INEQUIVALENT REPRESENTATIONS OF BIAS MATROIDS

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Abstract. Suppose that q is a prime power exceeding five. For every integer N there exists a 3-connected GF(q)-representable matroid, in particular, a free spike or a free swirl, that has at least N inequivalent GF(q)-representations. In contrast to this, Geelen, Oxley, Vertigan and Whittle have conjectured that, for any integer $r > 2$, there exists an integer $n(q, r)$ such that if M is a 3-connected GF(q)-representable matroid and M has no rank-r free-swirl or rank-r free-spike minor, then M has at most $n(q, r)$ inequivalent GF(q)-representations. The main result of this paper is a proof of this conjecture for Zaslavsky's class of bias matroids.

1. Introduction

Suppose that M is a matroid and that $\mathbf{F}$ is a field. If A is a matrix over $\mathbf{F}$, and the columns of A are bijectively labelled with the elements of M in such a way that a set of column labels is independent in M if and only if the corresponding set of columns is linearly independent, then A is an *$\mathbf{F}$-representation of M*. Two representations of M are *equivalent* if one can be obtained from the other by a sequence of the following operations: elementary row operations; multiplying a column by a non-zero scalar; permuting the columns, along with their labels; deleting or adjoining zero rows; and applying an automorphism of $\mathbf{F}$ to every entry of the matrix. The matroid M is *uniquely representable over* $\mathbf{F}$ if any two $\mathbf{F}$-representations of M are equivalent.

In 1988, while proving that 3-connected quaternary matroids are uniquely representable over GF(4), Kahn [7] made the following conjecture:

Conjecture 1.1 (Kahn's Conjecture)**.** *For every prime power q there is an integer $n(q)$ such that no 3-connected GF(q)-representable matroid has more than $n(q)$ inequivalent GF(q)-representations.*

At the time, the conjecture was known to be true when $q \in \{2, 3, 4\}$. Oxley, Vertigan and Whittle [12] showed that the conjecture is true if $q =$

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5, but provided counterexamples showing it is false for all prime powers exceeding five.

The two families of matroids introduced as counterexamples are known as the "free swirls" and the "free spikes". For any integer $r > 2$, the rank-r free swirl (denoted by Δ_r) is isomorphic to the matroid obtained by freely adding a point to each non-trivial line of the rank-r whirl, and then deleting those points that lie on the intersection of two non-trivial lines; the rank-r free spike (denoted by Λ_r) is produced by positioning r three-point lines as freely as possible in r-space so that they are concurrent, and then deleting the common point of intersection.

Let q be a prime power. We use the notation $n_q(M)$ to denote the number of inequivalent representations of the matroid M over the field GF(q). If $q > 5$, then either the entire family of free swirls or the entire family of free spikes is GF(q)-representable. In the first case, $n_q(\Delta_r)$ is a strictly increasing function of r, and, in the second, $n_q(\Lambda_r)$ increases strictly with r.

No other counterexamples to Kahn's conjecture have been forthcoming. Geelen, Oxley, Vertigan and Whittle [6] have conjectured that free swirls and free spikes are, in a sense, the only obstruction to Kahn's conjecture.

Conjecture 1.2. [6, Conjecture 2.9] *Let r be an integer exceeding two, and let q be a prime power. There exists an integer $n(q, r)$ such that if M is a 3-connected GF(q)-representable matroid, and M has no Δ_r- or Λ_r-minor, then $n_q(M) \leq n(q, r)$.*

Bias matroids were introduced by Zaslavsky [16, 17]. A "biased graph" is a graph with a family of distinguished, "balanced", cycles such that no theta subgraph contains exactly two balanced cycles. A "bicycle" is a minimal connected graph containing exactly two independent cycles. The bias matroid of a biased graph has the edge set of the graph as its ground set, and its circuits are the balanced cycles and the unbalanced bicycles.

If the set of balanced cycles in a biased graph contains every cycle, then the associated bias matroid is the polygon matroid of the graph. If the set of balanced cycles is empty, then the associated bias matroid is the "bicircular matroid" of the graph. Bicircular matroids were introduced by Simões-Pereira [13, 14] and were further studied in [1, 8, 15].

Conjecture 1.2 is known to be true for the class of matroids that have no $U_{3,6}$-minor [5] and the class produced by applying the truncation operator to bicircular matroids [9]. In this paper we prove that the conjecture holds for bias matroids.

Theorem 1.3. *Let r be an integer exceeding two, and let q be a prime power. There exists an integer $m(q, r)$ such that if M is a 3-connected GF(q)-representable bias matroid, and M has no Δ_r-minor, then $n_q(M) \leq m(q, r)$.*

To prove Theorem 1.3 we use "totally free matroids", which were introduced by Geelen, Oxley, Vertigan and Whittle in [6]. Let q be a prime power and let $\mathcal{M}$ be a family of GF(q)-representable matroids. If there exists an integer N such that $n_q(M) \leq N$ for every totally free matroid $M \in \mathcal{M}$, then $n_q(M') \leq N$ for every 3-connected matroid $M' \in \mathcal{M}$. Thus Theorem 1.3 follows from the following results:

Theorem 1.4. *Let r be an integer exceeding two, and let q be a prime power. There are only a finite number of GF(q)-representable bicircular matroids that are totally free and have no Δ_r-minor.*

Theorem 1.5. *Every totally free bias matroid is a bicircular matroid.*

Terminology and notation will follow that used by Oxley [10], with the following exceptions. The simple and cosimple matroids canonically associated with the matroid M will be denoted by si(M) and co(M) respectively. A *triangle* of a matroid is a 3-element circuit, and a *triad* is a 3-element cocircuit.

Graphs may contain loops and parallel edges. A *link* is a non-loop edge. If X is a set of edges of the graph G, then $G[X]$ denotes the subgraph induced by X. We shall frequently make no distinction between a set of edges and the subgraph that it induces.

2. Totally Free Matroids

In this section we introduce totally free matroids and review some of the main results of [6].

If e and e' are elements of the matroid M, and the function that exchanges e and e' and acts as the identity on $E(M) - \{e, e'\}$ is an automorphism of M, then e and e' are *clones in M*.

A *cyclic flat* of a matroid is a flat that is also a union of circuits.

Proposition 2.1. [6, Proposition 4.9] *Two elements e and e' are clones in M if and only if the set of cyclic flats containing e is equal to the set of cyclic flats containing e'.*

It is easy to see that if two elements are clones in M, then they are clones in M^*. It is also clear that the relation of being clones is an equivalence relation on the elements of M. The equivalence classes of this relation are known as *clonal classes*. A *clonal pair* (respectively *clonal triple*) is a pair (respectively triple) of elements contained in a clonal class.

The property of being clones is preserved in minors.

Proposition 2.2. [6, Proposition 4.3] *Suppose that e and e' are clones in M. If X and Y are disjoint subsets of $E(M) - \{e, e'\}$, then e and e' are clones in $M/X\backslash Y$.*

Let e be an element of a matroid M. If the matroid M' is obtained through extending M by the single element e' in such a way that e and e' are clones in M', then we say that M' is obtained from M by *cloning e with e'*. If $\{e, e'\}$ is independent in M', then we say that e has been *independently cloned with*

e'. If e is an element of a matroid M, and e cannot be independently cloned, then e is *fixed in* M. If e is fixed in M^*, then it is *cofixed in* M.

The next result can be deduced from the work of Duke [4, Corollary 3.5]

Proposition 2.3. *Let e be an element of the matroid M. Then e is fixed in M if and only if* $\mathrm{cl}(\{e\})$ *is in the modular cut generated by the cyclic flats containing e.*

Proposition 2.4. [6, Proposition 4.8] *Let e and e' be clones in M. If $\{e, e'\}$ is independent, then e is fixed in neither M nor $M\backslash e'$.*

A matroid M is *totally free* if:

(i) M is 3-connected;
(ii) $|E(M)| \geq 4$; and,
(iii) for any $e \in E(M)$, if e is fixed in M, then $\mathrm{co}(M\backslash e)$ is not 3-connected, and if e is cofixed in M, then $\mathrm{si}(M/e)$ is not 3-connected.

Note that the definition of a totally free matroid is self-dual, so that M is totally free if and only if M^* is totally free.

The usefulness of totally free matroids is shown by the following lemma.

Lemma 2.5. [6, Corollary 10.1] *Let q be a prime power and let $\mathcal{M}$ be a minor-closed class of GF(q)-representable matroids. Suppose that, for some integer k, every totally free matroid in $\mathcal{M}$ has at most k inequivalent GF(q)-representations. Then every* 3*-connected matroid in $\mathcal{M}$ has at most k inequivalent GF(q)-representations. In particular, if $\mathcal{M}$ contains only a finite number of totally free matroids, then, for some integer k', every* 3*-connected member of $\mathcal{M}$ has at most k' inequivalent GF(q)-representations.*

We will require several further properties of totally free matroids.

Proposition 2.6. [6, Lemma 8.8] *Any triangle or triad of a totally free matroid is a clonal triple.*

Corollary 2.7. *Let F and F' be two non-trivial lines of the totally free matroid M. If $F \neq F'$, then $F \cap F' = \varnothing$.*

The next two results show that totally free matroids cannot occur sporadically in a minor-closed class. They demonstrate that if M is totally free and contains more than four elements, then a totally free minor of M can be obtained by removing either one or two elements.

Proposition 2.8. [6, Proposition 8.9] *Let e be an element of the totally free matroid M. Then either $M\backslash e$ or M/e is* 3*-connected.*

Lemma 2.9. [6, Theorem 8.12] *Let M be a totally free matroid such that $|E(M)| \geq 5$. If e is an element of M such that either $M\backslash e$ or M/e is* 3*-connected but not totally free, then:*

(i) *e has a unique clone e' in M;*
(ii) *$M\backslash e/e' = M/e\backslash e'$ is totally free; and,*
(iii) *both $M\backslash e$ and M/e are* 3*-connected.*

Proposition 2.10. *Let M be a totally free matroid such that $|E(M)| \geq 5$. If e is fixed in M, then M/e is totally free.*

Proof. Since e is fixed $M\backslash e$ cannot be 3-connected, for $\mathrm{co}(M\backslash e)$ is not 3-connected. Therefore M/e must be 3-connected by Proposition 2.8. If M/e is not totally free, then e has a unique clone, e', in M by Lemma 2.9. In that case $\{e, e'\}$ is certainly independent in M, as M is 3-connected. But then Proposition 2.4 implies that e is not fixed in M. □

Proposition 2.11. *Suppose that C and C' are circuits of a totally free matroid M. Suppose also that $|C| = |C'| = 4$ and $r(C \cup C') = 4$. If C and C' meet in exactly two elements, e and e', then e and e' are clones.*

Proof. Assume that e and e' are not clones. By Proposition 2.1 we may assume, without loss of generality, that there is a cyclic flat, F, that contains e but not e'. Note that $\mathrm{cl}(C)$ and $\mathrm{cl}(C')$ are cyclic flats and that $\mathrm{cl}(C) \cap \mathrm{cl}(C') = \mathrm{cl}(\{e, e'\})$. Since $r(\mathrm{cl}(C) \cup \mathrm{cl}(C')) = r(C \cup C') = 4$, it follows that $(\mathrm{cl}(C), \mathrm{cl}(C'))$ is a modular pair. Therefore $\mathrm{cl}(\{e, e'\})$ is in the modular cut generated by the cyclic flats containing e. It is easy to check that $(\mathrm{cl}(\{e, e'\}), F)$ is a modular pair, so $\mathrm{cl}(\{e, e'\}) \cap F = \{e\}$ is in the same modular cut. Hence e is fixed by Proposition 2.3. Proposition 2.10 implies that M/e is totally free. Therefore all distinct non-trivial lines of M/e are disjoint by Corollary 2.7. But $C - e$ and $C' - e$ span distinct non-trivial lines of M/e that meet in e'. This contradiction completes the proof. □

We conclude this section by discussing free swirls and free spikes. Let r be an integer exceeding two, and let E_r be the set $\{a_1, b_1, \ldots, a_r, b_r\}$. For any integer $i \in \{1, \ldots, r\}$ we will let z_i denote either a_i or b_i. Both the *rank-r free swirl* (Δ_r) and the *rank-r free spike* (Λ_r) have E_r as their ground set. Both Δ_3 and Λ_3 are isomorphic to $U_{3,6}$. For $r > 3$ the non-spanning circuits of Δ_r are the sets $\{a_i, b_i, z_{i+1}, \ldots, z_{i+j-1}, a_{i+j}, b_{i+j}\}$ where $1 \leq i \leq r$ and $1 \leq j \leq r - 3$. (Subscripts are to be read modulo r.) The non-spanning circuits of Λ_r for $r > 3$ are sets of the form $\{a_i, b_i, a_j, b_j\}$ where $1 \leq i < j \leq r$. It is straightforward to verify that every free swirl and every free spike is totally free.

3. Bicircular Matroids

If a graph can be obtained from one of the graphs illustrated in Figure 1 by subdividing edges, then it is a *bicycle*.

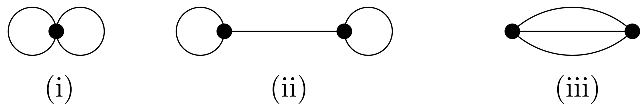

FIGURE 1

A bicycle that is obtained from graph (i) (respectively (ii)) is known as a *tight* (respectively *loose*) *handcuff*. A bicycle obtained from graph (iii) is a *theta graph*.

Let G be a graph. A *bicycle of* G is a set E' of edges such that $G[E']$ is a bicycle. The *bicircular matroid of* G, denoted by $B(G)$, has $E(G)$ as its ground set. The circuits of $B(G)$ are the bicycles of G. A matroid M is a *bicircular matroid* if there exists a graph G such that $M \cong B(G)$.

The following proposition can be derived from [17, Theorem 2.5].

Proposition 3.1. *Let e be an edge of the graph G. Then $B(G)\backslash e = B(G\backslash e)$. Furthermore, if e is a link of G, then $B(G)/e = B(G/e)$.*

Corollary 3.2. *Let G be a graph. If G' is a graph minor of G, then $B(G')$ is a matroid minor of $B(G)$.*

The next result is from Wagner [15], and was also proved by Matthews [8]. We restate it here as a consequence of [1, Proposition 3.1].

Proposition 3.3. *Let G be a graph such that G has no isolated vertices and $|V(G)| \geq 3$. The matroid $B(G)$ is 3-connected if and only if:*

(i) *G is 2-connected;*
(ii) *the minimum degree of G is at least three; and,*
(iii) *no vertex of G is incident with more than one loop.*

Let $\mathrm{st}_G(v)$ denote the set of edges in the graph G that are incident with the vertex v.

Proposition 3.4. *Suppose that v is a vertex of the 2-connected graph G. Let X be a non-empty subset of $\mathrm{st}_G(v)$ that contains no loops. If $n = |X| \geq 2$, then $B(G)$ has a $U_{2,n}$-minor on the set X.*

Suppose that G is a simple graph and that n is a positive integer. Let nG denote the graph which is obtained by replacing every edge of G with a parallel class of size n. We will let G° denote the graph obtained by adjoining a loop to every vertex of G.

Let C_r denote the cycle of length r, where $r > 2$. The rank-r whirl is denoted by $\mathcal{W}^r$. The next proposition can be verified using the description of free swirls and whirls via their non-spanning circuits.

Proposition 3.5. *Let r be an integer exceeding two. Then $B(2C_r) \cong \Delta_r$ and $B(C_r^\circ) \cong \mathcal{W}^r$.*

We will use the following result from extremal graph theory (see, for example [11, Proposition 3.2]).

Lemma 3.6. *For every positive integer t there is an integer $M(t)$ such that if G is a 2-connected graph and $|V(G)| > M(t)$, then G contains either a vertex incident with t non-loop edges or a cycle of length at least t.*

The goal of this section is to prove Theorem 1.4, which we restate here.

Theorem 3.7. *Let r be an integer exceeding two, and let q be a prime power. There are only a finite number of GF(q)-representable bicircular matroids that are totally free and have no Δ_r-minor.*

Theorem 3.7 will follow immediately from the next result.

Lemma 3.8. *Let r be an integer exceeding two, and let q be a prime power. There exists an integer $N(q, r)$ such that if G is a graph with the following properties:*

- *G has no isolated vertices;*
- *$|V(G)| > N(q, r)$; and,*
- *$B(G)$ is a 3-connected GF(q)-representable matroid,*

then either $B(G)$ has a Δ_r-minor or a $\mathcal{W}^r$-minor. Furthermore, if $B(G)$ is totally free, then $B(G)$ has a Δ_r-minor.

We first show that Theorem 3.7 follows from Lemma 3.8. Suppose that M is a GF(q)-representable bicircular matroid that is totally free and has no Δ_r-minor. It is easy to see that Lemma 3.8 implies that $r(M) \leq N(q, r)$. Since M is simple and GF(q)-representable, it follows that

$$|E(M)| \leq \frac{q^{N(q,r)} - 1}{q - 1}.$$

Therefore there can be only a finite number of such matroids.

Proof of Lemma 3.8. Let G be a graph with no isolated vertices such that $B(G)$ is 3-connected. It follows from Proposition 3.3 that G is 2-connected. Let $t = 3(q+2)(r-1)+2$ and let $N(q, r) = M(t)$, where $M(t)$ is the integer supplied by Lemma 3.6. Suppose that $|V(G)| > N(q, r)$. Now $t > q+2$, so if a vertex of G is incident with at least t links, then $B(G)$ has a $U_{2,q+2}$-minor by Proposition 3.4. In this case $B(G)$ is not GF(q)-representable. Thus we may assume that G contains a cycle, C, of length at least t.

Let W be the set of vertices in C that are not incident with a loop. The proof will be structured as follows. We first show that if $B(G)$ has no $\mathcal{W}^r$-minor, or if $B(G)$ is totally free, then $|W| \geq (q+2)(r-1)$. We complete the proof by showing that this implies that either $B(G)$ has a Δ_r-minor, or $B(G)$ is not GF(q)-representable.

Let m be the number of vertices in C that are incident with loops, and suppose that $m \geq r$. We may delete every edge in $E(G)-C$ that is not a loop incident with a vertex of C, and then contract edges to obtain a C_r°-minor of G. Therefore $B(G)$ has a $\mathcal{W}^r$-minor by Corollary 3.2 and Proposition 3.5. Thus, if $B(G)$ has no $\mathcal{W}^r$-minor, it follows that $m < r$, and hence

$$|W| \geq 3(q+2)(r-1) + 2 - (r-1) > (q+2)(r-1).$$

We now assume that $B(G)$ is totally free. Suppose that v_1, v_2 and v_3 are consecutive vertices in C and that, for all $i \in \{1, 2, 3\}$, the vertex v_i is incident with the loop l_i. The set containing l_1, l_2 and the edge v_1v_2 is a loose handcuff of G, and hence a triangle of $B(G)$. Therefore it spans a non-trivial line of $B(G)$. Similarly, the set containing l_2, l_3 and the edge v_2v_3 spans a non-trivial line of $B(G)$. These lines are distinct, but not disjoint, and this contradicts Corollary 2.7. Thus, if $v \in C$ is incident with a loop, at most one of the neighbours of v in C is incident with a loop.

Suppose that $v_1, \ldots, v_p$ is the vertex sequence of C, where $p \geq 3(q+2)(r-1)+2$. Let p' be the greatest multiple of three that does not exceed p. The vertices $v_1, \ldots, v_{p'}$ can be partitioned into subsets, each consisting of three consecutive vertices. By the discussion in the previous paragraph at least one vertex from each of these 3-element subsets is not incident with a loop. Thus $|W| \geq \frac{1}{3}p' \geq (q+2)(r-1)$.

Let $W = \{w_1, \ldots, w_n\}$. By Proposition 3.3 every vertex in G is incident with at least three edges. Thus every vertex $w_i \in W$ is incident with a link, e_i, that is not in C. If e_i joins w_i to another vertex of C, then let the path P_i consist of the edge e_i. Otherwise e_i joins w_i to a vertex u_i not in C. Select an arbitrary vertex $v \neq w_i$ in C. Since G is 2-connected, there is a path P joining u_i to v that does not pass through the vertex w_i. Let P_i be the path formed by adjoining the edge e_i to P. Thus for every vertex $w_i \in W$ we have a path P_i, not contained in C, joining w_i to another vertex of C.

We produce the minor G_0 by deleting every edge of $E(G) - C$ that is not in a path P_i. For $i \in \{1, \ldots, n\}$ we inductively describe G_i. In G_{i-1}, P_i is a path from w_i to a vertex of C. Suppose that v is the first vertex in P_i (other than w_i) that is in C. Contract all but one of the edges of P_i that lie between w_i and v. Of the edges of P_i that lie after v, delete all those that are not in C and not in another path P_j where $j > i$.

Note that C is a cycle in G_n. Furthermore, in G_n every vertex in W is joined to another vertex of C by a single edge that is not in C. Let the set of these edges be E'. Then $|E'| \geq \frac{1}{2}|W| \geq \frac{1}{2}(q+2)(r-1)$.

We derive a minor, G'_n, of G_n by contracting, in turn, all those edges that are in C and that have no parallel edge, until no such edge remains. If G'_n contains at least r vertices then, since every edge of C that is in G'_n has a parallel edge, we can obtain a $2C_r$-minor. In this case $B(G)$ has a Δ_r-minor by Proposition 3.5. We now assume that G'_n has at most $r-1$ vertices. The edge set of G'_n contains all the edges of E'. Therefore the average degree of the vertices in G'_n is at least

$$\frac{2|E'|}{(r-1)} \geq \frac{(q+2)(r-1)}{(r-1)} = q+2.$$

Thus there is a vertex of G'_n with degree at least $q+2$. Furthermore, C is a Hamiltonian cycle of G'_n, hence G'_n is 2-connected. Also, G'_n has no loops by construction. It follows from Proposition 3.4 that $B(G)$ is not GF(q)-representable. This completes the proof. □

4. Bias Matroids

Suppose that G is a graph and that $\mathcal{A}$ is a class of cycles of G. If no theta subgraph of G contains exactly two members of $\mathcal{A}$, then $\mathcal{A}$ is a *linear class*. Equivalently, $\mathcal{A}$ is a linear class if, whenever C and C' are two distinct members of $\mathcal{A}$ and e is an edge in $C \cap C'$, then $(C \cup C') - e$ contains either a member of $\mathcal{A}$, or a bicycle. A *biased graph* is a pair, $(G, \mathcal{A})$, where G is a graph and $\mathcal{A}$ is a linear class of cycles. If $(G, \mathcal{A})$ is a biased graph and $C \in \mathcal{A}$, then C is a *balanced cycle of* $(G, \mathcal{A})$.

Let $(G, \mathcal{A})$ be a biased graph. The *bias matroid of* $(G, \mathcal{A})$, denoted by $M(G, \mathcal{A})$, has $E(G)$ as its ground set. The circuits of $M(G, \mathcal{A})$ are the members of $\mathcal{A}$ and the bicycles of G that contain no balanced cycle. Thus, for any graph G, $M(G, \varnothing) = B(G)$, and if $\mathcal{C}$ is the family of cycles of G, then $M(G, \mathcal{C}) = M(G)$. A matroid M is a *bias matroid* if there exists a biased graph $(G, \mathcal{A})$ such that $M \cong M(G, \mathcal{A})$.

Zaslavsky [18] characterised bias matroids as follows: M is a bias matroid if and only if there exists a matroid, M', on the ground set $E(M) \cup B$, where $E(M) \cap B = \varnothing$, such that $M'|E(M) = M$, and B is a basis of M' having the property that for any element $e \in E(M)$, the unique circuit of M' that is contained in $B \cup e$ contains at most three elements.

Dowling geometries [3] can be described via bias matroids. Let H be a non-trivial group, and consider the graph $G = (|H|K_n)^\circ$. Assign directions to the links of G in such a way that parallel edges have the same direction. Now bijectively label each parallel class of G with the elements of H. We will define a linear class, $\mathcal{A}$, of cycles. To determine whether a cycle belongs to $\mathcal{A}$, traverse the edges in cyclic order, and take the product of the edge labels in that order, except that if the direction on an edge is contrary to the cyclic order, use the inverse of the edge label in the product instead. The cycle belongs to $\mathcal{A}$ if and only if the product thus produced is the identity of H. The corresponding bias matroid, $M(G, \mathcal{A})$, is isomorphic to the rank-n Dowling geometry on the group H. A proof of this equivalence appears in [2].

Let e be an edge of the biased graph $(G, \mathcal{A})$. Define $\mathcal{A}\backslash e$ to be $\{C \in \mathcal{A} \mid e \notin C\}$ and $\mathcal{A}/e$ to be $\{C - e \mid C \in \mathcal{A} \text{ and } C - e \text{ is a cycle of } G/e\}$.

The following proposition generalises Proposition 3.1.

Proposition 4.1. [17, Theorem 2.5] *Let $(G, \mathcal{A})$ be a biased graph and suppose that e is an edge of G. Then $M(G, \mathcal{A})\backslash e = M(G\backslash e, \mathcal{A}\backslash e)$. If e is a link, then $M(G, \mathcal{A})/e = M(G/e, \mathcal{A}/e)$, Furthermore, every minor of a bias matroid is itself a bias matroid.*

The next two results can be derived from parts (i) and (j) of [17, Theorem 2.1]

Proposition 4.2. *Suppose that $(G, \mathcal{A})$ is a biased graph and that v is a vertex of G. If $\mathrm{st}_G(v)$ is non-empty, then $\mathrm{st}_G(v)$ contains a cocircuit of $M(G, \mathcal{A})$.*

Proposition 4.3. *Suppose that* $(G, \mathcal{A})$ *is a biased graph such that* G *is connected. If* $\mathcal{A}$ *contains every cycle of* G*, then* $r(M(G, \mathcal{A})) = |V(G)| - 1$*. Otherwise* $r(M(G, \mathcal{A})) = |V(G)|$*.*

Lemma 4.4. *Let* $(G, \mathcal{A})$ *be a biased graph, and suppose that* $C \in \mathcal{A}$*. If an edge* $e \notin C$ *is in the closure of* C *in* $M(G, \mathcal{A})$*, then* $C \cup e$ *is a bicycle and every cycle in* $C \cup e$ *is balanced. Furthermore,* e *is fixed in* $M(G, \mathcal{A})$*.*

Proof. It is easy to verify that if $e \in \mathrm{cl}(C)$, then $C \cup e$ is a bicycle and every cycle of $C \cup e$ is balanced. It remains to show that e is fixed in $M(G, \mathcal{A})$. If e is a loop of G, then the cycle consisting only of e is in $\mathcal{A}$, and so e is a loop of $M(G, \mathcal{A})$. In this case e is certainly fixed. Thus we assume that e is a link.

Let C_1 and C_2 be the two cycles of $C \cup e$ that contain e. For any $i \in \{1, 2\}$ let n_i be the size of C_i and let C_i' be the closure of C_i in $M(G, \mathcal{A})$. Since C_i is a balanced cycle, the rank of C_i' is $n_i - 1$.

Suppose that $e' \in (C_1' \cap C_2') - e$. Since $e' \in \mathrm{cl}(C_1)$ the first part of the lemma implies that every cycle in $C_1 \cup e'$ is balanced. In particular, if e' is a loop of G, then e' is a loop of $M(G, \ \mathcal{A})$. Assume that e' is a link. Then e' must join two vertices that are in both C_1 and C_2. The only vertices C_1 and C_2 have in common are the end vertices of e, so e' is parallel to e. In this case the cycle consisting of e and e' is balanced, so $e' \in \mathrm{cl}(\{e\})$. We have shown that $r(C_1' \cap C_2') = 1$. Also,

$$r(C_1' \cup C_2') = r(C_1 \cup C_2) = r(C \cup e) = |C| - 1 = n_1 + n_2 - 3.$$

It follows that $r(C_1') + r(C_2') = n_1 + n_2 - 2 = r(C_1' \cup C_2') + r(C_1' \cap C_2')$. Since C_1' and C_2' are cyclic flats and $C_1' \cap C_2' = \mathrm{cl}(\{e\})$, the element e is fixed in $M(G, \mathcal{A})$ by Proposition 2.3. □

Suppose that two matroids, M and M', have identical ground sets. If every independent set in M is independent in M' we shall say that M' *is freer than* M', and we shall indicate this with the notation $M' \geq M$.

The next two propositions are easily proved.

Proposition 4.5. *Let* $(G, \mathcal{A}')$ *and* $(G, \mathcal{A})$ *be two biased graphs.* $M(G, \mathcal{A}') \geq M(G, \mathcal{A})$ *if and only if* $\mathcal{A}' \subseteq \mathcal{A}$*.*

Proposition 4.6. *Suppose that* M *and* N *are two matroids on the same ground set such that* $M \geq N$ *and* $r(M) = r(N)$*. If* N *is* n*-connected, then* M *is* n*-connected.*

If M' is obtained from M by relaxing a circuit-hyperplane, then clearly $M' \geq M$.

Proposition 4.7. *Let* C *be a circuit-hyperplane of the matroid* M*. Let* M' *be produced by relaxing* C*. If* M' *is* n*-connected, and* (X, Y) *is a* k*-separation of* M *where* $k < n$*, then either* $C = X$ *or* $C = Y$*.*

Proposition 4.8. *Let* $(G, \mathcal{A})$ *be a biased graph such that* G *is connected and* $\mathcal{A}$ *does not contain every cycle of* G*. If* $C \in \mathcal{A}$ *is a circuit-hyperplane of* $M(G, \mathcal{A})$*, then the matroid obtained from* $M(G, \mathcal{A})$ *by relaxing* C *is* $M(G, \mathcal{A} - \{C\})$*.*

Proof. Under the hypotheses, the rank of $M(G, \mathcal{A})$ is $|V(G)|$, so C must be a Hamiltonian cycle. Suppose that $\mathcal{A} - \{C\}$ is not a linear class of cycles. There must exist a theta subgraph G' of G such that $C \subseteq G'$ and every cycle of G' is in $\mathcal{A}$. Because C is a Hamiltonian cycle of G, there can be only one edge of G' not in C. This edge is in the closure of C in $M(G, \mathcal{A})$, contradicting the hypothesis that C is a flat. Thus $(G, \mathcal{A} - \{C\})$ is a biased graph, and the result follows easily. □

The next proposition follows from [17, Theorem 2].

Proposition 4.9. $U_{3,7}$ *is not a bias matroid.*

We have now assembled enough machinery to prove Theorem 1.5, which we restate here.

Theorem 4.10. *Every totally free bias matroid is a bicircular matroid.*

This will follow in a straightforward manner from the following result.

Lemma 4.11. *Suppose that* $(G, \mathcal{A})$ *is a biased graph such that* G *has no isolated vertices and* $|V(G)| \geq 4$*. If* $M(G, \mathcal{A})$ *is totally free, then* $\mathcal{A} = \varnothing$*.*

Proof. Let $M = M(G, \mathcal{A})$ be a counterexample with $|E(G)|$ minimal. By [6, Corollary 8.6], every totally free matroid is non-binary and hence non-graphic. Thus $\mathcal{A}$ does not contain every cycle of G. Clearly G is connected. Therefore $r(M) = |V(G)| = r(B(G))$. Now $B(G) \geq M$, and hence $B(G)$ is 3-connected by Proposition 4.6. Thus by Proposition 3.3 we have the following sublemma.

4.11.1. G *is* 2*-connected and has minimum degree at least three. Furthermore, no vertex of* G *is incident with more than one loop.*

Let C be a balanced cycle of $(G, \mathcal{A})$. We prove that C is a Hamiltonian cycle of G.

4.11.2. *Any link of* G *joins two vertices of* C*.*

Proof. Suppose otherwise, and let e be a link incident with at most one vertex of C. First assume that $|V(G)| > 4$. If either $M/e = M(G/e, \mathcal{A}/e)$ or $M\backslash e = M(G\backslash e, \mathcal{A}\backslash e)$ is totally free then, since C is in both $\mathcal{A}/e$ and $\mathcal{A}\backslash e$, the minimality of our counterexample is contradicted. Therefore we suppose that neither M/e nor $M\backslash e$ is totally free. The fact that $r(M) > 4$ implies that $|E(M)| \geq 5$. By Proposition 2.8 and Lemma 2.9 there is a unique clone e' of e in M, and $M/e\backslash e'$ is totally free. If e' is in C, then the cyclic flat spanned by C contains e' but not e. Therefore $e' \notin C$, so C is a member of $(\mathcal{A}/e)\backslash e'$. Again we have a contradiction to the minimality of our counterexample.

We must suppose that $|V(G)| = 4$. Now C does not meet every vertex of G, and since M contains no circuit with fewer than three elements, it follows that the length of C is three. Hence C is a triangle of M. Let any cyclic flat of M that has a non-empty intersection with C, but does not contain C, be known as a "bad" cyclic flat. C is a clonal triple of M by Proposition 2.6, and so M contains no bad cyclic flats. Suppose that the vertices and edges of C are labelled in order v_1, e_1, v_2, e_2, v_3, e_3. Let u be the vertex incident with e that is not in C. Since G is 2-connected u must be adjacent to at least two vertices of C. Without loss of generality suppose that u is joined to v_1 by the edge f_1, and to v_2 by the edge f_2. The cycle $\{e_1, f_1, f_2\}$ cannot be balanced, for then it would span a bad cyclic flat. The vertex u is not incident with a loop, for that loop and the edges e_1, f_1, f_2 would form a bicycle that spanned a bad cyclic flat. Similarly there are no edges parallel to f_1 or f_2. Since u has degree at least three, it is adjacent to v_3. It is easy to show that u and v_3 are joined by exactly one edge. A loop incident with a vertex of C, or an edge parallel to an edge of C, would create a bicycle that spans a bad cyclic flat. Thus G is isomorphic to K_4. Furthermore, we can show that C must be the only balanced cycle. Now it is easily seen that $M = M(G, \mathcal{A})$ is not 3-connected. This contradiction completes the proof of the sublemma and shows that C meets every vertex of G. □

Since C was chosen arbitrarily from the balanced cycles of $(G, \mathcal{A})$, we have in fact proved the following result.

4.11.3. *Every balanced cycle of* $(G, \mathcal{A})$ *is a Hamiltonian cycle.*

4.11.4. *C is a circuit-hyperplane of M.*

Proof. C is certainly a circuit of rank $|V(G)| - 1 = r(M) - 1$. Suppose that e is an edge of G not in C and that $e \in \mathrm{cl}_M(C)$. Lemma 4.4 implies that $C \cup e$ is a bicycle and that every cycle in $C \cup e$ is balanced. We first consider the case where $|V(G)| = 4$. Since M contains no circuits of size less than three, it follows that e must be a link, and that e must join two diagonally opposite vertices of C. But then e lies in the intersection of two balanced cycles of G that are of length three, and hence e is in the intersection of two distinct non-trivial lines of M. This is a contradiction by Corollary 2.7. Therefore we may assume that $|V(G)| > 4$.

Lemma 4.4 implies that e is fixed in M. Therefore M/e is totally free by Proposition 2.10. Let C' be one of the cycles of $G[C \cup e]$ that contain e. The number of edges in C' is greater than one, for otherwise e is a loop of M. Thus $C' - e$ is a balanced cycle of $(G/e, \mathcal{A}/e)$, and, since $|V(G/e)| \geq 4$, the minimality of our counterexample is contradicted. □

Since C was chosen arbitrarily, we have again proved a more general result.

4.11.5. *Every balanced cycle of* $(G, \mathcal{A})$ *is a circuit-hyperplane.*

4.11.6. *Every vertex of G is incident with at least four edges.*

Proof. Suppose that v is incident with exactly three edges. Proposition 4.2 implies that $\mathrm{st}_G(v)$ contains a cocircuit. Since M is 3-connected, $\mathrm{st}_G(v)$ must be a triad of M. Therefore $\mathrm{st}_G(v)$ is a clonal triple of M by Proposition 2.6. But C is a cyclic flat of M that meets exactly two edges in $\mathrm{st}_G(v)$, so Proposition 2.1 implies that $\mathrm{st}_G(v)$ cannot be a clonal triple. □

4.11.7. $|V(G)| > 4$.

Proof. Suppose that G has exactly four vertices and that the vertices and edges of C are labelled in order $v_1, e_1, \ldots, v_4, e_4$. Note that 4.11.3 implies that every cycle of length at most three is unbalanced.

First suppose that some edge in C has two distinct parallel edges. This set of three parallel edges is a triangle of M, and therefore a clonal triple by Proposition 2.6. However, C is a cyclic flat that meets this triangle in exactly one edge. This contradiction shows that every edge in C has at most one parallel edge.

Let us suppose that there is an edge, e, joining v_1 to v_3, and an edge, e', joining v_2 to v_4. We will show that, in this case, none of the edges in C has a parallel edge. Suppose, without loss of generality, that e_1 is parallel to another edge, e_1'. Both $\{e_1, e_1', e_2, e\}$ and $\{e_1, e_1', e_4, e'\}$ are circuits of M, so Proposition 2.11 implies that e_1 and e_1' are clones. However C is a cyclic flat that contains e_1 and not e_1'. Therefore no edge in C has a parallel edge.

Assume that G contains a loop. Without loss of generality v_1 is incident with a loop, l. Both $\{e_1, e_2, e, l\}$ and $\{e_3, e_4, e, l\}$ are circuits. Again we can use Proposition 2.11 to show that e and l must be clones. But $\{e_1, e_4, e', l\}$ is a circuit, and the closure of this circuit contains l but not e. Therefore G contains no loops.

Since the minimum degree of G is at least four, it follows that both e and e' must have at least one parallel edge. Let f be a parallel edge of e. Observe that $\{e_1, e_2, e, f\}$ is a circuit. Since C meets this circuit in $\{e_1, e_2\}$, e_1 and e_2 must be clones. But if we let f' be a parallel edge of e', we see that $\{e_1, e_4, e', f'\}$ is a circuit, and the closure of this circuit contains e_1 but not e_2.

Our assumption that there were edges joining v_1 to v_3, and v_2 to v_4 has lead to a contradiction. Therefore we will assume that v_1 and v_3 are not adjacent.

Suppose that v_1 is incident with a loop, l. There are at least four edges incident with v_1, and there can be at most one loop incident with v_1 by 4.11.1 so either e_1 or e_4 has a parallel edge. We will assume the former. Let e_1' be a parallel edge of e_1. The set $\{e_1, e_1', l\}$ is a triangle of M and hence a clonal triple by Proposition 2.6. But C is a cyclic flat of M that contains e_1 but not e_1'. Therefore v_1 is incident with no loops.

Since v_1 is incident with no loops, it follows that e_1 and e_4 must both have exactly one parallel edge. By applying the arguments of the last two paragraphs to v_3 we can show that e_2 and e_3 must also have exactly one parallel edge each.

For $1 \leq i \leq 3$ let e'_i be the parallel edge of e_i. The sets $\{e_1, e'_1, e_2, e'_2\}$ and $\{e_2, e'_2, e_3, e'_3\}$ are both circuits of M, so we may again apply Proposition 2.11 to show that e_2 and e'_2 are clones. This is a contradiction, as C contains e_2 but not e'_2. □

4.11.8. *If $e \in C$, then e has a unique clone in M.*

Proof. First suppose that there exists a balanced cycle $C' \neq C$ of $(G, \mathcal{A})$ such that $e \notin C'$. Now C' is a member of $\mathcal{A}\backslash e$, and $C - e$ is a member of $\mathcal{A}/e$. Since G has more than four vertices, if either $M\backslash e = M(G\backslash e, \mathcal{A}\backslash e)$ or $M/e = M(G/e, \mathcal{A}/e)$ is totally free, then we have a contradiction to the minimality of our counterexample. Hence, by Proposition 2.8 and Lemma 2.9, there must be some unique clone e' of e in M.

Therefore we assume that e is in every balanced cycle of $(G, \mathcal{A})$. Let $\mathcal{A} = \{C_1, \ldots, C_n\}$. By 4.11.3 and 4.11.5 every balanced cycle is Hamiltonian and a circuit-hyperplane of M.

We wish to show that M/e is 3-connected. We begin by demonstrating that $B(G/e)$ is 3-connected. Firstly, G/e is 2-connected for it contains a Hamiltonian cycle. Furthermore, no vertex of G/e is incident with more than one loop, for in that case one of the following situations must occur in G:

(i) there are two edges parallel to e.
(ii) there is one edge parallel to e, and a loop incident with an end vertex of e; or,
(iii) each end vertex of e is incident with a loop.

In any of these cases we can find a triangle of M that meets the cyclic flat C in exactly one edge. This is a contradiction by Proposition 2.6. Since G has minimum degree at least four, it follows that the minimum degree of G/e is at least four. Thus it follows by Proposition 3.3 that $B(G/e)$ is 3-connected.

The only balanced cycles of $(G/e, \mathcal{A}/e)$ are the cycles $\{C_1 - e, \ldots, C_n - e\}$ and each of these is a circuit-hyperplane of M/e. The rank of M/e is $r(M) - 1 = |V(G)| - 1 = |V(G/e)|$, so $\mathcal{A}/e$ cannot contain every cycle of G/e. Thus, by Proposition 4.8, we can obtain $B(G/e)$ by relaxing in turn each of the n circuit-hyperplanes $\{C_1 - e, \ldots, C_n - e\}$. Let $M_0 = M/e$ and for $i \in \{1, \ldots, n\}$ let M_i be the matroid obtained by relaxing $C_i - e$ in M_{i-1}. Then $M_n = B(G/e)$. If M/e is not 3-connected, then there must be some integer j such that M_{j-1} is not 3-connected, but M_j is. Let (X, Y) be a k-separation of M_{j-1} where $k < 3$. Then, without loss of generality, $X = C_j - e$ by Proposition 4.7.

Since every vertex of G/e is incident with at least four edges, every vertex of G/e is incident with at least two edges not in $C_j - e$. If $G/e\backslash(C_j - e)$ is connected, then let D be a spanning tree. Otherwise, since every vertex has degree at least two in $G/e\backslash(C_j - e)$, for each connected component we can find a spanning set that contains exactly one cycle. In this case let D be the union of such spanning sets. In either case D is independent in M_{j-1}, for the only balanced cycles of M_{j-1} are Hamiltonian cycles, and D contains no such cycle and no bicycles. Therefore the rank of D is at least $|V(G/e)| - 1 \geq 3$. But $D \subseteq Y$, so $r(Y) \geq 3$. Now $r(X)+r(Y) = r(C_j - e)+r(Y) \geq r(M_{j-1})+2$, contradicting the fact that (X, Y) is a k-separation of M_{j-1} where $k < 3$. Thus we must assume that M/e is 3-connected.

If $M/e = M(G/e, \mathcal{A}/e)$ is totally free, then we have a contradiction to our assumption of minimality, for $C - e$ is a member of $\mathcal{A}/e$. Therefore M/e is 3-connected and not totally free. Hence e has a unique clone in M by Lemma 2.9. □

Suppose that e is an edge of C. Since C is a cyclic flat that contains e, the unique clone of e must also be in C.

Let e and e' be a clonal pair contained in C. Let P be one of the paths contained in C that has e and e' as its end edges. Furthermore, assume that e and e' have been chosen from the clonal pairs in C so that P is as short as possible. Let the internal vertices of P be $u_1, \ldots, u_s$. Let R be the path obtained by deleting P from C, and let the vertices of R be $v_1, \ldots, v_t$ (so e joins u_1 to v_1, and e' joins u_s to v_t). Note that our assumption about the length of P means that $t \geq s$.

The remainder of the proof of Lemma 4.11 will consist of finding a bicycle that spans a cyclic flat containing exactly one of e and e'. For the sake of brevity we will refer to such a bicycle as a *bad bicycle*. Since e and e' are clones, showing that a bad bicycle must exist will lead to a contradiction and complete the proof of the lemma. Throughout the rest of the proof we will be making use of the fact that any non-Hamiltonian cycle must be unbalanced.

First, suppose that u_1 is incident with a loop, l. Since u_1 is incident with at least four edges, at most one of which is a loop, u_1 is incident with a link, $f \notin C$.

Assume that f joins u_i to another vertex $u_i \in \{u_1, \ldots, u_s\}$. Let P_1 be the path contained in P that joins u_1 to u_i. The path P_1 is non-empty, as f is not a loop. The set containing P_1, f and l is a tight handcuff, so it spans a cyclic flat. This cyclic flat meets C exactly in P_1, so if g is an edge in P_1, the unique clone of g must also be in P_1. But then the path joining g to its clone is shorter than P, as P_1 is properly contained in P. This contradicts our hypothesis on e and e' being chosen to make P as short as possible.

We will now assume that f joins u_1 to some vertex $v_i \in \{v_1, \ldots, v_t\}$. Let P_2 be the path contained in R that joins v_1 to v_i. Note that f cannot be parallel to e or e', as this would create a triangle of M that meets, but is not contained in, C. The set containing P_2, e, f and l is a bad bicycle. Therefore we will assume that u_1 is not incident with a loop.

Since u_1 is incident with at least four edges, there are links, f and f', that are incident with u_1 and are not in C.

Suppose that f and f' join u_1 to vertices in $\{u_2, \ldots, u_s\}$. We may assume, without loss of generality, that there exist integers $1 < i \leq j \leq s$ such that f is incident with u_i and f' is incident with u_j. Let P_3 be the path contained in P that joins u_1 to u_j. The set $\{P_3, f, f'\}$ is a bicycle and the cyclic flat spanned by this bicycle meets C exactly in P_3. Since P_3 is non-empty and properly contained in P, we may again find a clonal pair in C that is joined by a path shorter than P.

We will now assume that exactly one of $\{f, f'\}$ joins u_1 to a vertex in $\{u_1, \ldots, u_s\}$. Assume that f joins u_1 to u_i, and that f' joins u_1 to v_j. Let P_4 be the path contained in P that joins u_1 to u_i, and let P_5 be the path contained in R that joins v_1 to v_j. The set $\{P_4, P_5, e, f, f'\}$ is a bad bicycle unless $i = s$ and $j = t$. In this case $\{P_4, P_5, e, f, f'\}$ spans M, so its closure certainly contains e'. However, if this is the case, then $\{P_4, e', f, f'\}$ is a bad bicycle.

We must assume that both f and f' join u_1 to vertices in $\{v_1, \ldots, v_t\}$. We may assume without loss of generality that f joins u_1 to v_i and that f' joins u_1 to v_j where $i \leq j$. Let P_6 be the path contained in R that joins v_1 to v_j. The set $\{P_6, e, f, f'\}$ is a bad bicycle unless $s = 1$ and $j = t$. Let us assume that this is the case. Note that e and e' are adjacent in C, and that f' is parallel to e'. Let P_7 be the path contained in R that joins v_i to v_t. If $i \neq 1$ then $\{P_7, e', f, f'\}$ is a bad bicycle. Therefore we will assume that $i = 1$.

We have shown that e and e' are adjacent in C, that f is parallel to e, and that f' is parallel to e'. Note that $|V(G)| = t + s \geq 5$, and $s = 1$, so $t \geq 4$. Each vertex of G is incident with at least four edges, at most one of which is a loop, so v_2 is incident with a link h. If h joins v_2 to u_1 or to v_1, then the set containing e, f, h and the edge v_1v_2 is a bad bicycle. Therefore we assume that h joins v_2 to some vertex $v_i \in \{v_3, \ldots, v_t\}$. Let P_8 be the path contained in R that joins v_1 to v_i. We observe that if $i \neq t$ then $\{P_8, e, f, h\}$ is a bad bicycle. However, if $i = t$, then we may let P_9 be the path contained in R that joins v_2 to v_t. We conclude the proof of Lemma 4.11 by noting that $\{P_9, e', f', h\}$ is a bad bicycle. □

Proof of Theorem 4.10. Suppose that $M = M(G, \mathcal{A})$ is a totally free bias matroid. If the rank of M is at least four, then $|V(G)| \geq 4$, and so $\mathcal{A}$ is empty by Lemma 4.11. Thus $M = B(G)$. Therefore we need only consider the case that $r(M) \leq 3$. The only rank-2 totally free matroids are the uniform matroids with at least four elements, and these are all bicircular, so we may assume that $r(M) = 3$.

All non-trivial lines of M are disjoint by Corollary 2.7. It is now straightforward to show that either M has a $U_{3,7}$-minor, which contradicts Proposition 4.9; or M is a restriction of a matroid whose ground set can be partitioned into three lines. In this case, M is easily seen to be bicircular. □

Theorem 3.7 and Theorem 4.10 enable us to prove the main result.

Proof of Theorem 1.3. Let $\mathcal{M}$ be the family of GF(q)-representable bias matroids that have no Δ_r-minor. By Theorem 4.10 any totally free matroid in $\mathcal{M}$ is a bicircular matroid, and by Theorem 3.7 there can be only a finite number of such matroids. The theorem now follows by Lemma 2.5. □

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