslash z$, a contradiction. Thus $M\backslash y\backslash z$ is 3-connected, and it is easy to see that $M\backslash z$ is 3-connected. □

Subcase 2.1. Statement (i) of Lemma 6.7 applies.

In this case M is a single-element extension of Δ_r for some $r \geq 4$. Then Lemma 7.2 implies that M is isomorphic to either C_{11} or $M_{4,11}$, a contradiction.

SUBCASE 2.2. Statement (ii) of Lemma 6.7 applies.

In this case M is a 3-connected coextension of Δ_r for some $r \geq 4$. We apply Corollary 7.4. It cannot be the case that M contains a triad or a four-element circuit-cocircuit, so we deduce that M is isomorphic to $M_{5,11}$, a contradiction.

SUBCASE 2.3. Statement (iii) of Lemma 6.7 applies.

This case immediately leads to a contradiction, as no Möbius matroid has both a triangle and a triad.

SUBCASE 2.4. Statement (iv) of Lemma 6.7 applies.

We use Claim 7.15. It cannot be the case that $M_0 = M/x$, for then statement (ii) applies, and we have already considered that possibility.

Suppose that (ii) of Claim 7.15 holds. Then $M\backslash y$ is a 3-connected coextension of Δ_r for some $r \geq 4$ and there exists an allowable triangle T of Δ_r such that x is contained in a triad of $M\backslash y$ with two elements of T. Now Corollary 7.7 tells us that M is isomorphic to $M^a_{5,12}$ or $M^b_{5,12}$.

This contradiction means that we must assume (iii) of Claim 7.15 holds. Therefore $M\backslash y\backslash z$ is a 3-connected single-element coextension of Δ_r for some $r \geq 4$, and there is an allowable triangle T of Δ_r such that x is in a triad of $M\backslash y\backslash z$ with two elements of T. Suppose that $r = 4$. Then $N \cong \Delta_4$, and by the maximality of N it follows that M has no minor isomorphic to $M^a_{5,12}$ or $M^b_{5,12}$. Thus Lemma 7.8 tells us that M is isomorphic to either Δ_5 or $M_{5,13}$. In either case we have a contradiction. Similarly, if $r \geq 5$ then Lemma 7.9 asserts that M is isomorphic to Δ_{r+1}, and again we have a contradiction.

SUBCASE 2.5. Statement (v) of Lemma 6.7 applies.

Consider Claim 7.16. If (i) applies then we are in Subcase 2.1. If (ii) holds then $M\backslash y$ is a coextension of Δ_r by the element x. Since $M\backslash y$ cannot have a four-element circuit-cocircuit Corollary 7.4 tells us that there is a allowable triangle T of Δ_r such that x is in a triad with two elements of T in $M\backslash y$. Since y is in a triangle of M with x Corollary 7.7 tells us that either M has an $M(K_{3,3})$-minor, or M is isomorphic to $M^a_{5,12}$ or $M^b_{5,12}$. Thus we have a contradiction.

Suppose that (iii) of Claim 7.16 holds. We can derive a contradiction by using Lemmas 7.8 and 7.9.

SUBCASE 2.6. Statement (vi) of Lemma 6.7 applies.

In this case we immediately derive a contradiction from Lemmas 7.10 and 7.12.

SUBCASE 2.7. Statement (vii) of Lemma 6.7 applies.

In this case Lemma 7.13 shows that we have a contradiction.

SUBCASE 2.8. Statement (viii) of Lemma 6.7 applies.

In this case Lemma 7.14 shows that we have a contradiction.

SUBCASE 2.9. Statement (ix) of Lemma 6.7 applies.

In this case there is an element $x \in E(M)$ such that $M\backslash x$ contains three cofans, $\{x_1, \ldots, x_5\}$, $\{y_1, \ldots, y_5\}$, and $\{z_1, \ldots, z_5\}$, where $x_5 = y_1$, $y_5 = z_1$, and $z_5 = x_1$, and $M_0 = M\backslash x/x_1/y_1/z_1$. It is easy to see that $\{x_2, x_3, x_4\}$, $\{y_2, y_3, y_4\}$, and $\{z_2, z_3, z_4\}$ are triangles of M_0, and that $\{x_3, x_4, y_2, y_3\}$, $\{y_3, y_4, z_2, z_3\}$, and $\{z_3, z_4, x_2, x_3\}$ are cocircuits.

Thus we seek pairwise disjoint triangles T_1, T_2, and T_3 in M_0 such that $T_i \cup T_j$ contains a cocircuit C^*_{ij} of size four for $1 \leq i < j \leq 3$. We can reconstruct M from M_0 by coextending with the element e so that e forms a triad with $T_1 \cap C^*_{12}$, then coextending by f so that $(T_2 \cap C^*_{23}) \cup f$ is a triad, coextending by g so that $(T_3 \cap C^*_{13}) \cup g$ is a triad, and finally extending by x so that $\{e, f, g, x\}$ is a circuit.

It follows from this discussion and Corollary 2.28 that each of T_1, T_2, and T_3 is an allowable triangle in M_0. It is simple to prove by induction on r that when $r \geq 4$, the only four-element cocircuits in Δ_r are sets of the form $\{a_i, b_i, b_{i-1}, e_i\}$, for $2 \leq i \leq r-1$, and $\{a_1, b_1, b_{r-1}, e_1\}$. Since the only allowable triangles of Δ_r are those triangles that contain spoke elements, it is now straightforward to see that M_0 is isomorphic to Δ_4.

By the symmetries of Δ_4 we can assume that $T_1 = \{b_1, e_1, e_2\}$, $T_2 = \{a_2, a_3, b_2\}$, and $T_3 = \{a_1, b_3, e_3\}$. Now it is a simple matter to construct a representation of M and check that it has an $M(K_{3,3})$-minor. ✠30

This contradiction means that we have completed the case-checking when M_0 is a triangular Möbius matroid.

CASE 3. M_0 is isomorphic to one of the sporadic matroids listed in Appendix B.

SUBCASE 3.1. Statement (i) of Lemma 6.7 applies.

A computer check reveals that all the 3-connected binary single-element extensions of the matroids in Appendix B either have $M(K_{3,3})$-minors, or are isomorphic to other sporadic matroids. Thus we have a contradiction. ✠31

SUBCASE 3.2. Statement (ii) of Lemma 6.7 applies.

In this case M is a single-element coextension of one of the matroids listed in Appendix B. For each of the matroids in Appendix B, other than $M_{9,18}$ and $M_{11,21}$, we can use a computer to generate all the 3-connected single-element coextensions that belong to $\mathcal{EX}(M(K_{3,3}))$. The only vertically 4-connected matroids we produce in this way are isomorphic to T_{12}/e and T_{12}. ✠32

The matroids $M_{9,18}$ and $M_{11,21}$ are large enough that generating all their single-element coextensions is non-trivial, so we provide an inductive argument to show that they have no vertically 4-connected single-element coextensions in $\mathcal{EX}(M(K_{3,3}))$. Using a computer we can generate all the 3-connected coextensions of $M_{7,15}$ by the element e such that the resulting

matroid is in $\mathcal{EX}(M(K_{3,3}))$. There are 12 such coextensions, ignoring isomorphisms. In each of these coextensions e is contained in a cocircuit C^*
✠33 such that there is a triangle T of $M_{7,15}$ with the property that $C^* \subseteq T \cup e$. In every such coextension it must be the case that $|C^*| \geq 3$, and it is easy to see that $C^* \cup T$ is a vertical 3-separator. Therefore no such coextension is vertically 4-connected.

We prove inductively that if M' is a 3-connected coextension of $M_{9,18}$ or (respectively) $M_{11,21}$ by the element e, such that $M' \in \mathcal{EX}(M(K_{3,3}))$, then there is a cocircuit C^* of M' and a triangle T of $M_{9,18}$ or (respectively) $M_{11,21}$, such that $e \in C^*$ and $C^* \subseteq T \cup e$.

Recall that if $\mathcal{T}_0$ is a set of triangles in F_7, and t_e stands for the number of triangles in $\mathcal{T}_0$ that contain e for each element $e \in F_7$, then we obtain M_1 by adding $t_e - 1$ parallel elements to each element e in F_7. We then let $\mathcal{T}$ be a set of pairwise disjoint triangles in M_1 corresponding in the natural way to triangles in $\mathcal{T}_0$; and finally we perform Δ-Y operations on each of the triangles in $\mathcal{T}$ to obtain the matroid $\Delta(F_7; \mathcal{T}_0)$. Then $\nabla(F_7^*; \mathcal{T}_0)$ is the dual of $\Delta(F_7; \mathcal{T}_0)$. We know that $M_{7,15}$, $M_{9,18}$, and $M_{11,21}$ are isomorphic respectively to $\nabla(F_7^*; \mathcal{T}_5)$, $\nabla(F_7^*; \mathcal{T}_6)$, and $\nabla(F_7^*; \mathcal{T}_7)$, where $\mathcal{T}_5$, $\mathcal{T}_6$, and $\mathcal{T}_7$ are sets of five, six, and seven pairwise distinct triangles in F_7.

Suppose that $M_{9,18}^*$ is obtained from M_1 by performing Δ-Y operations on the triangles in $\mathcal{T}$, where the triangles in $\mathcal{T}$ correspond naturally to the triangles in $\mathcal{T}_6$. Let T' be any triangle in $\mathcal{T}$. Then T' is also a triangle in $M_{9,18} = \nabla(F_7^*; \mathcal{T}_6)$. Since $\mathcal{T}_6$ contains six triangles it follows that every element in T' is contained in a non-trivial parallel class in M_1. From this fact and Proposition 2.27 we deduce that

$$M_{9,18}^* \backslash T' = \Delta(F_7; \mathcal{T}_6) \backslash T' = \Delta(F_7; \mathcal{T}_5) = M_{7,15}^*. \tag{7.1}$$

Thus $M_{7,15} = M_{9,18}/T'$.

Let M' be a coextension of $M_{9,18}$ by the element e such that M' is a 3-connected member of $\mathcal{EX}(M(K_{3,3}))$. Let $M'' = M'/T'$. By Equation (7.1) it follows that M'' is a coextension of $M_{7,15}$ by the element e. If M'' is not 3-connected then e is contained in a cocircuit of size at most two in M''. But this implies that M' contains a cocircuit of size at most two, a contradiction. Thus M'' is a 3-connected single-element coextension of $M_{7,15}$ that belongs to $\mathcal{EX}(M(K_{3,3}))$. Our earlier argument tells us that in M'' the element e belongs to a cocircuit C^* and that there is a triangle T of $M_{7,15}$ such that $C^* \subseteq T \cup e$. Now $M_{7,15}$ has exactly five triangles, and these triangles are exactly the members of $\mathcal{T}_5$. It follows that T is also a triangle of $M_{9,18}$, and C^* is certainly a cocircuit of $M_{9,18}$, so our claim holds for $M_{9,18}$. We can use exactly the same argument to prove that the claim holds for $M_{11,21}$. Thus no 3-connected single-element extension of $M_{9,18}$ or $M_{11,21}$ can be a vertically 4-connected member of $\mathcal{EX}(M(K_{3,3}))$. This completes the proof that statement (ii) cannot hold.

Subcase 3.3. Statement (iii) of Lemma 6.7 applies.

There is only one matroid listed in Appendix B that contains both a triad and a triangle, so we assume that M_0 is isomorphic to $M_{5,11}$. Thus M contains elements x and y such that $M/x\backslash y$ is isomorphic to $M_{5,11}$, and there is a triangle of M that contains both x and y. Suppose that $M\backslash y$ is not 3-connected. Then x is contained in a cocircuit of size at most two in $M\backslash y$, and this implies that M contains a cocircuit of size at most three, a contradiction. Therefore $M\backslash y$ is 3-connected. However, a computer check reveals that $M_{5,11}$ has no 3-connected single-element coextensions in $\mathcal{EX}(M(K_{3,3}))$. ✠34

Subcase 3.4. Statement (iv) of Lemma 6.7 applies.

If statement (i) of Claim 7.15 holds then the case reduces to Subcase 3.2. So we suppose that either (ii) or (iii) of Claim 7.15 holds.

For each of the matroids in Appendix B other than $M_{7,15}$, $M_{9,18}$ and $M_{11,21}$, we perform the following procedure: we generate all single-element coextensions, and then extend by either one or two more elements, at each step restricting to those 3-connected matroids that belong to $\mathcal{EX}(M(K_{3,3}))$. The only internally 4-connected matroids uncovered by this procedure are $M_{5,11}$, T_{12}/e, $M^b_{5,12}$, $M_{5,13}$, T_{12}, and $M_{7,15}$. ✠35

We give an inductive argument to cover the possibility that M_0 is isomorphic to $M_{7,15}$, $M_{9,18}$ or $M_{11,21}$. The base case of our argument is centered on $M^a_{5,12}$. Recall that $M^a_{5,12}$ is isomorphic to $\nabla(F_7^*; \mathcal{T}_4^a)$, where $\mathcal{T}_4^a$ is a set of four triangles in F_7, three of which contain a common point. Proposition 2.37 tells us that each of the triangles in $\mathcal{T}_4^a$ corresponds to an allowable triangle in $\nabla(F_7^*; \mathcal{T}_4^a)$, and Proposition C.2 implies that the allowable triangles on $M^a_{5,12}$ are precisely the triangles that arise in this way.

Sub-subcase 3.4.1. Statement (ii) of Claim 7.15 holds.

We verify the next claim by constructing each of the coextensions of $M^a_{5,12}$ by the element x so that x is in a triad with two elements of an allowable triangle, and then using a computer to check every 3-connected single-element extension in $\mathcal{EX}(M(K_{3,3}))$, ignoring isomorphisms. ✠36

Claim 7.17. Let T be an allowable triangle in $M^a_{5,12}$. Suppose that we construct M' by coextending with x so that it forms a triad T_x with two elements of T, and then extending by the element y so that y is in a triangle T_{xy} with x. If M' is a 3-connected member of $\mathcal{EX}(M(K_{3,3}))$, then either:

(i) T_{xy} meets T; or,
(ii) there exists an allowable triangle T' of $M^a_{5,12}$ such that $T' \neq T$, $|T_{xy} \cap T'| = 1$, and $T_{xy} \cup T'$ contains a cocircuit.

If (i) holds in Claim 7.17 then $T_{xy} \cup \{x, y\}$ is a vertical 3-separator. Suppose that (ii) holds. Then T' is a triangle in M', for otherwise $T' \cup x$ is a circuit in $M'\backslash y$ which meets the triad T_x in one element, x. Now $T_{xy} \cup T'$ is a vertical 3-separator as $r_{M'}(T_{xy} \cup T') = 4$ and $T_{xy} \cup T'$ contains a cocircuit.

Thus Claim 7.17 shows that any matroid obtained from $M^a_{5,12}$ in this way can not be vertically 4-connected.

Claim 7.17 provides the base case for our inductive argument. Recall that $M_{7,15}$ is isomorphic to $\nabla(F_7^*;\, \mathcal{T}_5)$, where $\mathcal{T}_5$ is a set of five triangles in F_7. Proposition 2.37 and Proposition C.4 show that the allowable triangles of $M_{7,15}$ correspond exactly to the members of $\mathcal{T}_5$. Let the allowable triangles of $M_{7,15}$ be $T_1, \ldots, T_5$.

Suppose that M' is obtained from $M_{7,15}$ by coextending with the element x so that it is in a triad T_x with two elements of an allowable triangle (which we will assume to be T_1 by relabeling if necessary), and then extending by the element y so that it is in a triangle T_{xy} with x. Assume that M' is a 3-connected member of $\mathcal{EX}(M(K_{3,3}))$. We wish to show that either T_{xy} meets T_1, or that there is an allowable triangle T' of $M_{7,15}$ such that $T' \neq T_1$, T_{xy} meets T' in exactly one element, and $T_{xy} \cup T'$ contains a cocircuit. If T_{xy} meets T_1 then obviously we are done, so we assume that $T_{xy} \cap T_1 = \varnothing$.

For the element e in F_7 we let t_e be the number of triangles of $\mathcal{T}_5$ that contain e. Then M_1 is the matroid obtained by adding $t_e - 1$ elements in parallel to e, for every element e. If $\mathcal{T}$ is a set of pairwise disjoint triangles in M_1 which correspond to the triangles in $\mathcal{T}_5$, then $M^*_{7,15}$ is isomorphic to the matroid obtained by performing Δ-Y operations on the triangles in $\mathcal{T}$. It is easy to see that $t_e > 1$ for every element e, and from this fact and Proposition 2.27 we deduce that for $i \in \{2, \ldots, 5\}$ we have

$$\Delta(F_7;\, \mathcal{T}_5)\backslash T_i = \nabla_{T_i}(\Delta(F_7;\, \mathcal{T}_5))\backslash T_i = \Delta(F_7;\, \mathcal{T}_5 - T_i). \tag{7.2}$$

Therefore $\nabla(F_7^*;\, \mathcal{T}_5)/T_i = \nabla(F_7^*;\, \mathcal{T}_5 - T_i)$. It is easy to see that there is at most one triangle T_i in $\mathcal{T}_5$ such that $\mathcal{T}_5 - T_i$ is not the same arrangement of triangles as $\mathcal{T}^a_4$. Therefore we will assume that $M_{7,15}/T_2$ and $M_{7,15}/T_3$ are both isomorphic to $M^a_{5,12}$.

Next we argue that T_{xy} is the only triangle of M' that contains y. Suppose that T_y is some other triangle of M' such that $y \in T_y$. Then $(T_y \cup T_{xy}) - y$ is a circuit in $M\backslash y$, and it meets the triad T_x in the element x. Therefore it contains exactly one elements of $T_x - x$, and since $T_x - x$ is contained in T_1 this means that T_y meets T_1. Now T_1 and $(T_y \cup T_{xy}) - \{x,\, y\}$ are both triangles in $M'/x\backslash y = M_{7,15}$. But this is a contradiction, as all triangles of $M_{7,15}$ are disjoint.

Since T_{xy} cannot meet both T_2 and T_3 we assume without loss of generality that T_{xy} does not meet T_2. Therefore there is no triangle of M' that both contains y and meets T_2. Let M'' be M'/T_2. Then $M''/x\backslash y = M^a_{5,12}$. Now T_x is a triad of $M'\backslash y$, and hence of $M''\backslash y$. Furthermore, if T_{xy} is not a triangle of M'' then T_{xy} has a non-empty intersection with T_2, contrary to our assumption. Thus T_{xy} is a triangle of M'' which contains both x and y. By applying Claim 7.17 to M'' we deduce that either T_{xy} meets T_1, or that there is an allowable triangle T' in $M^a_{5,12}$ other than T_1 such that T_{xy} meets T' in one element, and $T_{xy} \cup T'$ contains a cocircuit of M''. We have assumed that $T_{xy} \cap T_1 = \varnothing$, so the latter holds.

The allowable triangles of $M_{5,12}^a = M''/x\backslash y$ are exactly T_1, T_3, T_4, T_5. Hence T' is also an allowable triangle in $M_{7,15}$. Furthermore, the cocircuit which is contained in $T_{xy} \cup T'$ is also a cocircuit of M'. We have shown that Claim 7.17 also holds when $M_{5,12}^a$ is replaced with $M_{7,15}$. By the same argument as used earlier, no matroid obtained from $M_{7,15}$ in the way described can be vertically 4-connected.

But we can use the inductive argument again, and show that Claim 7.17 also holds when $M_{5,12}^a$ is replaced with $M_{9,18}$ or $M_{11,21}$. Therefore we have dealt with the case that (ii) of Claim 7.15 holds.

Sub-subcase 3.4.2. Statement (iii) of Claim 7.15 holds.

We use a computer to check all 3-connected members of $\mathcal{EX}(M(K_{3,3}))$ that are formed from $M_{5,12}^a$ by coextending with x so that it is in a triad with two elements of an allowable triangle T, and then extending by y and z, at each stage considering only those matroids which are 3-connected members of $\mathcal{EX}(M(K_{3,3}))$ and ignoring isomorphisms. The check reveals that in any such matroid, if x and y lie in a triangle T_{xy}, and x and z lie in a triangle T_{xz}, then it is not the case that T_{xy} meets T and T_{xz} avoids T. We summarize ✠37
this in the following claim.

Claim 7.18. Let T be an allowable triangle in $M_{5,12}^a$. Suppose that we construct M' by coextending with x so that it forms a triad T_x with two elements of T, and then extending by the element y so that y is in a triangle T_{xy} such that $x \in T_{xy}$ and $T_{xy} \cap T = \varnothing$, and then extending by the element z so that z is in a triangle T_{xz} such that $x \in T_{xz}$ and $T_{xz} \cap T \neq \varnothing$. If M', $M'\backslash z$, and $M'\backslash y\backslash z$ are all 3-connected then M' has an $M(K_{3,3})$-minor.

Let $T_1, \ldots, T_5$ be the allowable triangles of $M_{7,15}$. Suppose M' is obtained from $M_{7,15}$ by coextending with the element x so that x is in a triad T_x with two elements of T_1, and then extending by the elements y and z so that there are triangles T_{xy} and T_{xz} such that $x, y \in T_{xy}$, $x, z \in T_{xz}$, and $T_{xy} \cap T_1 = \varnothing$, while $T_{xz} \cap T_1 \neq \varnothing$. Suppose also that M', $M'\backslash z$, and $M'\backslash y\backslash z$ are all 3-connected.

Since there is exactly one triangle T_i in $\mathcal{T}_5$ such that $\mathcal{T}_5 - T_i$ is not the same configuration as $\mathcal{T}_4^a$ we can assume that $M_{7,15}/T_2$, $M_{7,15}/T_3$, and $M_{7,15}/T_4$ are all isomorphic to $M_{5,12}^a$. We can argue, as before, that T_{xy} is the only triangle of $M'\backslash z$ that contains y. Therefore any triangle of M' other than T_{xy} that contains y contains z. Therefore there can be at most two triangles of M' that contain y. Without loss of generality we can assume that there is no triangle of M' that contains y and that intersects T_2.

Next we note that T_{xz} has an empty intersection with T_2, since the members of T_{xz} are x, z, and a single element of T_1. Suppose that there is a triangle T_z of M' such that $z \in T_z$ and $T_z \cap T_2 \neq \varnothing$. It cannot be the case that $y \in T_z$, since we have assumed that no triangle of M' containing y meets T_2. Therefore $(T_z \cup T_{xz}) - z$ is a circuit of $M'\backslash y\backslash z$ that contains x. Since T_x is a triad of $M'\backslash y\backslash z$ and $x \in T_x$ it follows that $(T_z \cup T_{xz}) - z$

contains an element of $T_x - x \subseteq T_1$. This means that $(T_z \cup T_{xz}) - \{x, z\}$ is a triangle of $M'/x\backslash y\backslash z = M_{7,15}$ which meets T_1. As the triangles of $M_{7,15}$ are disjoint this is a contradiction, so no such triangle T_z exists.

Now we let M'' be M'/T_2. Then $M''/x\backslash y\backslash z$ is isomorphic to $M^a_{5,12}$. Moreover T_x is a triad of $M''\backslash y\backslash z$ and both T_{xy} and T_{xz} are triangles of M'', since they do not intersect T_2.

If $M''\backslash y\backslash z$ is not 3-connected, then x is in a cocircuit of size at most two in $M''\backslash y\backslash z$, and hence in $M'\backslash y\backslash z$. This is a contradiction as $M'\backslash y\backslash z$ is 3-connected. If $M''\backslash z$ is not 3-connected then y is contained in a circuit of size at most two in $M''\backslash z$. But this implies that y is contained in a triangle of $M'\backslash z$ which meets T_2, contrary to hypothesis. Therefore we assume that $M''\backslash z$ is 3-connected. Similarly, if M'' is not 3-connected, then z is contained in a circuit of size at most two in M'', and this leads to a contradiction, since z is contained in no triangle of M' which meets T_2.

We have shown that M'', $M''\backslash z$, and $M''\backslash y\backslash z$ are all 3-connected. Now Claim 7.18 implies that M'', and hence M', has an $M(K_{3,3})$-minor. Thus Claim 7.18 holds when $M^a_{5,12}$ is replaced with $M_{7,15}$. By using the same inductive argument we can show that Claim 7.18 holds when $M^a_{5,12}$ is replaced with $M_{9,18}$ or $M_{11,21}$. This finishes the case that statement (iv) of Lemma 6.7 holds.

Subcase 3.5. Statement (v) of Lemma 6.7 applies.

In this case M_0 contains a pair of intersecting triangles. Thus M_0 is isomorphic to one of the 13 matroids listed in Proposition C.7. Since M is not a single-element coextension of M_0, it follows from Claim 7.16 that we can find an isomorphic copy of M by constructing all the 3-connected single-element coextensions of M_0 in $\mathcal{EX}(M(K_{3,3}))$, and then extending by one or two more elements, at each step considering only the 3-connected members of $\mathcal{EX}(M(K_{3,3}))$. When we perform this procedure for each of the matroids listed in Proposition C.7 we find that the only internally 4-connected matroids we uncover are isomorphic to $M_{5,11}$, T_{12}/e, $M^b_{5,12}$, $M_{5,13}$, and
✠38 T_{12}.

Subcase 3.6. Statement (vi) of Lemma 6.7 applies.

In this case there is a triangle $\{x, y, z\}$ in M such that $M_0 = M/x/y\backslash z$. If $M/y\backslash z$ is not 3-connected then x is contained in a cocircuit of size at most two in $M/y\backslash z$. This means that there is a cocircuit of size at most three in M, a contradiction. Similarly, $M\backslash z$ is 3-connected. It follows from this argument that we can recover M from M_0 by coextending by two elements, and then extending by a single element, at each step considering only 3-connected binary matroids with no $M(K_{3,3})$-minor. We perform this procedure on the matroids in Appendix B, other than $M_{7,15}$, $M_{9,18}$, and $M_{11,21}$. The only internally 4-connected matroids produced are isomorphic
✠39 to $M_{5,11}$, T_{12}/e, T_{12}, and $M_{7,15}$.

Suppose that $M_0 = M_{7,15}$. Recall that $M_{7,15} = \nabla(F_7^*; \mathcal{T}_5)$, where $\mathcal{T}_5$ is a set of five distinct triangles in F_7. Let M_1 be the matroid derived from F_7 by adding $t_e - 1$ parallel elements to each element e in F_7. Then $\mathcal{T}$ is the set of disjoint triangles in M_1 which correspond to the triangles in $\mathcal{T}_5$. In this case $M_{7,15}^*$ is the matroid obtained from M_1 by performing Δ-Y operations on each of the triangles in $\mathcal{T}$ in turn.

Suppose that $T_1 = \{a_1, b_1, c_1\}$ and $T_2 = \{a_2, b_2, c_2\}$ are triangles of $M_{7,15}$. Proposition 2.37 and the fact that $M_{7,15}$ contains exactly five triangles means that T_1 and T_2 are members of $\mathcal{T}$. Let M' be the binary matroid obtained from $M_{7,15}$ by coextending with the elements x and y such that $\{b_1, c_1, x\}$ and $\{b_2, c_2, y\}$ are triads, and then extending with the element z so that $\{x, y, z\}$ is a triangle. Assume that M' is vertically 4-connected.

Suppose that a_1 and a_2 are parallel elements in M_1. Then $\{a_1, a_2\}$ is a circuit in $M_1 \backslash b_1 \backslash b_2$. But Proposition 2.27 implies that $M_1 \backslash b_1 \backslash b_2$ is isomorphic to $\Delta_{T_2}(\Delta_{T_1}(M_1))/b_1/b_2$ under the function that switches a_1 with c_1 and a_2 with c_2. This implies that $\{b_1, c_1, b_2, c_2\}$ contains a circuit in $\Delta_{T_2}(\Delta_{T_1}(M_1))$, and hence in $M_{7,15}^*$. Since $M_{7,15}$ contains no cocircuits of size less than four it follows that $\{b_1, c_1, b_2, c_2\}$ is a cocircuit of $M_{7,15}$. Since $\{b_1, c_1, x\}$ and $\{b_2, c_2, y\}$ are triads of $M' \backslash z$ it follows that x and y are in series in $M' \backslash z$, and that therefore M' contains a cocircuit of size at most three, a contradiction as M' is vertically 4-connected.

Now we assume that a_1 and a_2 are not in parallel in M_1. Suppose that we add y in parallel to a_1 and x in parallel to a_2 in M_1. Let z' be an element of M_1 such that $\{x, y, z'\}$ is a triangle. If z' is contained in a non-trivial parallel class of M_1 then we obtain M_2 by adding z in parallel to z', otherwise we simply relabel z' with z. Let T be the triangle of F_7 that corresponds to the triangle $\{x, y, z\}$ in M_2. It follows easily from the dual of Lemma 2.30 that $(M')^*$ is equal to the matroid obtained from M_2 by performing Δ-Y operations on all the triangles in $\mathcal{T} \cup \{\{x, y, z\}\}$. Thus M' is equal to $\nabla(F_7^*; \mathcal{T}_5 \cup \{T\})$.

If T does not belong to $\mathcal{T}_5$ then $\mathcal{T}_5 \cup T$ is equal to $\mathcal{T}_6$, a set of six distinct triangles in F_7, and therefore M' is isomorphic to $M_{9,18}$. If T is already contained in $\mathcal{T}_5$ then in M_2 there is a triangle T' such that T' and $\{x, y, z\}$ are disjoint but $r_{M_2}(T' \cup \{x, y, z\}) = 2$. It follows easily that in $(M')^*$ the rank of $T' \cup \{x, y, z\}$ is four. However T' and $\{x, y, z\}$ are both triads of $(M')^*$, and therefore $T' \cup \{x, y, z\}$ is a 3-separator of $(M')^*$. Hence M' is not internally 4-connected, so if $M_0 = M_{7,15}$ then M can only be isomorphic to $M_{9,18}$.

By using a similar argument, we can show that if $M_0 = M_{9,18}$, then M is equal to $M_{11,21}$. However, if $M_0 = M_{11,21}$, then $M_0 = \nabla(F_7^*; \mathcal{T}_7)$. As every triangle of F_7 is contained in $\mathcal{T}_7$ it follows that there is no internally 4-connected binary matroid obtained from M_0 by coextending with elements x and y so that they form triads with two elements from triangles and then extending by z so that $\{x, y, z\}$ is a triangle. This completes the case that statement (vi) applies.

Subcase 3.7. Statement (vii) of Lemma 6.7 applies.

In this case M_0 contains two intersecting triangles, and is therefore one of the matroids listed in Proposition C.7. There is a triangle $\{x,\ y,\ z\}$ in M such that $M_0 = M/x/y\backslash z$. As in Subcase 3.6, we can argue that $M\backslash z$ and $M/y\backslash z$ are 3-connected, so we can construct M from M_0 by coextending twice, and then extending once, at each step considering only 3-connected binary matroids with no $M(K_{3,3})$-minor. When we perform this procedure on the matroids in Proposition C.7 the only internally 4-connected matroids
✠40 produced are isomorphic to $M_{5,11}$, T_{12}/e, T_{12}, and $M_{7,15}$.

Subcase 3.8. Statement (viii) of Lemma 6.7 applies.

In this case we M_0 again contains two intersecting triangles. Furthermore, there is a triangle $\{x,\ y,\ z\}$ in M such that $M_0 = M/x/y\backslash z$. Therefore we can apply exactly the same arguments as in Subcase 3.7 and obtain a contradiction.

Subcase 3.9. Statement (ix) of Lemma 6.7 applies.

As in Subcase 2.9 we look for pairwise disjoint allowable triangles T_1, T_2, and T_3 in M_0 such that $T_i \cup T_j$ contains a cocircuit C^*_{ij} of size four for $1 \leq i < j \leq 3$. We call any such triple of allowable triangles a *good triple*. We then try to reconstruct M from M_0 by coextending in turn with the elements e, f, and g so that $(T_1 \cap C^*_{12}) \cup e$, $(T_2 \cap C^*_{23}) \cup f$, and $(T_3 \cap C^*_{13}) \cup g$, are triads, and finally extending by x so that $\{e,\ f,\ g,\ x\}$ is a circuit.

The proof sketched in Appendix C shows that only six of the matroids listed in Appendix B have allowable triangles. Proposition C.1 makes it clear that $M_{4,11}$ contains no good triple of allowable triangles. Let T_1, T_2, T_3, and T_4 be the four allowable triangles of $M^a_{5,12}$ in the order listed in Proposition C.2. Any pair of these triangles contains a cocircuit of size four. Omitting T_1, T_2, or T_3 from the set of allowable triangles leaves us with a good triple, but performing the procedure described above on this triple
✠41 results in a matroid with an $M(K_{3,3})$-minor. The only four-element cocircuit contained in $T_1 \cup T_2$ is $\{1,\ -1,\ -5,\ -6\}$ and the only four-cocircuit contained in $T_1 \cup T_3$ is $\{1,\ 5,\ -1,\ -4\}$. Thus if we try to perform the procedure on the triple $\{T_1,\ T_2,\ T_3\}$, we will be forced to coextend twice by an element so that it forms a triad with $\{1,\ -1\}$. This means that $M\backslash x$ will have a series pair, and that therefore M will have a cocircuit of size at most three, a contradiction. This finishes the case-check for $M^a_{5,12}$.

Proposition C.3 lists the four allowable triangles of $M_{6,13}$. Any triple of these is a good triple, but performing the procedure described above
✠42 produces a matroid with an $M(K_{3,3})$-minor.

Let $T_1, \ldots, T_5$ be the allowable triangles of $M_{7,15}$, in the order listed in Proposition C.4. Each pair contains a cocircuit of size four. The only four-element cocircuit contained in $T_1 \cup T_2$ is $\{1,\ -1,\ -5,\ -6\}$ and the only four-element cocircuit contained in $T_1 \cup T_3$ is $\{1,\ -1,\ -4,\ 5\}$, and both these cocircuits meet T_1 in $\{1,\ -1\}$. This means that performing the procedure on the triple $\{T_1, T_2, T_3\}$ results in a cocircuit of size at most three. Similarly, the only four-element cocircuit contained in $T_2 \cup T_4$ is $\{3, 6, 7, -6\}$ and the only four-element cocircuit contained in $T_2 \cup T_5$ is $\{3, -2, -3, -6\}$, and both these cocircuits meet T_2 in $\{3, -6\}$, so we can dismiss the triple $\{T_2, T_4, T_5\}$. Performing the procedure on any other triple of triangles in $\{T_1, \dots, T_5\}$ produces a matroid with an $M(K_{3,3})$-minor. ✠43

Suppose that $\{T_1, T_2, T_3\}$ is a triple of allowable triangles in $M_{9,18}$ such that $T_i \cup T_j$ contains a four-element cocircuit C^*_{ij} for $1 \le i < j \le 3$. Suppose that M' is produced from $M_{9,18}$ by coextending by e, f, and g, and then extending by x is the way described above. Then $M' \backslash x/g/f/e = M_{9,18}$.

Let T be a triangle of $M_{9,18}$ that is not in $\{T_1, T_2, T_3\}$. It is not difficult to see that T is a triangle in M', for if it is not then there is a circuit that meets one of the cocircuits $(T_1 \cap C^*_{12}) \cup \{e, x\}$, $(T_2 \cap C^*_{23}) \cup \{f, x\}$, or $(T_3 \cap C^*_{13}) \cup \{g, x\}$ in exactly one element. Let $M'' = M'/T$. Then Equation (7.1) tells us that $M'' \backslash x/g/f/e = M_{7,15}$.

Suppose that $\{e, f, g, x\}$ is not a circuit in M''. Then there is a circuit C of M' which has a non-empty intersection with both T and a proper subset of $\{e, f, g, x\}$. If $x \notin C$ then, by relabeling if necessary, we assume that $e \in C$, and therefore C meets the cocircuit $(T_1 \cap C^*_{12}) \cup \{e, x\}$ of M' in exactly one element. On the other hand, if $x \in C$, then we assume without loss of generality that $e \notin C$, and we reach the same contradiction. Thus $\{e, f, g, x\}$ is indeed a circuit of M''. Moreover, $(T_3 \cap C^*_{13}) \cup g$ is a triad in $M' \backslash x$, and hence in $M'' \backslash x$. We can use exactly the same arguments to show that $(T_2 \cap C^*_{23}) \cup f$ and $(T_1 \cap C^*_{12}) \cup e$ are triads of $M'' \backslash x/g$ and $M'' \backslash x/g/f$ respectively. If T_i is not a triangle of $M'' \backslash x/g/f/e$ for some $i \in \{1, 2, 3\}$, then $T_i \cup T$ has rank at most three in $M' \backslash x/g/f/e = M_{9,18}$. This would imply that $M_{9,18}$ contains a parallel pair, a contradiction. Thus T_i is a triangle of $M'' \backslash x/g/f/e$ for all $i \in \{1, 2, 3\}$. Similarly, C^*_{ij} is a cocircuit of $M'' \backslash x/g/f/e$ for $1 \le i < j \le 3$.

The above arguments show that M'' can be obtained from $M_{7,15}$ in exactly the same way that M' was obtained from $M_{9,18}$. By the computer checking described earlier this means that M'' either has an $M(K_{3,3})$-minor, or a cocircuit with at most three elements. If M'' has an $M(K_{3,3})$-minor then so does M', and similarly, if M'' contains a cocircuit of size at most three, then so does M'. These arguments mean that M_0 cannot be isomorphic $M_{9,18}$. By using the same arguments we can show that M_0 cannot be equal to $M_{11,21}$.

This completes the case that statement (ix) applies. With this we have completed the case-checking and we conclude that the counterexample M does not exist. Therefore Theorem 7.2 holds. □

APPENDIX A

Case-checking

The next proposition sketches the case-check needed to complete the proof of Corollary 5.3.

Proposition A.1. *Suppose that $M \in \mathcal{EX}(M(K_{3,3}))$ is a 3-connected matroid with a four-element circuit-cocircuit C^* such that M has a Δ_4-minor, but if $e \in E(M) - C^*$, then neither $M\backslash e$ nor M/e has a Δ_4-minor. Then M has a Δ_4^+-minor.*

Proof. Let $\mathcal{M}$ be the class of labeled 3-connected matroids in $\mathcal{EX}(M(K_{3,3}),\, \Delta_4^+)$. Assume that $M \in \mathcal{M}$ is a counterexample to the proposition. Thus M has both a four-element circuit-cocircuit C^* and a Δ_4-minor. Let us first suppose that there is no element $x \in E(M)$ such that M/x has a Δ_4-minor. It cannot be the case that $M \cong \Delta_4$, since Δ_4 has no four-element circuit-cocircuit. Thus $M\backslash y$ has a Δ_4-minor for some $y \in C^*$. But $C^* - y$ is a triad of $M\backslash y$ and Δ_4 has no triads. Hence we must contract some element $x \in C^* - y$ from $M\backslash y$ to obtain a Δ_4-minor. Thus there is an element $x \in C^*$ such that M/x has a Δ_4-minor.

Suppose that M/x has a 2-separation (X, Y) such that $|X|, |Y| \geq 3$. Then $x \in \mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$. Without loss of generality we will assume that X contains two elements of $C^* - x$. Thus $C^* \subseteq \mathrm{cl}_M(X)$, and the assumption $|X|$ and $|Y|$ means that $(X \cup C^*,\, Y - C^*)$ is a 2-separation of M, a contradiction. Thus if (X, Y) is a 2-separation of M/x then without loss of generality $|X| = 2$. In this case there is a triangle T of M which contains x. It must be the case that T contains a single element t of $E(M) - C^*$. But t is in a parallel pair in M/x, and as Δ_4 has no parallel pairs it follows that $M/x\backslash t$ has a Δ_4-minor. This is a contradiction as $t \notin C^*$. Therefore M/x is 3-connected.

Suppose that Δ_4 is represented over GF(2) by the matrix $[I_4|A]$, where A is the matrix shown in Figure 3.1. Consider all the binary matrices obtained by adding a column to $[I_4|A]$. Suppose that this new column is labeled by the element e. Let EX be the set of labeled matroids in $\mathcal{M}$ represented over GF(2) by such a matrix. Ignoring isomorphism, there are five matroids in EX (recall that the members of $\mathcal{M}$ are 3-connected). Every single-element ✠44
extension of Δ_4 in $\mathcal{M}$ is isomorphic to a matroid in EX. Similarly, let CO be the set of matroids in $\mathcal{M}$ that are represented over GF(2) by a matrix $[I_5|A']$, where A' is obtained by adding a row to A. Again we will assume

that every matroid in CO is a coextension of Δ_4 by the element e. There
✠45 are fifteen matroids in CO, ignoring isomorphisms.

None of the matroids in CO has a four-element circuit-cocircuit, so
✠46 $M/x \not\cong \Delta_4$. By Theorem 2.12 there is an element $y \in E(M/x)$ such that either $M/x\backslash y$ or $M/x/y$ is 3-connected with a Δ_4-minor. Then y must be in $C^* - x$. Contracting any element of $C^* - x$ in M/x creates a parallel pair, so $M/x\backslash y$ is 3-connected with a Δ_4-minor.

Assume that $M/x\backslash y \cong \Delta_4$. Then M/x is isomorphic to a member of EX, and this isomorphism takes y to e. Thus e appears in a triangle. In three of the five matroids in EX, e appears in four triangles, and in two e appears in three triangles. Given a binary matroid N and a triangle T of N, there is a unique coextension of N by the element x such that $T \cup x$ is a four-element circuit-cocircuit of the coextension. Thus for each matroid $N \in$ EX and each triangle of N that contains e we construct the corresponding unique coextension of N. One of the resulting matroids is isomorphic to M. However, each of these eighteen matroids has either an
✠47 $M(K_{3,3})$-minor or a Δ_4^+-minor. We conclude that $M/x\backslash y \not\cong \Delta_4$.

We will next suppose that there is some element $z \in C^* - \{x, y\}$ such that $M/x\backslash y/z \cong \Delta_4$. Then $M/x\backslash y$ is isomorphic to a member of CO. Thus, for each matroid $N \in$ CO we consider all the extensions of N by the element f such that the extension belongs to $\mathcal{M}$. If e and f appear together in a triangle in the resulting matroid we construct the unique coextension that creates a four-element circuit-cocircuit. One of the resulting matroids is isomorphic to M.

The fifteen members of CO each have either six or seven single-element extensions that belong to $\mathcal{M}$ (ignoring isomorphisms), and in each case, either five or six of these extensions have triangles containing both e and f.
✠48 In total there are 78 candidate to check, but each of these has a Δ_4^+-minor.

Next we assume that $M/x\backslash y\backslash z \cong \Delta_4$ for some element $z \in C^* - \{x, y\}$. For each matroid $N \in$ EX we consider the extensions of N belonging to $\mathcal{M}$ by the element f. If e and f are contained in a triangle of the resulting matroid we construct the corresponding unique coextension. Each of the five matroids in EX has four single-element extensions in $\mathcal{M}$ (ignoring isomorphisms). Three of the matroids in EX have two extensions each, in which the two new elements are contained in a triangle. Every extension of the other two matroids in EX has a triangle containing the two new elements. This leads to a total of 14 matroids to be checked. Each of the 14 has an
✠49 $M(K_{3,3})$-minor.

Let $C^* - \{x, y\} = \{z, w\}$. There are four remaining cases to check. In the first $M/x\backslash y/z$ is 3-connected and $M/x\backslash y/z/w \cong \Delta_4$. For each matroid $N \in$ CO we construct the coextensions by the element f that belong to $\mathcal{M}$. We then extend so that the new element makes a triangle with e and f, and then construct the unique coextension that creates a four-element circuit-cocircuit. The fifteen matroids in CO have either zero, six, or eight
✠50 coextensions in $\mathcal{M}$. There are 84 candidates to check.

In the second of the four cases $M/x\backslash y/z$ is 3-connected and $M/x\backslash y/z\backslash w \cong \Delta_4$. For each matroid $N \in \mathrm{EX}$ we construct the coextensions of N by the element f that belong to $\mathcal{M}$. In no such coextension are e and f contained in a triangle, so we proceed as in the previous paragraph. Three of the five matroids in EX have nine single-element coextensions in $\mathcal{M}$ each. The other two have no such coextensions. Thus there are 27 candidates to check. ✠51

In the next case we assume that $M/x\backslash y\backslash z$ is 3-connected and $M/x\backslash y\backslash z/w \cong \Delta_4$. For each matroid $N \in \mathrm{CO}$ we construct the extensions of N belonging to $\mathcal{M}$ by the element f. If e and f are contained in a triangle of the resulting matroid then we need proceed no further, since y is not contained in a parallel pair of M/x. Otherwise we proceed as in the previous two cases. The fifteen matroids in CO have six or seven extensions each that belong to $\mathcal{M}$. In each case either one or two of these extensions does not have a triangle containing both e and f. There are 21 candidates to check. ✠52

Finally we assume that $M/x\backslash y\backslash z$ is 3-connected and $M/x\backslash y\backslash z\backslash w \cong \Delta_4$. Each of the five matroids in EX has four single-element extensions in $\mathcal{M}$. In three of the five cases exactly two of the extensions do not have triangles containing both e and f. In the other two cases all of the extensions have triangles containing e and f. This leads to a total of 6 candidates to check. Each of these has either an $M(K_{3,3})$-minor or a Δ_4^+-minor. Thus we have ✠53
exhausted the supply of matroids that could potentially be isomorphic to M. We conclude that no counterexample exists and that the result holds. $\square$

APPENDIX B

Sporadic matroids

There are 27 internally 4-connected non-cographic matroids in $\mathcal{EX}(M(K_{3,3}))$ with rank at most 11 and ground sets of size at most 21. They are categorized in Tables 1 and 2. Five of these matroids are triangular Möbius matroids and four are triadic Möbius matroids (recall that Δ_3 and Υ_4 are isomorphic to the Fano plane and its dual respectively). The other 18 matroids are sporadic members of the class. All of the matroids listed in this appendix are vertically 4-connected, with the exception of $M_{5,11}$, which contains a single triad. ✠54

In the following pages we give matrix representations of these sporadic matroids. If A is one of the matrices displayed in Figures B.1 to B.9 then $[I_{r(M)}|A]$ is a GF(2)-representation of a sporadic matroid M.

size	3	4	5	6
16				Δ_6
15		PG(3, 2)		
14		$M_{4,14}$		
13		$M_{4,13}$	Δ_5, $M_{5,13}$	$M_{6,13}$
12		C_{12}, D_{12}	$M^a_{5,12}$, $M^b_{5,12}$	T_{12}
11		C_{11}, $M_{4,11}$	$M_{5,11}$, T_{12}/e	Υ_6
10		Δ_4, $M(K_5)$		
9				
8				
7	F_7	F_7^*		
	3	4	5	6 rank

Table 1: Internally 4-connected non-cographic matroids in $\mathcal{EX}(M(K_{3,3}))$.

size	7	8	9	10	11
21					$M_{11,21}$
20					
19	Δ_7			Υ_{10}	
18			$M_{9,18}$		
17					
16					
15	$M_{7,15}$	Υ_8			
	7	8	9	10	11 rank

Table 2: Internally 4-connected non-cographic matroids in $\mathcal{EX}(M(K_{3,3}))$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure B.1: Matrix representations of $M(K_5)$ and C_{11}.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Figure B.2: Matrix representations of $M_{4,11}$ and C_{12}.

C_{12} is isomorphic to the matroid produced by deleting a set of three collinear points from PG(3, 2).

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Figure B.3: Matrix representations of D_{12} and $M_{4,13}$.

D_{12} is isomorphic to the matroid produced by deleting a set of three non-collinear points from PG(3, 2).

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Figure B.4: Matrix representations of $M_{4,14}$ and PG(3, 2).

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Figure B.5: Matrix representations of $M_{5,11}$ and T_{12}/e.

$M_{5,11}$ is not vertically 4-connected. It contains exactly one triad: the set $\{4, 5, -6\}$, using the convention that if A is the matrix given so that $M_{5,11}$ is represented by $[I_5|A]$, then the columns of I are labeled by $1, \ldots, 5$, and the columns of A are labeled by $-1, \ldots, -6$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure B.6: Matrix representations of $M^a_{5,12}$ and $M^b_{5,12}$.

$M^a_{5,12}$ is isomorphic to $\nabla(F_7^*; \mathcal{T}_4^a)$, where $\mathcal{T}_4^a$ is a set of four triangles in the Fano plane, three of which contain a common point.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Figure B.7: Matrix representations of $M_{5,13}$ and T_{12}.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure B.8: Matrix representations of $M_{6,13}$ and $M_{7,15}$.

$M_{6,13}$ is isomorphic to $\nabla(F_7^*; \mathcal{T}_4^b)$, where $\mathcal{T}_4^b$ is a set of four triangles in the Fano plane, no three of which contain a common point, and $M_{7,15}$ is isomorphic to $\nabla(F_7^*; \mathcal{T}_5)$, where $\mathcal{T}_5$ is a set of five triangles in the Fano plane.

$$\begin{bmatrix} 1&0&0&0&1&1&0&0&0\\ 1&1&0&0&0&0&1&0&0\\ 0&1&1&0&0&1&0&0&0\\ 0&0&1&1&0&0&1&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&1&0&0&0&0&1&1&0\\ 0&1&1&0&1&1&0&1&0\\ 0&0&1&0&0&0&1&0&1\\ 0&1&1&0&1&1&0&0&1 \end{bmatrix} \qquad \begin{bmatrix} 1&0&0&0&1&1&0&0&0&0\\ 1&1&0&0&0&0&1&0&0&0\\ 0&1&1&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&1&0&0&0\\ 0&0&0&1&1&1&0&0&0&0\\ 0&1&0&0&0&0&1&1&0&0\\ 0&1&1&0&1&1&0&1&0&0\\ 0&0&1&0&0&0&1&0&1&0\\ 0&1&1&0&1&1&0&0&1&0\\ 0&1&0&0&1&1&1&0&0&1\\ 0&0&1&0&1&1&1&0&0&1 \end{bmatrix}$$

Figure B.9: Matrix representations of $M_{9,18}$ and $M_{11,21}$.

$M_{9,18}$ is isomorphic to $\nabla(F_7^*; \mathcal{T}_6)$, where $\mathcal{T}_6$ is a set of six triangles in the Fano plane, and $M_{11,21}$ is isomorphic to $\nabla(F_7^*; \mathcal{T}_7)$, where $\mathcal{T}_7$ is the set of all seven lines in the Fano plane.

APPENDIX C

Allowable triangles

Recall that if T is a triangle of the matroid $M \in \mathcal{EX}(M(K_{3,3}))$, and $\Delta_T(M)$ has no $M(K_{3,3})$-minor, then T is an allowable triangle of M. In this appendix we consider the sporadic matroids listed in Appendix B, and we give an outline of how a computer check can determine all their allowable triangles.

Suppose that a sporadic matroid M is represented in Appendix B by the matrix $[I|A]$. We adopt the convention that the columns of I are labeled by the positive integers $1, \ldots, r(M)$, while the columns of A are labeled by the negative integers $-1, \ldots, -r(M^*)$.

The next results can be checked by computer. ✠55

Proposition C.1. *There are* 13 *triangles in* $M_{4,11}$. *Of these,* 3 *are allowable:* $\{2, -3, -7\}$*,* $\{3, -2, -7\}$*, and* $\{4, -5, -7\}$.

Proposition C.2. *There are* 8 *triangles in* $M^a_{5,12}$. *Of these* 4 *are allowable:* $\{1, 2, -1\}$*,* $\{3, -5, -6\}$*,* $\{4, 5, -4\}$ *and* $\{-2, -3, -7\}$. *Any pair of these triangles contains a cocircuit of size four.*

Proposition C.3. *There are* 4 *triangles in* $M_{6,13}$*:* $\{1, 2, -1\}$*,* $\{3, 4, -2\}$*,* $\{5, 6, -3\}$*, and* $\{-4, -5, -6\}$. *Each of these is allowable. Any pair of these triangles contains a cocircuit of size four.*

Proposition C.4. *There are* 5 *triangles in* $M_{7,15}$*:* $\{1, 2, -1\}$*,* $\{3, -5, -6\}$*,* $\{4, 5, -4\}$*,* $\{6, 7, -8\}$*, and* $\{-2, -3, -7\}$. *Each of these is allowable. Any pair of these triangles contains a cocircuit of size four.*

Proposition C.5. *There are* 6 *triangles in* $M_{9,18}$*:* $\{1, 2, -1\}$*,* $\{3, -5, -6\}$*,* $\{4, 5, -4\}$*,* $\{6, 7, -8\}$*,* $\{8, 9, -9\}$*, and* $\{-2, -3, -7\}$. *Each of these is allowable. Any pair of these triangles contains a cocircuit of size four.*

Proposition C.6. *There are* 7 *triangles in* $M_{11,21}$*:* $\{1, 2, -1\}$*,* $\{3, -5, -6\}$*,* $\{4, 5, -4\}$*,* $\{6, 7, -8\}$*,* $\{8, 9, -9\}$*,* $\{10, 11, -10\}$*, and* $\{-2, -3, -7\}$. *Each of these is allowable. Any pair of these triangles contains a cocircuit of size four.*

Any sporadic matroid that is not listed in Propositions C.1 to C.6 contains no allowable triangles. We now sketch a proof of this fact. We will make repeated use of the fact that if the triangle T is not allowable in the matroid N, and N' has N as a minor, then T is not allowable in N'. This follows immediately from Proposition 2.26.

There are 12 triangles in C_{11}, and we can check by computer than none of
✠56 them is allowable. Any matroid produced from C_{12} by deleting an element is
✠57 isomorphic to C_{11}. Therefore any triangle of C_{12} is not allowable in a minor
of C_{12}, and hence not allowable in C_{12} itself. Thus C_{12} has no allowable
triangles.

Performing a Δ-Y operation on $M(K_5)$ produces a non-planar graphic matroid, since the Δ-Y and Y-Δ operations preserves planarity. Thus performing a Δ-Y operation on any triangle of $M(K_5)$ produces a matroid with an $M(K_{3,3})$-minor. Hence $M(K_5)$ has no allowable triangles.

There are 17 triangles in D_{12}, and we can check by computer that none is
✠58 allowable. Deleting any element other than 4 from $M_{4,13}$ produces a matroid
✠59 isomorphic to D_{12}. It follows that $M_{4,13}$ has no allowable triangles. Delet-
✠60 ing any element at all from $M_{4,14}$ produces a minor isomorphic to $M_{4,13}$.
Therefore there are no allowable triangles in $M_{4,13}$. Similarly, it is obviously
true that deleting any element from PG(3, 2) produces a minor isomorphic
to $M_{4,14}$. Thus PG(3, 2) has no allowable triangles.

There are four triangles in $M_{5,11}$, and five in T_{12}/e. None of these is
✠61 allowable. Next we note that $M^b_{5,12}$ is an extension of T_{12}/e by the element
-7. Therefore any triangle that does not contain -7 is not allowable. The
two triangles that contain -7 are $\{1, -6, -7\}$ and $\{-2, -4, -7\}$. Neither
✠62 of these is allowable.

The matroid produced from $M_{5,13}$ by deleting any of the elements in
✠63 $\{1, 5, -7, -8\}$ is isomorphic to $M^b_{5,12}$. Any triangle avoids at least one of
these elements, so $M_{5,13}$ has no allowable triangles.

✠64 Finally, T_{12} has no triangles.

✠65 The next result can be checked by computer.

Proposition C.7. *The only matroids listed in Appendix* B *that contain a pair of intersecting triangles are:* $M(K_5)$, C_{11}, $M_{4,11}$, C_{12}, D_{12}, $M_{4,13}$, $M_{4,14}$, PG(3, 2), $M_{5,11}$, T_{12}/e, $M^a_{5,12}$, $M^b_{5,12}$, *and* $M_{5,13}$.

The only sporadic matroids not listed in this previous result are: T_{12}, $M_{6,13}$, $M_{7,15}$, $M_{9,18}$, $M_{11,21}$.

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