THE STRUCTURE OF GRAPHS WITH A VITAL LINKAGE OF ORDER 2

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ABSTRACT. A *linkage of order* k of a graph G is a subgraph with k components, each of which is a path. A linkage is *vital* if it spans all vertices, and no other linkage connects the same pairs of end vertices. We give a characterization of the graphs with a vital linkage of order 2: they are certain minors of a family of highly structured graphs.

1. INTRODUCTION

Robertson and Seymour [4] defined a *linkage* in a graph G as a subgraph in which each component is a path. The *order* of a linkage is the number of components. A linkage L of order k is *unique* if no other collection of paths connects the same pairs of vertices, it is *spanning* if $V(L) = V(G)$, and it is *vital* if it is both unique and spanning. Graphs with a vital linkage are well-behaved. For instance, Robertson and Seymour proved the following:

Theorem 1.1 (Robertson and Seymour [4, Theorem 1.1])**.** *There exists an integer* w*, depending only on* k*, such that every graph with a vital linkage of order* k *has tree width at most* w*.*

Note that Robertson and Seymour use the term p-linkage to denote a linkage with p terminals. Robertson and Seymour's proof of this theorem is surprisingly elaborate, and uses their structural description of graphs with no large clique-minor. Recently Kawarabayashi and Wollan [2] proved a strengthening of this result. Their shorter proof avoids using the structure theorem.

Our interest in linkages, in particular those of order 2, stems from quite a different area of research: matroid theory. Truemper [5] studied a class of binary matroids that he calls *almost regular*. His proofs lean heavily on a class of matroids that are single-element extensions of the cycle matroids of graphs with a vital linkage of order 2. These matroids turned up again in the excluded-minor characterization of matroids that are either binary or ternary, by Mayhew et al. [3].

Truemper proves that an almost regular matroid can be built from one of two specific matroids by certain $\Delta - Y$ operations. This is a deep result, but it does not yield bounds on the branch width of these matroids. In a forthcoming paper the authors of this paper, together with Chun, will give an explicit structural description of the class of almost regular matroids [1]. The main result of this paper will be of use in that project.

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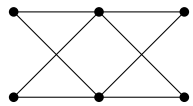

FIGURE 1. The graph $K_{2,4}$.

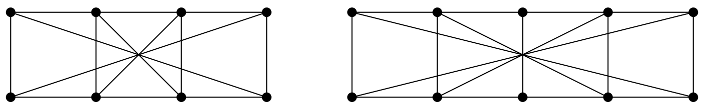

FIGURE 2. The graphs $\ddot{U}_4$ and $\ddot{U}_5$.

To state our main result we need a few more definitions. Fix a graph G and a spanning linkage L of order k. A *path edge* is a member of $E(L)$; an edge in $E(G) \setminus E(L)$ is called a *chord* if its endpoints lie in a single path, and a *rung edge* otherwise. If L is vital, then G cannot have any chords.

A *linkage minor* of G with respect to a (chordless) linkage L is a minor H of G such that all path edges in $E(G) \setminus E(H)$ have been contracted, and all rung edges in $E(G) \setminus E(H)$ have been deleted. If the linkage L is clear from the context we simply say that H is a linkage minor of G. Moreover, let G be a graph with a chordless 2-linkage L. If G has a linkage minor isomorphic to $K_{2,4}$, such that the terminals of L are mapped to the degree-2 vertices of $K_{2,4}$, we say that G has an *XX* linkage minor (cf. Figure 1).

For each integer n, the graph $\ddot{U}_n$ is the graph with $V(\ddot{U}_n) = \{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\}$, and

$$\begin{aligned} E(\ddot{U}_n) = &\{v_i v_{i+1} \mid i = 1, \dots, n-1\} \cup \{u_i u_{i+1} \mid i = 1, \dots, n-1\} \cup \\ &\{u_i v_i \mid i = 1, \dots, n\} \cup \{u_i v_{n+1-i} \mid i = 1, \dots, n\}. \end{aligned} \tag{1}$$

We denote by L_n the linkage of $\ddot{U}_n$ consisting of all edges $v_i v_{i+1}$ and $u_i u_{i+1}$ for $i = 1, \dots, n-1$. In Figure 2 the graphs $\ddot{U}_4$ and $\ddot{U}_5$ are depicted.

Finally, we say that G is a *Truemper graph* if G is a linkage minor of $\ddot{U}_n$ for some n. The main result of this paper is the following:

Theorem 1.2. *Let G be a graph. The following statements are equivalent:*

(1) *G has a vital linkage of order 2;*
(2) *G has a chordless spanning linkage of order 2 with no XX linkage minor;*
(3) *G is a Truemper graph.*

Robertson and Seymour [4] commented, without proof, that graphs with a vital linkage with $k \leq 5$ terminal vertices have path width at most k. A weaker claim is the following:

Corollary 1.3. *Let G be a graph with a vital linkage of order* 2. *Then G has path width at most 4.*

Another consequence of our result is that graphs with a vital linkage of order 2 embed in the projective plane:

Corollary 1.4. *Let G be a graph with a vital linkage of order 2. Then G can be embedded on a Möbius strip.*

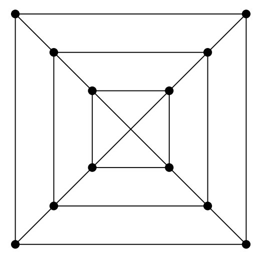

FIGURE 3. The graph $\ddot{U}_6$. The linkage is formed by the two diagonally drawn paths.

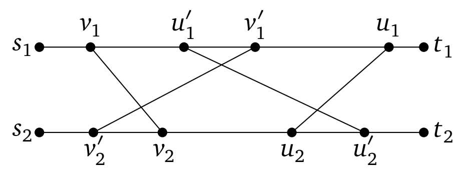

FIGURE 4. Detail of the proof of Lemma 2.2.

Both corollaries can be seen to be true by considering an alternative depiction of $\ddot{U}_{2n}$, analogous to Figure 3.

2. Proof of Theorem 1.2

We start with a few more definitions. Suppose L is a linkage of order 2 with components P_1 and P_2, such that the terminal vertices of P_1 are s_1 and t_1, and those of P_2 are s_2 and t_2. We order the vertices on the paths in a natural way, as follows. If v and w are vertices of P_i, then we say that v is *(strictly) to the left* of w if the graph distance from s_i to v in the subgraph P_i is (strictly) smaller than the graph distance from s_i to w. The notion *to the right* is defined analogously.

We will frequently use the following elementary observation, whose proof we omit.

Lemma 2.1. *Let G be a graph with a chordless spanning linkage L of order 2. Let P_1 and P_2 be the components of L, with terminal vertices respectively s_1, t_1 and s_2, t_2. Let H be a linkage minor of G. If v and w are on P_i, and v is to the left of w, then the vertex corresponding to v in H is to the left of the vertex corresponding to w in H.*

Without further ado we dive into the proof, which will consist of a sequence of lemmas. The first deals with the equivalence of the first two statements in the theorem.

Lemma 2.2. *Let G be a graph with a chordless spanning linkage L of order 2. Then L is vital if and only if G has no XX linkage minor with respect to L.*

Proof. First we suppose that there exists a graph G with a non-vital chordless spanning linkage L of order 2 such that G has no XX linkage minor. Let P_1, P_2 be the paths of L, where P_1 runs from s_1 to t_1, and P_2 runs from s_2 to t_2. Let P_1', P_2' be different paths connecting the same pairs of vertices. Without loss of generality,

$P_1' \neq P_1$. But then P_1' must meet P_2, so $P_2' \neq P_2$. Let $e = v_1v_2$ be an edge of P_1' such that the subpath $s_1 - v_1$ of P_1' is also a subpath of P_1, but e is not an edge of P_1. Let $f = u_2u_1$ be an edge of P_1' such that the subpath $u_1 - t_1$ of P_1' is also a subpath of P_2, but f is not an edge of P_2. Similarly, let $e' = v_2'v_1'$ be an edge of P_2' such that the subpath $s_2 - v_2'$ of P_2' is also a subpath of P_2, but e' is not an edge of P_2. Let $f' = u_1'u_2'$ be an edge of P_2' such that the subpath $u_2' - t_2$ of P_2' is also a subpath of P_2, but f' is not on P_2. See Figure 4.

Since P_1' and P_2' are vertex-disjoint, v_2' must be strictly to the left of v_2 and u_2. For the same reason, v_1' must be strictly between v_1 and u_1. Likewise, u_2' must be strictly to the right of v_2 and u_2, and u_1' must be strictly between v_1 and u_1. Now construct a linkage minor H of G, as follows. Contract all edges on the subpaths $s_1 - v_1$, $v_1' - u_1'$, and $u_1 - t_1$ of P_1, contract all edges on the subpaths $s_2 - v_2'$, $v_2 - u_2$, and $u_2' - t_2$ of P_2, delete all rung edges but $\{e, f, e', f'\}$, and contract all but one of the edges of each series class in the resulting graph. Clearly H is isomorphic to XX, a contradiction.

Conversely, suppose that G has an XX linkage minor, but that L is unique. Clearly having a vital linkage is preserved under taking linkage minors. But XX has two linkages, a contradiction. □

Next we show that the third statement of Theorem 1.2 implies the second.

Lemma 2.3. *For all n, $\ddot{U}_n$ has no XX linkage minor with respect to L_n.*

Proof. The result holds for $n \leq 2$, because then $|V(\ddot{U}_n)| < |V(XX)|$. Suppose the lemma fails for some $n \geq 3$, but is valid for all smaller n. Every edge of XX is incident with exactly one of the four end vertices of the paths. Hence all rung edges incident with at least two of the four end vertices are not in any XX linkage minor. But after deleting those edges from $\ddot{U}_n$ the end vertices have degree one, and hence the edges incident with them will not be in any XX linkage minor. Contracting these four edges produces $\ddot{U}_{n-2}$, a contradiction. □

Reversing a path P_i means exchanging the labels of vertices s_i and t_i, thereby reversing the order on the vertices of the path.

Lemma 2.4. *Let G be a graph, and L a chordless spanning linkage of order 2 of G consisting of paths P_1, running from s_1 to t_1, and P_2, running from s_2 to t_2. If G has no XX linkage minor, then G is a linkage minor of $\ddot{U}_n$ with respect to L_n for some integer n, such that L is a contraction of L_n.*

Proof. Suppose the statement is false. Let G be a counterexample with as few edges as possible. If some end vertex of a path, say s_1, has degree one (with $e = s_1v$ the only edge), then we can embed G/e in $\ddot{U}_n$ for some n. Let G' be obtained from $\ddot{U}_n$ by adding four vertices s_1', t_1', s_2', t_2', and edges $s_1'v_1, s_1's_2', s_1't_2', s_2'u_1, s_2't_1', v_nt_1', u_nt_2', t_1't_2'$. Then G' is isomorphic to $\ddot{U}_{n+2}$, and G' certainly has G as linkage minor.

Hence we may assume that each end vertex of P_1 and P_2 has degree at least two. Suppose no rung edge runs between two of these end vertices. Then it is not hard to see that G has an XX minor, a contradiction. Therefore some two end vertices must be connected. By reversing paths as necessary, we may assume there is an edge $e = s_1s_2$.

By our assumption, $G\backslash e$ can be embedded in $\ddot{U}_n$ for some n. Again, let G' be obtained from $\ddot{U}_n$ by adding four vertices s_1', t_1', s_2', t_2', and edges $s_1'v_1$, $s_1's_2'$, $s_1't_2'$,

s'_2u_1, $s'_2t'_1$, $v_nt'_1$, $u_nt'_2$, $t'_1t'_2$. Then G' is isomorphic to $\ddot{U}_{n+2}$, and G' certainly has G as linkage minor, a contradiction. □

As an aside, it is possible to prove a stronger version of the previous lemma. We say a partition (A, B) of the rung edges is *valid* if the edges in A are pairwise non-crossing, and the edges in B are pairwise non-crossing after reversing one of the paths. One can show:

- Each Truemper graph has a valid partition.
- For every valid partition (A, B) of a Truemper graph G, some $\ddot{U}_n$ has G as linkage minor in such a way that (A, B) extends to a valid partition of $\ddot{U}_n$.

Now we have all ingredients of our main result.

Proof of Theorem 1.2. From Lemma 2.2 we learn that (1)⇔(2). From Lemma 2.3 we learn that (3)⇒(2), and from Lemma 2.4 we conclude that (2)⇒(3). □

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