EXCLUDING KURATOWSKI GRAPHS AND THEIR DUALS FROM BINARY MATROIDS

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Abstract. We consider some applications of our characterisation of the internally 4-connected binary matroids with no $M(K_{3,3})$-minor. We characterise the internally 4-connected binary matroids with no minor in $\mathcal{M}$, where $\mathcal{M}$ is a subset of $\{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\}$ that contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. We also describe a practical algorithm for testing whether a binary matroid has a minor in $\mathcal{M}$. In addition we characterise the growth-rate of binary matroids with no $M(K_{3,3})$-minor, and we show that a binary matroid with no $M(K_{3,3})$-minor has critical exponent over GF(2) at most equal to four.

1. Introduction

Earlier, we proved the following theorem.

Theorem 1.1 ([10, Theorem 1.1]). *An internally* 4*-connected binary matroid M has no minor isomorphic to $M(K_{3,3})$ if and only if M is either:*

(i) *cographic,*
(ii) *isomorphic to a triangular or triadic Möbius matroid, or*
(iii) *isomorphic to one of* 18 *sporadic matroids.*

The 18 sporadic matroids appearing in Theorem 1.1 have ground sets of cardinality at most 21, and have rank at most 11. Their matrix representations appear in Appendix B of [10]. Möbius matroids are single-element extensions of the cographic matroids corresponding to two families of graphs: The *cubic Möbius ladder* CM_{2n} is obtained from an even cycle with vertex sequence $v_0, \ldots, v_{2n-1}$ by joining each vertex v_i to the antipodal vertex v_{i+n}. (Indices are read modulo $2n$.) The *quartic Möbius ladder* QM_{2n+1} is obtained from an odd cycle with vertex sequence $v_0, \ldots, v_{2n}$ by joining each vertex v_i to the two antipodal vertices v_{i+n} and v_{i+n+1}. (Indices are read modulo $2n+1$.)

Let $r > 2$ be an integer and let $\{e_1, \ldots, e_r\}$ be the standard basis of $\mathrm{GF}(2)^r$. For $1 \le i \le r-1$ let a_i be the sum of e_i and e_r, and for $1 \le i \le r-2$ let b_i be the sum of e_i and e_{i+1}. Let b_{r-1} be the sum of e_1, e_{r-1}, and e_r. The rank-r *triangular Möbius matroid*, denoted by Δ_r, is represented by the set $\{e_1, \ldots, e_r, a_1, \ldots, a_{r-1}, b_1, \ldots, b_{r-1}\}$. (We also take this set to be the ground set of Δ_r.) Deleting e_r from Δ_r produces a copy of $M^*(CM_{2r-2})$. It is easy to see that if $r \ge 4$, then Δ_r has Δ_{r-1} as a minor.

Now let $r \geq 4$ be an even integer, and again let $\{e_1, \ldots, e_r\}$ be the standard basis of $\mathrm{GF}(2)^r$. For $1 \leq i \leq r-2$ let c_i be the sum of e_i, e_{i+1}, and e_r. Let c_{r-1} be the sum of e_1, e_{r-1}, and e_r. The rank-r *triadic Möbius matroid*, denoted by Υ_r, is represented by the set $\{e_1, \ldots, e_r, c_1, \ldots, c_{r-1}\}$. If $r \geq 4$ is an even integer then $\Upsilon_r \backslash e_r$ is isomorphic to $M^*(QM_{r-1})$. If $r > 4$, then Υ_r has Υ_{r-2} as a minor.

This sequel explores various applications of Theorem 1.1. If $\mathcal{M}$ is a set of binary matroids, then $\mathrm{EX}_2(\mathcal{M})$ is the set of binary matroids that have no minor isomorphic to a member of $\mathcal{M}$. Throughout this introduction, we let $\mathcal{M}$ be some subset of $\{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\}$ that contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. First of all, by using Theorem 1.1, and the classical results by Hall and Wagner on graphs with no $K_{3,3}$- or K_5-minor, we can obtain additional characterisations. In Section 2 we list descriptions of the internally 4-connected matroids in $\mathrm{EX}_2(\mathcal{M})$. Thus we characterise the internally 4-connected members in twelve different families of binary matroids. Only the smallest of these classes has been characterised before [13].

The Graph Minors Project of Robertson and Seymour showed that there is a polynomial-time algorithm for testing whether a graph contains a fixed minor [14]. Similarly, the Matroid Minors Project of Geelen, Gerards, and Whittle [2] is expected to show that the following problem has a polynomial-time solution for each $\mathrm{GF}(q)$-representable matroid N: Given a matrix, A, over the field $\mathrm{GF}(q)$, decide whether the matroid $M[A]$ has an N-minor. However, the existence proofs of these algorithms are very non-constructive. In Section 3 we present algorithms that could actually be implemented. In particular, we present an algorithm that will decide whether $M[A]$ has a minor in $\mathcal{M}$, where A is a matrix over $\mathrm{GF}(2)$. The algorithm runs in $O(n^{13})$ steps, where n is the number of columns in A.

A very well-known example due to Seymour [16] shows that an oracle algorithm for testing whether a matroid is binary cannot run in polynomial time relative to the size of the ground set. As we discuss in Section 3, the same example shows that there is no polynomial-time oracle algorithm for testing whether a matroid is binary with no minor in $\mathcal{M}$. However, this difficulty vanishes when we restrict ourselves to internally 4-connected matroids: There is a polynomial-time oracle algorithm that tests whether an internally 4-connected matroid (not necessarily binary) belongs to $\mathrm{EX}_2(\mathcal{M})$. We conjecture that this is a general phenomenon: the problems created by Seymour's examples can be eliminated with higher connectivity.

Conjecture 1.2. *There is a polynomial-time oracle algorithm for deciding if an internally* 4*-connected matroid is binary.*

This is an ambitious conjecture. The next is somewhat more modest.

Conjecture 1.3. *There is a polynomial-time oracle algorithm for deciding whether an internally* 4*-connected matroid belongs to any given proper minor-closed class of binary matroids.*

The matroids in Seymour's example are known as binary spikes. Spikes are a notorious source of difficulty in matroid theory. A *spike-like flower* of order n in a 3-connected matroid M is a partition $(P_1, \ldots, P_n)$ of the ground set of M such that, for every proper subset J of $\{1, \ldots, n\}$ the partition $(\cup_{j \in J} P_j, E(M) - \cup_{j \in J} P_j)$ is an exact 3-separation of M; and, for all distinct i and j in $\{1, \ldots, n\}$ we have $r(P_i \cup P_j) = r(P_i) + r(P_j) - 1$. A rank-$n$ spike contains a spike-like flower of order n. We believe the existence of large spike-like flowers is at the heart of the difficulty of recognising binary matroids. This belief is encapsulated by the next conjecture, which is a strengthening of Conjecture 1.2.

Conjecture 1.4. *Let k be a fixed positive integer. There is a polynomial-time oracle algorithm for deciding if a 3-connected matroid with no spike-like flower of order k is a binary matroid.*

In the final two sections of the paper, we consider growth-rates and critical exponents. In Section 4 we use Theorem 1.1 to determine the maximum size of a simple rank-r binary matroid with no $M(K_{3,3})$-minor. Moreover, we characterise the matroids that obtain this upper bound. This completely resolves a question studied by Kung [8]. He showed that a simple rank-r binary matroid M without an $M(K_{3,3})$-minor has at most $10r$ elements. Theorem 4.2 shows that, in fact, $|E(M)| \leq 14r/3 - \alpha(r)$, where $\alpha(r)$ assumes one of three values depending on the residue of r modulo 3. Any matroid meeting this bound can be obtained by starting with either $\mathrm{PG}(1,2)$, $\mathrm{PG}(2,2)$, or $\mathrm{PG}(3,2)$, and then repeatedly adding copies of $\mathrm{PG}(3,2)$ via parallel connections along points.

If M is a matroid, then its characteristic polynomial, $\chi(M;t)$, is a polynomial in the variable t, and naturally generalises the chromatic polynomial of a graph. If M is loopless and representable over $\mathrm{GF}(q)$, then the critical exponent of M over q, denoted $c(M;q)$, is the smallest positive integer k such that $\chi(M;q^k) \neq 0$. The material in Section 4 shows that $|E(M)| \leq 5r(M)$, for every simple binary matroid, M, with no $M(K_{3,3})$-minor. It therefore follows from Lemma 7.5 in [9] that the critical exponent of such a matroid is at most 5. Kung had already shown that the critical exponent is at most 10 [8]. In Section 5 we improve these bounds by showing that any loopless binary matroid with no $M(K_{3,3})$-minor has a critical exponent over $\mathrm{GF}(2)$ of at most 4. This result cannot be improved: we also characterise the matroids with critical exponent exactly equal to 4: They are precisely those with a 3-connected component isomorphic to $\mathrm{PG}(3,2)$.

2. Additional classes

Kung initiated the study of binary matroids that have no minor isomorphic to one of the graphic matroids of the Kuratowski graphs, $K_{3,3}$ and K_5 [8]. We extend his programme here. Recall that if $\mathcal{M}$ is a set of binary matroids, then $\mathrm{EX}_2(\mathcal{M})$ is the class of binary matroids that have no minors isomorphic to members of $\mathcal{M}$. Thus Theorem 1.1 gives a structural characterisation of $\mathrm{EX}_2(\{M(K_{3,3})\})$. Let $\mathcal{M}$ be a subset of the collection $\{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\}$ such that $\mathcal{M}$ contains either $M(K_{3,3})$ or its dual. There are exactly twelve such subsets, leading to twelve classes of the form $\mathrm{EX}_2(\mathcal{M})$. By using Theorem 1.1, and classical results by Hall [3] and Wagner [17], we obtain characterisations of the internally 4-connected matroids in each of these twelve classes. To do so, we occasionally need to check whether certain binary matroids have particular minors. We accomplish this task by using the matroid capabilities of the Sage mathematics package (`www.sagemath.org`). We start with some preliminary lemmas.

Lemma 2.1. *The triangular and triadic Möbius matroids have no $M(K_5)$-minors.*

Proof. Lemma 3.8 of [10] states that the only internally 4-connected non-cographic minors of Möbius matroids are themselves Möbius matroids. Thus if a Möbius matroid had an $M(K_5)$-minor it would imply that $M(K_5)$ is a Möbius matroid. It is easily seen that this is not the case. □

The next lemma follows from Wagner's characterisation of graphs with no K_5-minor (see [5, Theorem 1.6]).

Lemma 2.2. *If M is an internally 4-connected cographic matroid with no minor isomorphic to $M^*(K_5)$ then either $M = M^*(G)$, where G is a planar graph, or M is isomorphic to either $M^*(K_{3,3})$ or $M^*(CM_8)$.*

Now to the characterisations of the twelve families. All sporadic matroids are described in Appendix B of [10]. The next result follows from Theorem 1.1, Lemma 2.1, and a simple computer check.

Theorem 2.3. *An internally 4-connected matroid M belongs to $\mathrm{EX}_2(\{M(K_{3,3}), M(K_5)\})$ if and only if M is either:*

(i) *cographic,*
(ii) *isomorphic to a triangular or triadic Möbius matroid, or*
(iii) *isomorphic to one of the sporadic matroids C_{11}, C_{12}, $M^a_{5,12}$, $M_{6,13}$, $M_{7,15}$, $M_{9,18}$, or $M_{11,21}$.*

It is easy to check that Δ_6 has an $M^*(K_5)$-minor, and therefore Δ_r has an $M^*(K_5)$-minor for all $r \geq 6$. On the other hand Δ_r has no $M^*(K_5)$-minor if $r \in \{3, 4, 5\}$. Similarly Υ_r has an $M^*(K_5)$-minor if $r \geq 6$, but Υ_4 has no $M^*(K_5)$-minor. We note that $\Delta_3 \cong F_7$ and $\Upsilon_4 \cong F_7^*$. The next theorem follows from these facts, and by applying Theorem 1.1, Lemma 2.2, and some computer tests. The sporadic matroid T_{12} was introduced in [6].

Theorem 2.4. *An internally 4-connected matroid M belongs to $\mathrm{EX}_2(\{M(K_{3,3}), M^*(K_5)\})$ if and only if M is either:*

(i) *planar graphic,*
(ii) *isomorphic to one of the cographic matroids $M^*(K_{3,3})$ or $M^*(CM_8)$,*
(iii) *isomorphic to one of the Möbius matroids F_7, F_7^*, Δ_4, Δ_5, or*

(iv) *isomorphic to one of the* 18 *sporadic matroids of* Theorem 1.1*, other than* T_{12}*.*

Next we consider $\mathrm{EX}_2(\{M(K_{3,3}), M^*(K_{3,3})\})$. A result due to Hall [3] implies that the only 3-connected cographic matroids with no $M^*(K_{3,3})$-minor are $M^*(K_5)$, and cycle matroids of planar graphs. The only Möbius matroids with no $M^*(K_{3,3})$-minor are Δ_3, Υ_4, and Υ_6. Next we consider the sporadic matroids. As Δ_4 has an $M^*(K_{3,3})$-minor, we need only consider sporadic matroids with no Δ_4-minor. By a result in [6] the matroid T_{12} has a transitive automorphism group, so $T_{12}\backslash e$ and T_{12}/e are well-defined. Corollary 2.15 of [10] says that the only internally 4-connected non-cographic matroids in $\mathrm{EX}_2(\{M(K_{3,3}), \Delta_4\})$ are F_7, F_7^*, $M(K_5)$, $T_{12}\backslash e$, T_{12}/e, and T_{12}. None of these matroids has an $M^*(K_{3,3})$-minor. Both T_{12} and T_{12}/e are among the sporadic matroids of Theorem 1.1, while $F_7 \cong \Delta_3$, $F_7^* \cong \Upsilon_4$, and $T_{12}\backslash e \cong \Upsilon_6$ are all Möbius matroids. The next result follows.

Theorem 2.5. *An internally* 4*-connected matroid* M *belongs to* $\mathrm{EX}_2(\{M(K_{3,3}), M^*(K_{3,3})\})$ *if and only if* M *is either:*

(i) *planar graphic,*
(ii) *isomorphic to the cographic matroid* $M^*(K_5)$*,*
(iii) *isomorphic to one of the Möbius matroids* F_7*,* F_7^**, or* $T_{12}\backslash e$*, or*
(iv) *isomorphic to one of the sporadic matroids* $M(K_5)$*,* T_{12}/e*, or* T_{12}*.*

The next theorems are easy consequences of results stated above.

Theorem 2.6. *An internally* 4*-connected matroid* M *belongs to* $\mathrm{EX}_2(\{M(K_{3,3}), M(K_5), M^*(K_5)\})$ *if and only if* M *is either:*

(i) *planar graphic,*
(ii) *isomorphic to one of the cographic matroids* $M^*(K_{3,3})$ *or* $M^*(CM_8)$*,*
(iii) *isomorphic to one of the Möbius matroids* F_7*,* F_7^**,* Δ_4*,* Δ_5*, or*
(iv) *isomorphic to one of the sporadic matroids* C_{11}*,* C_{12}*,* $M^a_{5,12}$*,* $M_{6,13}$*,* $M_{7,15}$*,* $M_{9,18}$*, or* $M_{11,21}$*.*

Theorem 2.7. *An internally* 4*-connected matroid* M *belongs to* $\mathrm{EX}_2(\{M(K_{3,3}), M^*(K_{3,3}), M(K_5)\})$ *if and only if* M *is either:*

(i) *planar graphic,*
(ii) *isomorphic to the cographic matroid* $M^*(K_5)$*, or*
(iii) *isomorphic to one of the Möbius matroids* F_7*,* F_7^**, or* $T_{12}\backslash e$*.*

Finally, we have the following characterisation, which has already been proved by Qin and Zhou [13].

Theorem 2.8. *An internally* 4*-connected matroid* M *belongs to* $\mathrm{EX}_2(\{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\})$ *if and only if* M *is either:*

(i) *planar graphic, or*
(ii) *isomorphic to one of* F_7 *or* F_7^**.*

Theorem 1.1 and Theorems 2.3 to 2.8 gives us characterisations of seven families. Dualising these theorems gives us five additional characterisations (since Theorems 2.5 and 2.8 characterise self-dual classes).

3. Polynomial-time algorithms

Let $\mathcal{M}$ be a set of binary matroids. We consider the following computational problem.

> Membership of $\mathrm{EX}_2(\mathcal{M})$
> Input: A GF(2) matrix representing the matroid, M.
> Question: Does M belong to the class $\mathrm{EX}_2(\mathcal{M})$?

The fact that this problem has a polynomial-time solution is expected to follow from the Matroid Minors Project of Geelen, Gerards, and Whittle (see [2]). However, the proofs in that project are highly non-constructive, and the algorithms that follow from them are not implementable. In this section we will describe a practical algorithm for solving Membership of $\mathrm{EX}_2(\mathcal{M})$ when $\mathcal{M}$ is a subset of $\{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\}$ that contains $M(K_{3,3})$ or $M^*(K_{3,3})$. We start with some preliminary material.

The symmetric difference of sets Z_1 and Z_2 is denoted by $Z_1 \triangle Z_2$. Suppose that M is a binary matroid. A *cycle* of M is either the empty set, or a set that can be partitioned into circuits. Binary matroids are characterised by the fact that the symmetric difference of any two cycles is another cycle [11, Theorem 9.1.2].

Let M_1 and M_2 be two binary matroids on the ground sets E_1 and E_2 respectively. Let $\mathcal{Z}$ be the collection

$$\{Z_1 \triangle Z_2 \colon Z_i \text{ is a cycle of } M_i \text{ for } i = 1, 2, \text{ and } Z_1 \cap E_2 = Z_2 \cap E_1\}.$$

Then $\mathcal{Z}$ is the collection of cycles of a binary matroid on the ground set $E_1 \triangle E_2$ (see [15] or [11, Lemma 9.3.1]). We denote this matroid $M_1 \triangle M_2$.

Proposition 3.1 ([15, (4.4)])**.** *Suppose that M_1 and M_2 are binary matroids on the sets E_1 and E_2 respectively. If I and J are disjoint subsets of $E_1 - E_2$ then $(M_1 \triangle M_2)/I \backslash J = (M_1/I \backslash J) \triangle M_2$.*

Next we define 1-, 2-, and 3-sums of binary matroids, following the path taken by Seymour [15]. This means that the sums defined here are slightly different from those in [11]. If E_1 and E_2 are disjoint, and neither E_1 nor E_2 is empty, then $M_1 \triangle M_2$ is the 1-*sum* of M_1 and M_2, denoted $M_1 \oplus_1 M_2$. If $E_1 \cap E_2 = \{p\}$, where p is neither a loop nor a coloop in M_1 or M_2, and $|E_1|, |E_2| \geq 3$, then $M_1 \triangle M_2$ is the 2-*sum* of M_1 and M_2, denoted $M_1 \oplus_2 M_2$. We say that p is the *basepoint* of the 2-sum. Finally, suppose that $E_1 \cap E_2 = T$ and assume that the following conditions hold:

(i) T is a triangle in both M_1 and M_2,
(ii) T contains a cocircuit in neither M_1 nor M_2, and
(iii) $|E_1|, |E_2| \geq 7$.

In this case $M_1 \triangle M_2$ is the 3-*sum* of M_1 and M_2, denoted $M_1 \oplus_3 M_2$.

Proposition 3.2 ([15, (2.1)])**.** *If (X_1, X_2) is a 1-separation of the binary matroid M, then $M = (M|X_1) \oplus_1 (M|X_2)$. Conversely, if $M = M_1 \oplus_1 M_2$, then $(E(M_1), E(M_2))$ is a 1-separation of M.*

The next result is easy, and also follows from [11, Proposition 4.2.20] and [11, (2.1)].

Proposition 3.3. *Let $M = M_1 \oplus_1 M_2$ and let N be a connected matroid. Then M has an N-minor if and only if either M_1 or M_2 has an N-minor.*

Proposition 3.4 ([15, (2.6)])**.** *If (X_1, X_2) is an exact 2-separation of the binary matroid M, then there are binary matroids M_1 and M_2 on the ground sets $X_1 \cup p$ and $X_2 \cup p$, where $p \notin X_1 \cup X_2$, such that $M = M_1 \oplus_2 M_2$. Conversely, if $M = M_1 \oplus_2 M_2$ then $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ is a 2-separation of M.*

We can deduce the next result from [11, Proposition 8.3.5] and [15, (2.6)].

Proposition 3.5. *Let $M = M_1 \oplus_2 M_2$ and let N be a 3-connected matroid. Then M has an N-minor if and only if either M_1 or M_2 has an N-minor.*

Proposition 3.6 ([15, (2.9)])**.** *Suppose that (X_1, X_2) is an exact 3-separation of the binary matroid M such that $\min\{|X_1|, |X_2|\} \geq 4$. Then there are binary matroids M_1 and M_2 on the ground sets $X_1 \cup T$ and $X_2 \cup T$ respectively, where T is disjoint from $X_1 \cup X_2$, such that $M = M_1 \oplus_3 M_2$. Conversely, if $M = M_1 \oplus_3 M_2$, then $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ is an exact 3-separation of M.*

Proposition 3.7 ([15, (4.1)])**.** *Let M be the binary matroid $M_1 \oplus_3 M_2$. If M is 3-connected then M_1 and M_2 are isomorphic to minors of M.*

Proposition 3.8. *Let M be the binary matroid $M_1 \oplus_3 M_2$. Assume M is 3-connected and let N be an internally 4-connected binary matroid such that $|E(N)| \geq 4$ and N has no triad. Then M has an N-minor if and only if either M_1 or M_2 has an N-minor.*

Proof. Let E_1 and E_2 be the ground sets of M_1 and M_2 respectively, so that $|E_1|, |E_2| \geq 7$. We will assume that $E_1 \cap E_2 = T$, where T is a coindependent triangle in both M_1 and M_2. The 'if' direction of the proof follows from Proposition 3.7. To prove the 'only if' direction, we assume that neither M_1 nor M_2 has an N-minor, and yet M does. Amongst such counterexamples, assume M has been chosen so that $|E(M)|$ is as small as possible.

It cannot be the case that M is isomorphic to N, or else Proposition 3.6 would imply that N is not internally 4-connected. Therefore M has a proper N-minor. Furthermore, N is not a wheel, since all wheels have triads. Therefore we can apply Seymour's Splitter Theorem [15, (7.3)]. There is a 3-connected minor, M', of M such that M' has an N-minor, and $|E(M)| - |E(M')| = 1$. Let e be the element in $E(M) - E(M')$. Without loss of generality, we can assume that e is in $E_1 - T$. If $M' = M \backslash e$, then let $M_1' = M_1 \backslash e$, and if $M' = M/e$, let $M_1' = M_1/e$. Proposition 3.1 implies $M' = M_1' \triangle M_2$. Since neither M_1' nor M_2 has an N-minor, and yet M' does, it follows that $M_1' \triangle M_2$ is not the 3-sum of M_1' and M_2, or else the minimality of M is contradicted. Therefore, either T is not a triangle in

M_1', or T contains a cocircuit in M_1', or $|E(M_1')| < 7$. We eliminate these possibilities one by one.

3.8.1. *T is a triangle in M_1'.*

Proof. If T is not a triangle, then M_1' must be M_1/e, and e must be parallel to an element, $x \in T$, in M_1. Let C be a circuit of M_2 such that $C \cap T = \{x\}$ (C exists because T is coindependent in M_2). It is easy to see that $(C - x) \cup e$ is a circuit of $M = M_1 \triangle M_2$. Now $(E_1 - T, E_2 - T)$ is a 3-separation of M, by Proposition 3.6, and e is in $E_1 \cap \mathrm{cl}_M(E_2)$. Thus $(E_1 - (T \cup e), E_2 - T)$ is a 2-separation of $M' = M/e$, and this contradicts the fact that M' is 3-connected. □

3.8.2. *T does not contain a cocircuit in M_1'.*

Proof. Certainly T does not contain a cocircuit in M_1. If it contains a cocircuit in M_1', then $M_1' = M_1 \backslash e$, and there is a cocircuit, C_1^*, of M_1 such that $C_1^* \subseteq T \cup e$ and $e \in C_1^*$. The intersection $T \cap C_1^*$ contains exactly two elements [11, Theorem 9.1.2]. Let $T = \{x_1, x_2, x_3\}$, and assume that $T \cap C_1^* = \{x_1, x_2\}$.

In M_2, consider a basis, B, that contains x_2 and x_3. Then $\mathrm{cl}_{M_2}(B - x_2)$ is a hyperplane that intersects T exactly in x_3. Thus there is a cocircuit, C_2^*, of M_2 such that $C_2^* \cap T = \{x_1, x_2\}$. Now $M_1^* \triangle M_2^* = M^*$ (see [15, p. 319]). From this we can deduce that $(C_2^* - \{x_1, x_2\}) \cup e$ is a cocircuit in M. Hence $(E_1 - T, E_2 - T)$ is a 3-separation of M, and e is in $E_1 \cap \mathrm{cl}_M^*(E_2)$. Therefore $(E_1 - (T \cup e), E_2 - T)$ is a 2-separation of $M' = M \backslash e$, a contradiction. □

By 3.8.1 and 3.8.2, we must now assume that $|E(M_1')| < 7$, and hence $|E_1| = 7$. From [15, (4.3)], we know that M_1 is 3-connected, except that there may exist parallel classes of size two that contain elements of T. In particular, M_1 contains no series pair and no coloop. Any 7-element binary matroid with rank at least 4 that contains a triangle also contains a series pair or coloop, so $2 \leq r(M_1') \leq r(M_1) \leq 3$.

Assume $r(M_1') = 3$, so the complement of T in M_1' is a cocircuit of size at most three. Let C^* be this cocircuit. From the fact that $(M_1')^* \triangle M_2^* = (M')^*$, we can see that C^* is a cocircuit of M'. Since N has no triad, there is an element, $x \in C^*$, such that $M'/x = (M_1'/x) \triangle M_2$ has an N-minor. Therefore either $r(M_1') = 2$ and $M_1' \triangle M_2$ has an N-minor, or $r(M_1'/x) = 2$, and $(M_1'/x) \triangle M_2$ has an N-minor. In either case it is easy to see that $M_1' \triangle M_2$ or $(M_1'/x) \triangle M_2$ is obtained from M_2 by possibly deleting elements of T and adding parallel elements to elements of T. As N has no parallel pairs, it now follows that M_2 has an N-minor. This is a contradiction that completes the proof of the proposition. □

Now we prove the main result of this section.

Theorem 3.9. *Let $\mathcal{M}$ be a subset of $\{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\}$ that contains $M(K_{3,3})$ or $M^*(K_{3,3})$. There is an algorithm that solves* Membership of $\mathrm{EX}_2(\mathcal{M})$ *in time bounded by $O(|E(M)|^{13})$.*

Proof. Let n be $|E(M)|$. A representation of M^* can be produced in $O(n^3)$ steps. Therefore we can replace M with M^* if necessary, so we lose no generality in assuming that $\mathcal{M}$ contains $M^*(K_{3,3})$. Henceforth we also assume that M has no loops or coloops.

We sketch the procedure for finding an exact k-separation (when k is 1 or 2), or an exact 3-separation with at least 4 elements on each side. A more complete description is in [1]. The algorithm involves considering all pairs of disjoint k-element subsets (when k is 1 or 2) or 4-element subsets (when $k = 3$). This requires looping at most $O(n^8)$ times. We attempt to extend each such pair to a k-separation. This involves examining each remaining element of $E(M)$ (looping $O(n)$ times), and calculating the rank of a submatrix for each such element (which can be done in $O(n^3)$ steps). Thus it takes at most $O(n^{12})$ steps to search for a separation certifying that M can be decomposed via a 1-, 2-, or 3-sum.

Every loopless rank-r binary matroid can be considered as a multiset of points in the projective space $P = \mathrm{PG}(r-1, 2)$. If $X \subseteq E(M)$, we use $\mathrm{cl}_P(X)$ to denote the span of X in P. Suppose that (X_1, X_2) is an exact k-separation of M for some $k \in \{1, 2, 3\}$ with the property that if $k = 3$ then $|X_1|, |X_2| \geq 4$ and $r_M(X_1), r_M(X_2) \geq 3$. Let $Z = \mathrm{cl}_P(X_1) \cap \mathrm{cl}_P(X_2)$, and for $i = 1, 2$ let M_i be the binary matroid represented by the multiset $X_i \cup Z$. Then $M \cong M_1 \oplus_k M_2$. It follows easily that by solving a system of equations (which takes $O(n^3)$ steps), we can produce representations of M_1 and M_2.

Imagine a binary tree with nodes labelled by matroids. The root is labelled M. If a node is labelled M', we are allowed to label the children of that node with M_1 and M_2 if M' can be expressed as $M' = M_1 \oplus_k M_2$ for some $k \in \{1, 2, 3\}$. For every node, we assume the decomposition of M' into $M_1 \oplus_k M_2$ has been chosen so that k is as small as possible. Even so, the decomposition need not be unique. Now $|E(\mathrm{si}(M_1))| + |E(\mathrm{si}(M_2))| \leq |E(\mathrm{si}(M))| + 6$. It is easy to prove by induction that the binary tree has at most $\max\{1, |E(\mathrm{si}(M))| - 6\}$ leaves, and therefore at most $O(n)$ internal nodes. Each internal node corresponds to finding a k-separation. It follows that we can construct such a tree in time bounded by $O(n)(O(n^{12}) + O(n^3)) = O(n^{13})$. Note that the matroids labelling leaves are internally 4-connected (except that they may contain parallel pairs).

Let $\mathcal{M}_\Delta$ be the subset of $\mathcal{M}$ containing matroids that have triangles. By our earlier assumption, $M^*(K_{3,3})$ is in $\mathcal{M}_\Delta$. Note that no matroid in $\mathcal{M}_\Delta$ contains a triad. We have insisted that we decompose a matroid along a 3-sum only if it is 3-connected. Therefore we can apply Propositions 3.3, 3.5 and 3.8. From these results we deduce that M has no minor in $\mathcal{M}_\Delta$ if and only the simplification of each of the matroids labelling a leaf has no such minor. The basic classes of internally 4-connected binary matroids with no minor in $\mathcal{M}_\Delta$ are described either by a theorem in Section 2, or the dual of such a theorem. Therefore we now check that the simplification of each leaf matroid belongs to one of these basic classes. There are $O(n)$

leaves. Producing a representation of a dual can be done in $O(n^3)$ steps. We will show that we can test whether a matroid is cographic, isomorphic to a sporadic matroid, or a Möbius matroid in at most $O(n^7)$ steps, so we can complete this part of the algorithm with another $O(n)O(n^3)O(n^7)$ steps.

Assume that M' is the simplification of a leaf matroid. Checking that M' is isomorphic to a specific sporadic matroid can be done in constant time. There is an algorithm running in at most $O(n^3)$ calls to an independence oracle which will produce a graph that represents M', or certify that no such graph exists [1]. This also allows us to check whether M' is cographic or planar graphic. Each call to an independence oracle can be simulated in $O(n^3)$ operations on the matrix, so the total time required to check whether M' is graphic, cographic, or planar graphic, is $O(n^6)$.

To check if M' is a triangular Möbius matroid, we consider each matroid of the form $M' \backslash e$, and produce a graph G (if possible), such that $M^*(G) = M' \backslash e$. We then check each such graph to see if it is a cubic Möbius ladder. We can do this by finding all 4-cycles (in time $O(n^4)$), and checking that the edges lying in exactly one such cycle form a Hamiltonian cycle. The remaining edges must then join opposite vertices in the cycle. Assuming this is the case, we check that e forms a circuit with the set of edges not in the Hamiltonian cycle. This entire process can be completed in $O(n)(O(n^6) + O(n^4)) = O(n^7)$ steps. To check if M' is a triadic Möbius matroid, we go through a similar process, except that we find the 3-cycles of G. The edges that lie in exactly one 3-cycle must form a Hamiltonian cycle, and e must be in a circuit with the set of edges not in this cycle.

We have now shown that it is possible to test in $O(n^{13})$ steps whether M has a minor isomorphic to a member of $\mathcal{M}_\Delta$. If M has such a minor, then we halt the algorithm. If $\mathcal{M} = \mathcal{M}_\Delta$, then again, we can halt. Therefore we now assume that M has no minor in $\mathcal{M}_\Delta$ but that $\mathcal{M} - \mathcal{M}_\Delta$ is non-empty. We next produce a decomposition tree for M^*, and we test the matroids corresponding to leaves of this tree to see whether they belong to the basic classes of matroids with no minor in $\{N^* \colon N \in \mathcal{M}\}$. Note that the matroids in $\{N^* \colon N \in \mathcal{M} - \mathcal{M}_\Delta\}$ have no triads. We can again use the results from earlier in this section to deduce that M^* has no minor in $\{N^* \colon N \in \mathcal{M} - \mathcal{M}_\Delta\}$ if and only if the leaf matroids belong to the basic classes. We already have assumed that M^* has no minor in $\{N^* \colon N \in \mathcal{M}_\Delta\}$. Therefore we can complete the algorithm with another $O(n^{13})$ steps. $\square$

Oracle algorithms. Historically, matroid computation has often been discussed in terms of oracle algorithms. In this case our computational model is a deterministic Turing Machine equipped with an oracle which can, in unit time, return the rank of a specified subset of the ground set. Let $\mathcal{M}$ be a subset of the family $\{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\}$ that contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. We now briefly discuss the difficulty of testing membership in $\mathrm{EX}_2(\mathcal{M})$ via oracle algorithms.

A well-known example of Seymour's [16] shows that the number of calls to the oracle needed to test whether a matroid is binary is exponential in terms of the size of the ground set. We will use the same example to show that an exponential number of calls is required to test whether a matroid belongs to $\mathrm{EX}_2(\mathcal{M})$. For $r \geq 3$ let $\{e_1, \ldots, e_r\}$ be the standard basis of the vector space over GF(2) with dimension r. Let d be the sum of $e_1, \ldots, e_r$, and for $1 \leq i \leq r$ let d_i be the sum of d and e_i. The binary matroid represented by the set $\{e_1, \ldots, e_r, d_1, \ldots, d_r\}$ is known as the rank-r *binary spike*. We will denote this matroid by N_r. If H is a subset of $E(N_r)$ such that $|H \cap \{d_1, \ldots, d_r\}|$ is odd and $|H \cap \{e_i, d_i\}| = 1$ for $1 \leq i \leq r$, then H is a circuit-hyperplane of N_r. Let $N_r(H)$ be the matroid obtained from N_r by relaxing H. It is not difficult to prove by induction on r that N_r has no minor in $\mathcal{M}$. However, $N_r(H)$ is non-binary, so certainly does not belong to $\mathrm{EX}_2(\mathcal{M})$. In the worst case, an oracle algorithm will have to check each of the 2^{r-1} candidate sets, H, to decide whether the matroid it is considering is isomorphic to N_r or $N_r(H)$. Therefore testing whether a matroid belongs to $\mathrm{EX}_2(\mathcal{M})$ requires exponentially many calls to the oracle relative to the size of the ground set.

The matroid N_r contains many 3-separations. If we restrict our attention to internally 4-connected matroids the situation changes dramatically. Seymour [16] shows that there is an algorithm which, given a matroid M (not necessarily binary), will either output a graph G such that $M = M(G)$, or decide that no such graph exists, using a polynomial number of calls to the oracle. Using a similar strategy to that in the proof of Theorem 3.9 we can decide whether a matroid M is isomorphic to a Möbius matroid, using only a polynomial number of calls to an oracle. Since it is obviously possible to decide whether a matroid M is isomorphic to one of a finite number of sporadic matroids using a constant number of oracle calls, it follows that we can decide in a polynomial number of calls to the oracle whether an internally 4-connected matroid (not necessarily binary) belongs to $\mathrm{EX}_2(\mathcal{M})$.

4. The growth-rate of $\mathrm{EX}(\{M(K_{3,3})\})$

Kung [8] investigated simple rank-r matroids of maximum size in $\mathrm{EX}_2(\{M(K_{3,3})\})$. He showed that if N and N' are such matroids and $r(N) = r(N') + 1$, then $|E(N)| - |E(N')| \leq 10$. It then follows by induction that $|E(M)| \leq 10r(M)$ for any simple matroid $M \in \mathrm{EX}_2(\{M(K_{3,3})\})$. Using our structure theorem, we show that $|E(N)| - |E(N')|$ is 4, 8, or 2, depending on the residue of $r(N)$ modulo 3. The average of these three numbers is 14/3, so we can prove that $|E(M)| \leq 14r(M)/3$ for any simple matroid $M \in \mathrm{EX}_2(\{M(K_{3,3})\})$. Moreover, we characterise the simple rank-r matroids of maximum size in the class.

When $r \in \{2, 3, 4\}$, we define the class $\mathcal{P}_r$ to be $\{\mathrm{PG}(r-1, 2)\}$. When $r > 4$ we recursively define $\mathcal{P}_r$ to be the class $\{P(M, \mathrm{PG}(3, 2))\colon M \in \mathcal{P}_{r-3}\}$, where $P(M, \mathrm{PG}(3, 2))$ is a parallel connection of M and PG(3, 2) (see [11, Section 7.1]) along an arbitrary basepoint. Note that starting with $r = 8$, the class $\mathcal{P}_r$ contains non-isomorphic matroids. It is well known that parallel connections can be expressed as 2-sums by adding a parallel element. Therefore the next result follows easily from Proposition 3.5 and induction.

Proposition 4.1. *Let $r \geq 2$ be an integer. If M is in $\mathcal{P}_r$, then M has no $M(K_{3,3})$-minor.*

For an integer $r \geq 2$, define $h(r)$ to be the size of matroids in $\mathcal{P}_r$. Therefore

$$h(r) = \begin{cases} \frac{14}{3}r - 7 & \text{if } r \equiv 0 \pmod 3 \\ \frac{14}{3}r - \frac{11}{3} & \text{if } r \equiv 1 \pmod 3 \\ \frac{14}{3}r - \frac{19}{3} & \text{if } r \equiv 2 \pmod 3 \end{cases}$$

Let $\alpha(r)$ be 7, 11/3, or 19/3 according to whether r is equivalent to 0, 1, or 2 modulo 3. Thus $h(r) = 14r/3 - \alpha(r)$.

Theorem 4.2. *Let M be a simple member of $\mathrm{EX}_2(\{M(K_{3,3})\})$ with rank $r \geq 2$. Then $|E(M)| \leq h(r)$ and equality holds if and only if $M \in \mathcal{P}_r$.*

Lemma 4.3. *Assume M is a 3-connected member of $\mathrm{EX}_2(\{M(K_{3,3})\})$ with rank $r \geq 2$. Either $|E(M)| \leq 4r - 5$, or M is isomorphic to one of the rank-4 sporadic matroids $\mathrm{PG}(3,2)$, $M_{4,14}$, $M_{4,13}$, C_{12}, or D_{12}.*

Proof. Assume that M is a counterexample with the smallest possible rank. Therefore M is a 3-connected simple matroid in $\mathrm{EX}_2(\{M(K_{3,3})\})$, with $r \geq 2$ and $|E(M)| > 4r - 5$, while M is not isomorphic to any of the five sporadic matroids listed in the statement. It is easy to see that $r \geq 3$.

Assume that M is internally 4-connected. All the sporadic matroids from Theorem 1.1 satisfy the bound $|E(M)| \leq 4r(M) - 5$ (with the exceptions of $\mathrm{PG}(3,2)$, $M_{4,14}$, $M_{4,13}$, C_{12}, and D_{12}). Therefore M is either cographic or isomorphic to a Möbius matroid. If M is a Möbius matroid then $|E(M)|$ is either $3r - 2$ or $2r - 1$. If M is cographic then $|E(M)| \leq 3r - 3$ [11, Lemma 14.10.2]. Since $r \geq 3$, both $3r - 2$ and $2r - 1$ are bounded above by $4r - 5$. Thus we have a contradiction in any case, so M is not internally 4-connected.

By Proposition 3.6 we can express M as $M_1 \oplus_3 M_2$. Let T be $E(M_1) \cap E(M_2)$. Let r_i be the rank of M_i for $i = 1, 2$. Proposition 3.6 implies $r_1 + r_2 - r = 2$. Certainly $r_1, r_2 > 2$, or else T contains a cocircuit in M_1 or M_2. Hence $r_1, r_2 < r$. We see from [15, (4.3)] that $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ are 3-connected. Moreover $\mathrm{si}(M_i)$ has no $M(K_{3,3})$-minor for $i = 1, 2$, by Proposition 3.7. Therefore the lemma holds for $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ by our inductive assumption.

Assume that $\mathrm{si}(M_2)$ is isomorphic to $\mathrm{PG}(3,2)$, $M_{4,14}$, $M_{4,13}$, C_{12}, or D_{12}. If $\mathrm{cl}_{M_1}(T)$ is not coindependent in M_1, then $\mathrm{cl}_{M_1}(T)$ is 2-separating in M_1, which contradicts [15, (4.3)]. Therefore we can deduce that $\mathrm{si}(M_1)$ has corank at least three. Now we can apply [12, Theorem 3.6], which tells us that M_1 has a minor, M_1', isomorphic to $M(K_4)$ and containing the triangle

T. Now $M_1'\triangle M_2$ is a minor of M, by Proposition 3.1, and $M_1'\triangle M_2$ is obtained from M_2 by performing a Δ-Y operation on the triangle T. It follows that M has, as a minor, a matroid obtained from $\mathrm{PG}(3,2)$, $M_{4,14}$, $M_{4,13}$, C_{12}, or D_{12} by performing a Δ-Y operation. But any such matroid contains a $M(K_{3,3})$-minor, by results from [10, Appendix C]. From this contradiction and induction we deduce that $|E(\mathrm{si}(M_2))| \leq 4r_2 - 5$. Symmetrically, $|E(\mathrm{si}(M_1))| \leq 4r_1 - 5$.

The only parallel classes of M_i have size two and contain an element of T [15, (4.3)]. Note that no element in T can be in a parallel pair in both M_1 and M_2, for that would imply that $M = M_1 \oplus_3 M_2$ has a parallel pair. Let m be the number of elements in T that are contained in a parallel pair in either M_1 or M_2. Then $|E(M)| = |E(\mathrm{si}(M_1))| + |E(\mathrm{si}(M_2))| - m - 2(3-m) \leq |E(\mathrm{si}(M_1))| + |E(\mathrm{si}(M_2))| - 3$. Thus

$$\begin{aligned}|E(M)| &\leq |E(\mathrm{si}(M_1))| + |E(\mathrm{si}(M_2))| - 3\\ &\leq (4r_1 - 5) + (4r_2 - 5) - 3 = 4(r_1 + r_2 - 2) - 5 = 4r - 5\end{aligned}$$

and M is not a counterexample after all. □

Proof of Theorem 4.2. Assume that M is a counterexample with rank r, where r is as small as possible. Thus M is a simple member of $\mathrm{EX}_2(\{M(K_{3,3})\})$, and either $|E(M)| > h(r)$, or $|E(M)| = h(r)$ and M does not belong to $\mathcal{P}_r$. Certainly M is no larger than the projective geometry $\mathrm{PG}(r-1,2)$, so the result holds if $r \leq 4$. Hence $r > 4$, and therefore $14r/3 - 7 > 4r - 5$. As $|E(M)| \geq h(r) \geq 14r/3 - 7$, Lemma 4.3 now implies M is not 3-connected.

Assume that $M = M_1 \oplus_1 M_2$, so M_1 and M_2 belong to $\mathrm{EX}_2(\{M(K_{3,3})\})$ by Proposition 3.3. Suppose that $r_i = r(M_i)$ for $i = 1, 2$, so that $r = r_1 + r_2$. Since M is simple, $r_i > 0$ and hence $r_i < r$ for $i = 1, 2$. Therefore we can apply the inductive hypothesis and conclude that

$$\begin{aligned}|E(M)| = |E(M_1)| + |E(M_2)| &\leq (14r_1/3 - \alpha(r_1)) + (14r_2/3 - \alpha(r_2))\\ &= 14r/3 - (\alpha(r_1) + \alpha(r_2)).\end{aligned}$$

But $\alpha(r_1) + \alpha(r_2) > 7$, regardless of the residue classes of r_1 and r_2 modulo 3, so $|E(M)| < 14r/3 - 7 \leq h(r)$, contradicting our earlier statement. Therefore M is connected, but not 3-connected.

Now M can be expressed as $M_1 \oplus_2 M_2$. Let p be the basepoint of the 2-sum. Let $r_i = r(M_i)$ for $i = 1, 2$. By Proposition 3.4 we see $r_1 + r_2 - r = 1$. As M has no parallel pairs, $r_1, r_2 > 1$, so $r_1, r_2 < r$. By Proposition 3.5, neither M_1 nor M_2 has an $M(K_{3,3})$-minor, so we can apply the inductive hypothesis to $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$. We can assume that either M_1 or M_2 is non-simple, since otherwise we could add a parallel element to p in M_1, and obtain a simple matroid that has one more element than M despite having no $M(K_{3,3})$-minor (since adding parallel elements and taking a 2-sum cannot create a $M(K_{3,3})$-minor). However, it cannot be the case that both

M_1 and M_2 are non-simple, for then M would be non-simple. Therefore $|E(M)| = |E(\mathrm{si}(M_1))| + |E(\mathrm{si}(M_2))| - 1$.

Assume that $r_1, r_2 \equiv 0 \pmod 3$, so $r \equiv 2 \pmod 3$ and

$$\begin{aligned} h(r) = \tfrac{14}{3}r - \tfrac{19}{3} \leq |E(M)| &= |E(\mathrm{si}(M_1))| + |E(\mathrm{si}(M_2))| - 1 \\ &\leq \tfrac{14}{3}r_1 + \tfrac{14}{3}r_2 - 15 = \tfrac{14}{3}(r_1 + r_2 - 1) - \tfrac{31}{3} = \tfrac{14}{3}r - \tfrac{31}{3}, \end{aligned}$$

which is impossible. We reach a similar contradiction if the residues of r_1 and r_2 are $(0,2)$, $(2,0)$, or $(2,2)$, so at least one of r_1 and r_2 is equivalent to 1 modulo 3.

We consider the case that $r_1 \equiv 1 \pmod 3$ and $r_2 \equiv 0 \pmod 3$, so that $r \equiv 0 \pmod 3$ and

$$\begin{aligned} h(r) = \tfrac{14}{3}r - 7 \leq |E(M)| &= |E(\mathrm{si}(M_1))| + |E(\mathrm{si}(M_2))| - 1 \\ &\leq \tfrac{14}{3}r_1 + \tfrac{14}{3}r_2 - \tfrac{11}{3} - 7 - 1 = \tfrac{14}{3}(r_1 + r_2 - 1) - \tfrac{21}{3} = \tfrac{14}{3}r - \tfrac{21}{3}. \end{aligned}$$

As equality holds throughout, we deduce that $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ belong to $\mathcal{P}_{r_1}$ and $\mathcal{P}_{r_2}$ respectively. It is easy to see that the parallel connection is an associative operation. As $\mathrm{si}(M_1)$ is formed by taking the parallel connection of multiple copies of $\mathrm{PG}(3,2)$, and $\mathrm{si}(M_2)$ is similarly formed from copies of $\mathrm{PG}(3,2)$ and a single copy of $\mathrm{PG}(2,2)$, it now follows that M belongs to $\mathcal{P}_r$, so M is not a counterexample after all. In all the other possible cases, we reach a contradiction in exactly the same way. □

5. Critical exponents

Let M be a loopless rank-r $\mathrm{GF}(q)$-representable matroid. Then M can be considered as a multiset of points in the projective geometry $\mathrm{PG}(r-1,q)$. The *critical exponent* of M over q, denoted by $c(M;q)$, is the smallest integer k such that there is a set of hyperplanes, $H_1, \ldots, H_k$, in $\mathrm{PG}(r-1,q)$ with the property that $H_1 \cap \cdots \cap H_k$ contains no points of $E(M)$.

The critical exponent depends only on M and q, and not on the particular representation chosen. We can deduce this fact from a formulation in terms of the *characteristic polynomial*, denoted $\chi(M;t)$. Assume M is a matroid on the ground set E. Then

$$\chi(M;t) = \sum_{A \subseteq E} (-1)^{|A|} t^{r(M)-r(A)}.$$

Now $\chi(M;q^k)$ is the number of k-tuples of hyperplanes, $(H_1, \ldots, H_k)$, in $\mathrm{PG}(r-1,q)$ satisfying $H_1 \cap \cdots \cap H_k \cap E(M) = \emptyset$ [9, Theorem 4.1]. From this it follows that $\chi(M;q^k) \geq 0$ for all positive integers k, and $c(M;q)$ is the least positive integer k such that $\chi(M;q^k) > 0$. Note that, if $k \geq c(M;q)$, then $\chi(M;q^k) > 0$. It is obvious that if M has a loop, then $\chi(M;t)$ is identically zero. If $e \in E(M)$, then $c(M \backslash e;q) \leq c(M;q)$.

Kung [8] looked at the critical exponent over $\mathrm{GF}(2)$ of binary matroids with no $M(K_{3,3})$-minor. He showed that if $M \in \mathrm{EX}_2(\{M(K_{3,3})\})$ is loopless then $c(M;2) \leq 10$. By using Theorem 4.2 as well as [7, Lemma 3.1], we can improve this to $c(M;2) \leq 5$, since $|E(M)| \leq 5r(M)$ for every simple matroid $M \in \mathrm{EX}_2(\{M(K_{3,3})\})$. In this section we improve this further to $c(M;2) \leq 4$, and show that this bound cannot be improved. In particular, we show that if $c(M;2) = 4$, then M has a 3-connected component isomorphic to $\mathrm{PG}(3,2)$.

Lemma 5.1. *Let M be an internally 4-connected binary matroid with no $M(K_{3,3})$-minor. Then $c(M;2) \leq 4$, and if $c(M;2) = 4$, then M is isomorphic to* $\mathrm{PG}(3,2)$.

Proof. It is easy to see that $\mathrm{PG}(3,2)$ has critical exponent 4 over $\mathrm{GF}(2)$ (see [9, Section 8.1]) so we let M be an internally 4-connected member of $\mathrm{EX}_2(\{M(K_{3,3})\})$ other than $\mathrm{PG}(3,2)$. Assume that $M = M^*(G)$ for some graph G. Because M is connected, G has no isthmus. Jaeger showed that G has a nowhere-zero 8-flow [4]. The number of such flows is $\chi(M;8)$ [9, Theorem 4.6]. Hence $\chi(M;8) > 0$ and thus $c(M;2) \leq 3$. Consider the $\mathrm{GF}(2)$-representation of Δ_r discussed in Section 1. Each point of $\mathrm{PG}(r-1,2)$ corresponds to a vector $(x_1,\ldots,x_r)$. Let H_1, H_2, and H_3 be the hyperplanes of $\mathrm{PG}(r-1,2)$ defined, respectively, by the equations $x_r = 0$, $x_1 + \cdots + x_r = 0$, and $\sum x_i = 0$, where the final sum is taken over all odd indices in $\{1,\ldots,r-1\}$. It is easy to see that no point of M is contained in $H_1 \cap H_2 \cap H_3$, so $c(\Delta_r;2) \leq 3$. Similarly, no point of Υ_r is contained in the hyperplane defined by $x_1 + \cdots + x_r = 0$, so $c(\Upsilon_r;2) \leq 1$.

Now we can assume that M is neither cographic nor a Möbius matroid, so M is isomorphic to one of the sporadic matroids in Theorem 1.1. The largest such matroid with rank 4 is $\mathrm{PG}(3,2)$, and it known that every proper minor of this matroid has critical exponent at most three over $\mathrm{GF}(2)$ [9, Section 8.1]. Thus we will assume that $r(M) \geq 5$. The sporadic matroid T_{12} has rank 6. By examining the matrix representation of T_{12} in [10, Appendix B], we see that no point of T_{12} is contained in the hyperplane defined by $x_1 + \cdots + x_6 = 0$. Thus $c(T_{12};2) \leq 1$. Let A be the matrix in [10, Appendix B] such that $[I_5|A]$ represents the rank-5 sporadic matroid $M^a_{5,12}$. If

$$H^a_{5,12} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

then the matrix product $H^a_{5,12}[I_5|A]$ contains no zero columns. This means that no point of $M^a_{5,12}$ is contained in all three of the hyperplanes defined by $x_1 + x_2 = 0$, $x_1 + x_3 + x_5 = 0$, and $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Thus $c(M^a_{5,12};2) \leq 3$.

In the same way we can show that $M_{5,13}$, $M_{6,13}$, $M_{7,15}$, $M_{9,18}$, and $M_{11,21}$ all have critical exponent at most three by examining the matrices

$$H_{5,13} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \ H_{6,13} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$H_{7,15} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \; H_{9,18} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and

$$H_{11,21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

As every sporadic matroid in Theorem 1.1 can be produced from one of $\mathrm{PG}(3,2)$, $M^a_{5,12}$, $M_{5,13}$, T_{12}, $M_{6,13}$, $M_{7,15}$, $M_{9,18}$, or $M_{11,21}$ by deleting elements, the proof is complete. $\square$

Next we come to the main result of this section.

Theorem 5.2. *Let M be a loopless binary matroid with no $M(K_{3,3})$-minor. Then $c(M;2) \leq 4$, and if $c(M;2) = 4$, then either*

(i) *M is isomorphic to* $\mathrm{PG}(3,2)$*, or*

(ii) *M can be expressed as the* 1- *or* 2-*sum of M_1 and M_2, where M_1, M_2 belong to* $\mathrm{EX}_2(\{M(K_{3,3})\})$*, and either* $c(M_1;2) = 4$ *or* $c(M_2;2) = 4$.

Proof. Let M be a minor-minimal counterexample to the theorem. Lemma 5.1 shows that M cannot be internally 4-connected. Assume that M is not connected, so that M can be expressed as $M_1 \oplus_1 M_2$. Clearly M_1 and M_2 are loopless members of $\mathrm{EX}_2(\{M(K_{3,3})\})$. It is well known, and easy to verify, that $\chi(M;t) = \chi(M_1;t)\chi(M_2;t)$. By the inductive hypothesis, $c(M_i;2) \leq 4$, meaning that $\chi(M_i;16) > 0$ for $i = 1,2$. Hence $\chi(M;16) > 0$, so $c(M;2) \leq 4$. Since M is a counterexample, $c(M;2) = 4$, so $\chi(M;8) = 0$. Therefore $\chi(M_i;8) = 0$ for some $i \in \{1,2\}$. This implies that $c(M_i;2) = 4$ for some $i \in \{1,2\}$. However, M now satisfies statement (ii) so it is not a counterexample at all.

Now we must assume that M is connected. Assume M can be expressed as $M_1 \oplus_2 M_2$, where p is the basepoint of the 2-sum. Again, M_1 and M_2 are loopless members of $\mathrm{EX}_2(\{M(K_{3,3})\})$. Walton and Welsh [18, (7)] note the following relation:

$$\chi(M;t) = \frac{\chi(M_1;t)\chi(M_2;t)}{t-1} + \chi(M_1/p;t)\chi(M_2/p;t). \tag{1}$$

Note that if M_i/p is loopless, then, by earlier discussion, $\chi(M_i/p;k) \geq 0$ for all positive integers k. The same statement holds if M_i/p has a loop, for then $\chi(M_i/p;t)$ is identically zero. Since M_1 and M_2 are isomorphic to proper minors of M it follows that $\chi(M_i;16) > 0$ for $i = 1,2$. Now (1) implies that $\chi(M;16) > 0$, so $c(M;2) \leq 4$. Therefore it must be the case that $c(M;2) = 4$, so that $\chi(M;8) = 0$. It follows that either $\chi(M_1;8) = 0$ or $\chi(M_2;8) = 0$. Then M satisfies statement (ii) of the theorem, and we again have a contradiction.

Finally, we assume that M is 3-connected, so $M = M_1 \oplus_3 M_2$ for some matroids M_1 and M_2. Let $\{a,b,c\}$ be $E(M_1) \cap E(M_2)$. Let P be the generalised parallel connection of M_1 and M_2, so that $M = P \backslash \{a,b,c\}$.

Proposition 3.7 implies that M_1 and M_2 are isomorphic to proper minors of M. Moreover M_1 and M_2 are loopless, and both $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ are 3-connected by [15, (4.3)]. The following equality is from Walton and Welsh [18].

$$\chi(M;t) = \frac{\chi(M_1;t)\chi(M_2;t)}{(t-1)(t-2)} + \chi(P\backslash a\backslash b/c;t) + \chi(P\backslash a/b;t) + \chi(P/a;t). \tag{2}$$

Since P is binary, our earlier discussion means that the evaluations $\chi(P\backslash a\backslash b/c;16)$, $\chi(P\backslash a/b;16)$, and $\chi(P/a;16)$ are all non-negative. On the other hand, by the minimality of M, $\chi(M_1;16)$ and $\chi(M_2;16)$ are positive. We deduce that $c(M;2) \leq 4$. As M is a counterexample, $c(M;2) = 4$, so $\chi(M;8) = 0$. The terms in (2) must be zero when $t = 8$, so we can assume by relabeling that $\chi(M_1;8) = 0$. Therefore $c(M_1;2) = 4$.

The critical exponent of $\mathrm{si}(M_1)$ is precisely the critical exponent of M_1. Since $\mathrm{si}(M_1)$ is 3-connected and obeys the theorem, it follows that $\mathrm{si}(M_1) \cong \mathrm{PG}(3,2)$. Exactly as in the proof of Lemma 4.3, we can show that M has a minor isomorphic to the matroid produced from $\mathrm{PG}(3,2)$ by performing a Δ-Y operation on T. This matroid has an $M(K_{3,3})$-minor [10, Appendix C], so we have a contradiction that completes the proof. □

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